MATH G9905 RESEARCH SEMINAR IN NUMBER THEORY (FALL 2014)

LECTURE 6 (OCTOBER 14, 2014) GIOVANNI ROSSO

INTRODUCTION TO SPECIAL VALUES OF L-FUNCTIONS AND THEIR p-ADIC INTERPOLATION, I

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Abstract. The aim of these talks is to give an introduction to the theory of L-functions from an arithmetic point of view and to explain how to construct their p-adic avatars. We shall begin by explaining the values taken by Riemann's zeta function at integers. Subsequently, we shall try to give a conjectural description of the values at certain integers for other L-functions, such as the one of an elliptic curve.

In the second part, we study how the values of an L-function vary p-adically and we explain what the conjectural p-adic L-function should look like. We conclude with some general conjectures on the properties of p-adic L-functions, with special emphasis on the so-called trivial zeros.

This talk will be mostly complex-analytic in nature. p-adic L-functions will be introduced in the next talk.

Why do we study L-values?

Let $(a, b) \in \mathbb{Z}^2$. What is the probability that $\gcd(a, b) = 1$? Since 2 cannot divide both a and b, we have

$$P(2 \mid a \ \& \ 2 \mid b) = \frac{1}{4}$$

and similarly for $3, 5, \cdots$, so

$$P(\gcd(a, b) = 1) = \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right) \cdots = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

Recall the Riemann zeta function

$$\zeta(s) = \prod \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Theorem 1. *If $n > 0$, then*

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2m} (2\pi)^{2n}}{2(2n)!}$$

where $B_{2m} \in \mathbb{Q}$ is the Bernoulli number, given by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

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For the odd positive integers , we know $\zeta(3) \notin \mathbb{Q}$ and at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11) \notin \mathbb{Q}$. In general, we don't know much about $\zeta(2n+1)$.

For the negative integers, we have

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \in \mathbb{Q},$$

where $B_{2n+1} = 0$.

To explain the factor of $2\pi i$, we need a bit of geometry. Let

$$\mathbb{G}_m = \operatorname{Spec}(\mathbb{Z}[X, Y]/(XY - 1)).$$

If ℓ is a prime number, we let

$$\mathbb{Z}_\ell(1) = \varprojlim \mathbb{G}_m(\overline{\mathbb{Q}})[\ell^n] = \varprojlim \mu_{\ell^n}.$$

Then we will see that the corresponding L-function is

$$L(\mathbb{Z}_\ell(1), s) = \prod_q (1 - q^{-s-1}) = \zeta(s+1).$$

In general, for $\mathbb{Z}_\ell(m)$, we have

$$L(\mathbb{Z}_\ell(m), s) = \zeta(s+m).$$

Recall the real gamma function

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2).$$

Then we have the functional equation

$$\zeta(s)\Gamma_{\mathbb{R}}(s) = \Gamma_{\mathbb{R}}(1-s)\zeta(1-s)$$

for all $s \in \mathbb{C} - \{0, 1\}$.

Remark. If $s = -2n$ where $n > 0$, then $\Gamma_{\mathbb{R}}(-2n) = \infty$, so $\zeta(-2n) = 0$.

Consider $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$. If we consider $\mathbb{C}$ as a topological space, we have the singular homology

$$H_1(\mathbb{C}^\times) \simeq \mathbb{Z}$$

and de Rham cohomology

$$H^1_{\mathrm{dR}}(\mathbb{C}^\times) \simeq \mathbb{Z}$$

upon fixing a generator $\frac{dz}{z}$. These two groups are related by the isomorphism $H^1_{\mathrm{dR}}(\mathbb{C}^\times) \simeq (H_1(\mathbb{C}^\times))