MATH G9905 RESEARCH SEMINAR IN NUMBER THEORY (FALL 2014)

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INTRODUCTION TO SPECIAL VALUES OF L-FUNCTIONS AND THEIR p-ADIC INTERPOLATION, II

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Abstract. The aim of these talks is to give an introduction to the theory of L-functions from an arithmetic point of view and to explain how to construct their p-adic avatars. We shall begin by explaining the values taken by Riemann's zeta function at integers. Subsequently, we shall try to give a conjectural description of the values at certain integers for other L-functions, such as the one of an elliptic curve.

In the second part, we study how the values of an L-function vary p-adically and we explain what the conjectural p-adic L-function should look like. We conclude with some general conjectures on the properties of p-adic L-functions, with special emphasis on the so-called trivial zeros.

Last time we saw that the L-functions $L(M, s)$ are very nice, and for certain integers $n \in \mathbb{Z}$ (depending on M) we can find an almost canonical complex number $c^+(M, n)$ such that

$$\frac{L(M, n)}{c^+(M, n)} \in \mathbb{Q}.$$

Today we will study the p-adic properties of these rational numbers.

The values of the Riemann zeta function at the negative integers are

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \in \mathbb{Q}.$$

Since $\mathbb{N} \subset \mathbb{Z}_p$ is dense, we may ask the following question: does there exist $\zeta_p(s)$ ($s \in \mathbb{Z}_p$) such that $\zeta_p(n) = \zeta(-n)$ for all $n \in \mathbb{N}$? The answer is almost yes.

Recall $f : \mathbb{Z}_p \to \mathbb{Q}_p$ is continuous if and only if for every $M > 0$, there exists $N > 0$ such that

$$f(x) \equiv f(y) \pmod{p^M}$$

when $x \equiv y \pmod{p^N}$. This is just the usual ϵ-δ definition translated into congruences. We want to prove the congruence properties for $\frac{B_{n+1}}{n+1}$. Consider the Taylor expansion

$$\frac{te^{xt}}{e^t - 1} = \sum_{n \geq 0} B_n(x) \frac{t^n}{n!}$$

with the following properties:

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- $B_n(0) = B_n$ is the n-th Bernoulli number;
- $B_n(x)$ is a polynomial of degree n;
- $B_1(x) = x - \frac{1}{2}$.

We will use these to study the congruence properties.

Let $\mathcal{LC}(\mathbb{Z}_p)$ be the space of locally constant functions $\mathbb{Z}_p \to \mathbb{Q}_p$.

Definition 1. $\mu : \mathcal{LC}(\mathbb{Z}_p) \to \mathbb{Q}_p$ is a distribution if for all compact open U of $\mathbb{Z}_p$ with $U = \coprod_i U_i$, we have $\mu(U) = \sum \mu(U_i)$. μ is a measure if it is a bounded distribution, i.e. $|\mu(U)|_p \leq C$ for all U.

If μ is a measure and $f \in \mathcal{LC}(\mathbb{Z}_p, \mathbb{Q}_p)$, we define

$$\int_{\mathbb{Z}_p} f \, d\mu = \varinjlim \sum_{\substack{\coprod U = \mathbb{Z}_p \\ a_U \in U}} f(a_U)\mu(U).$$

Let

$$\mu_n(a + p^N\mathbb{Z}_p) = p^{N(n-1)} B_n\left(\frac{a}{p^N}\right).$$

Proposition 2. *For all $n \geq 1$ and $a \in \mathbb{Z}_p$,*

$$\mu_n(a + p^N\mathbb{Z}_p) = \sum_{b \equiv a \pmod{p^{N+1}}} \mu_n(b + p^{N+1}\mathbb{Z}_p)$$

This shows that μ_n is a distribution, but it is not a measure. The problem is that this function is not bounded at all. Since B_n is a polynomial of degree n,

$$B_n\left(\frac{a}{p^N}\right) \sim \left(\frac{a}{p^N}\right)^n$$

so μ_n can have arbitrarily large p-adic norm.

Take $\alpha \in \mathbb{Z}_p^\times \backslash \{n\}$, and let

$$\mu_{n,\alpha}(a + p^N\mathbb{Z}_p) = \mu_n(a + p^N\mathbb{Z}_p) - \alpha^{-n}\mu_n(\alpha(a + p^N\mathbb{Z}_p)).$$

For example,

$$\mu_{1,\alpha}(a + p^N\mathbb{Z}_p) = \frac{a}{p^N} - \frac{1}{2} - \alpha^{-1}\frac{[\alpha a]}{p^N} + \frac{1}{2}\alpha^{-1} = \frac{a - \alpha^{-1}[\alpha a]}{p^N} - \frac{1}{2}(1 - \alpha^{-1})$$

where $[\alpha a] \in \mathbb{Z}$ is such that $0 \leq [\alpha a] < p^N$ and $[\alpha a] \equiv \alpha a \pmod{p^N}$.

Theorem 3. *For all $n \geq 1$ and $\alpha \in \mathbb{Z}_p^\times \backslash \{1\}$, $\mu_{n,\alpha}$ is bounded. Moreover, we have $d_n \in \mathbb{N}$ such that $d_n\mu_{n,\alpha}$ has values in $\mathbb{Z}_p$, and*

$$d_n\mu_{n,\alpha}(a + p^N\mathbb{Z}_p) \equiv na^{n-1}d_n\mu_{1,\alpha}(a + p^N\mathbb{Z}_p) \pmod{p^N}.$$

This means we can more or less recover these measures from the first one.

Corollary 4.

$$\int_{\mathbb{Z}_p} d\mu_{n,\alpha} = n\int_{\mathbb{Z}_p} x^{n-1} \, d\mu_{1,\alpha}(x).$$

Note

$$\begin{aligned}\int_{\mathbb{Z}_p^\times} x^{n-1}\,d\mu_{1,\alpha} &= \int_{\mathbb{Z}_p} x^{n-1}\,d\mu_{1,\alpha} - \int_{p\mathbb{Z}_p} x^{n-1}\,d\mu_{1,\alpha}\\ &= \frac{B_n(0)}{n} - \alpha^{-n}\frac{B_n(0)}{n} - p^{n-1}\left(\frac{B_n(0)}{n} - \alpha^{-n}\frac{B_n(0)}{n}\right)\\ &= (1-\alpha^{-n})(1-p^{n-1})\frac{B_n}{n}\\ &= -(1-\alpha^{-n})(1-p^{n-1})\zeta(1-n).\end{aligned}$$

This tells us that to p-adically interpolate the Riemann zeta function, we have to remove the Euler factor at p.

Now we introduce the Mellin transform. Recall there is an isomorphism $1+p\mathbb{Z}_p \xrightarrow{\sim} \mathbb{Z}_p$: every $x \in 1+p\mathbb{Z}_p$ can be written as $x = (1+p)^\gamma$ with $\gamma \in \mathbb{Z}_p$. If μ is a measure,

$$\int_{\mathbb{Z}_p} (1+T)^\gamma\,d\mu(\gamma) = \sum_n \left(\int_{\mathbb{Z}_p} \binom{\gamma}{n}\,d\mu(\gamma)\right) T^n \in \mathbb{Q}_p \otimes \mathbb{Z}_p[[T]]$$

where $\binom{\gamma}{n} = \frac{\gamma(\gamma-1)\cdots(\gamma-n+1)}{n!}$. Mahler's theorem says that the binomial polynimals give a basis of locally constant functions. Thus we have an isomorphism

$$\text{Meas} \xrightarrow{\sim} \mathbb{Q}_p \otimes \mathbb{Z}_p[[T]] : \mu \mapsto F_\mu(T).$$

If $1+T = (1+p)^s$ where $s \in \mathbb{Z}_p$, then

$$F_\mu((1+p)^s - 1) = \int_{\mathbb{Z}_p} (1+p)^{\gamma s}\,d\mu(\gamma) = \int_{1+p\mathbb{Z}_p} x^s\,d\mu(x).$$

This is the p-adic Mellin transform. We say that a function $G(s)$ is an Iwasawa function if

$$G(s) = F((1+p)^s - 1)$$

with $F(T)$ a formal series with coefficients in $\mathbb{Z}_p$.

Recall we have a map $\omega : \mathbb{Z}_p^\times \to \mu_{p-1}$ which factors through $\mathbb{F}_p^\times \xrightarrow{\sim} \mu_{p-1}$, and every $x \in \mathbb{Z}_p^\times$ can be written as $\omega(x)\langle x\rangle$ where $\langle x\rangle \in 1+p\mathbb{Z}_p$.

The measure $\mu_{1,\alpha}$ gives $p-1$ Iwasawa functions: for $i \in [0,\cdots,p-2]$,

$$\zeta_{p,i}^\alpha(s) = \int_{\mathbb{Z}_p^\times} \omega(x)^{1-i}\langle x\rangle^{s-1}\,d\mu_{1,\alpha}(x)$$

for $s \in \mathbb{Z}_p$.

Theorem 5. *For all $n \equiv i \pmod{p-1}$ and $n \geq 0$, we have*

$$\zeta_{p,i}^\alpha(n) = (1-\alpha^{1-n})(1-p^{-n})\zeta(-n).$$

For $i \not\equiv 0$, $\frac{\zeta_{p,i}^\alpha(s)}{1-\alpha^{1-s}}$ is holomorphic. $\frac{\zeta_{p,0}^\alpha(s)}{1-\alpha^{1-s}}$ has a simple pole at $s=0$ with residue

$$\mathrm{Res}_{s=0}\left(\frac{\zeta_{p,0}^\alpha(s)}{1-\alpha^{1-s}}\right) = 1-p^{-1} = (1-p^{-1})\,\mathrm{Res}_{s=0}\,\zeta(s).$$

Thus we can call this function the p-adic avatar of the Riemann zeta function.

Corollary 6. *If* $\epsilon : 1 + p\mathbb{Z}_p \to \overline{\mathbb{Q}_p}$ *has finite order, then*

$$\int_{\mathbb{Z}_p^\times} \epsilon(x)\omega^{1-i}(x)\langle x\rangle^{n-1}\, d\mu_{1,\alpha} = (1-\alpha^{n-1})(1-(\epsilon\omega^{1-i})_0(p)p^n)L(-n,\epsilon\omega^{n-i})$$

where $L(-n, \epsilon\omega^{n-i}) = \sum_m \frac{\epsilon\omega^{n-i}(m)}{m^s}$.

This takes care the p-adic L-functions for Dirichlet characters.

For a motive M (M_{B}, M_{dR} and M_ℓ for all ℓ prime), we get an L-function $L(M,s)$. For m critical we get $c^+(M,n)$.

Conjecture 7.

$$\frac{L(M,n)}{c^+(M,n)} \in \mathbb{Q}.$$

Can we do a p-adic interpolation of $L(M,n)$? The first problem is we do not have many critical integers, by comparing $\Gamma(s-p)$ and $\Gamma(1-s+i-p)$. So we need to let the motive M vary.

Let $\epsilon : 1 + p\mathbb{Z}_p \to \overline{\mathbb{Q}}^\times$ be of finite order. Then we can form the twist $M \otimes \epsilon$. If

$$L(M,s) = \sum_m \frac{a_m}{m^s},$$

then

$$L(M\otimes\epsilon, s) = \sum_m \frac{a_m\epsilon(m)}{m^s}.$$

How does $\frac{L(M\otimes\epsilon,n)}{c^+(M,n)}$ vary p-adically, i.e. if $\epsilon_1 \equiv \epsilon_2$, then is

$$\frac{L(M\otimes\epsilon_1,n)}{c^+(M,n)} \equiv \frac{L(M\otimes\epsilon_2,n)}{c^+(M,n)}?$$

Example 8. Let $E/\mathbb{Q}$ be an elliptic curve with good ordinary reduction at p. Thus $p \nmid a_p(E)$, and the Hecke polynomial $X^2 - a_p(E)X + p = (X-\alpha_p)(X-\beta_p)$ with $v_p(\alpha_p) = 0$ and $v_p(\beta_p) = 1$.

Theorem 9. *For each choice of* $\delta = \alpha$ *or* β, *we have* $L_p^\delta(E,s)$ *such that*

$$L_p^\delta(E, \epsilon(1+p)-1) = G_\epsilon \delta^{-M}\frac{L(E\otimes\epsilon,1)}{\Omega_E^+}$$

where ϵ *has conductor* p^M. *In particular, if we take the trivial character,*

$$L_p^\delta(E,0) = (1-\delta^{-1})^2\frac{L(E,1)}{\Omega_E^+}.$$

We know that the L-function has a complex representation

$$L(E,s) = \int_0^{i\infty} f(y)y^s\, dy.$$

We can do something similar p-adically and obtain $L_p^\delta(E,s)$.

We have two p-adic L-functions for E, because we can make two different choices of "eigenvalues at p".

Let us know return to motives. We consider the Galois representation

$$M_p : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{Q}_p).$$

Suppose M_p is semistable (á la Fontaine). For a fixed integer $d \in \mathbb{N}$ (depending on M) choose d "good" eigenvalues $D = \{\alpha_1, \cdots, \alpha_d\}$ of the Frobenius at p for the representation M_p. D can be thought of as the p-adic equivalent of $\mathrm{Fil}^\bullet M_{\mathrm{dR}}$.

Conjecture 10 (Perrin-Riou, Coates)**.** *Associated to M and D we have a p-adic L-function $L_p^D(M,s)$ such that*

$$L_p^D(M, \epsilon(1+p)-1) = G_\epsilon C_D \frac{L(M\otimes\epsilon, n)}{c^+(M,n)}.$$

In the final part, I will give another way of constructing p-adic L-functions which does not directly use distributions.

Let $f \in S_k(\Gamma_0(N))$ and

$$L(f,s) = \prod_\ell [(1-\alpha_\ell \ell^{-s})(1-\beta_\ell \ell^{-s})]^{-1}.$$

We want to interpolate

$$L(\mathrm{Sym}^2 f, s) = \prod_\ell [(1-\alpha_\ell^2 \ell^{-s})(1-\alpha_\ell\beta_\ell \ell^{-s})(1-\beta_\ell^2 \ell^{-s})]^{-1}.$$

Fix a character ϵ. Then

$$L(\mathrm{Sym}^2 f \otimes \epsilon, 1) = \langle f, \Theta(\epsilon) E_k(\epsilon)\rangle$$

where

$$\Theta(\epsilon) = \sum_{\mathbb{Z}} \epsilon(n) q^{n^2}$$

and

$$E_k(\epsilon) = \sum_n L(0, \epsilon\sigma_n) P(n) q^n$$

where σ_n is the quadratic character associated with $\mathbb{Q}(\sqrt{n})/\mathbb{Q}$. For ϵ_1 and ϵ_2 such that $\epsilon_1|_{1+p^N\mathbb{Z}_p} = \epsilon_2|_{1+p^N\mathbb{Z}_p}$, we have

$$\Theta(\epsilon_1)E_k(\epsilon_1) \equiv \Theta(\epsilon_2)E_k(\epsilon_2) \pmod{p^N}.$$

Let

$$\Theta E_k(s) = \left(\sum_{\substack{n\in\mathbb{Z}\\(n,p)\neq 1}} n^s q^{n^2}\right)\left(\sum_n L_p(s,\sigma_n) P_n(n^s) q^n\right),$$

which is a formal series in q with Iwasawa functions as coefficients.

Define $\ell_f : M_k(\Gamma_0(N)) \to \mathbb{C}$ by

$$g \mapsto \frac{\langle f, g\rangle}{\langle f, f\rangle}.$$

ℓ_f is defined over a number field K, so we have $\ell_f : M_k(\Gamma_0(N), K) \to K$. Then ℓ_f extends to a linear form $M_k(\Gamma_0(N), K_p) \to K_p$ where K_p is the local field of K at a p-adic place.

Hypothesis. f is of finite slope for U_p if f has level divisible by p and

$$f|U_p = \sum a_{np}(f)q^n = \alpha f$$

for some $\alpha \neq 0$.

We can now define the p-adic L-function

$$L_p^\alpha(\mathrm{Sym}^2 f, s) = \ell_f(\Theta E_k(s)).$$

For $s = \epsilon(p+1) - 1$,

$$L_p^\alpha(\mathrm{Sym}^2 f, \epsilon(1+p) - 1) = G_\epsilon \alpha^{-2 \operatorname{cond} \epsilon} \frac{L(\mathrm{Sym}^2 f \otimes \epsilon, 1)}{\langle f, f \rangle}.$$

We have

$$\frac{\langle f, f \rangle}{c^+(\mathrm{Sym}^2 f, 1)} \in \overline{\mathbb{Q}}.$$

This construction also works in families, where we get a big L-function in two variables.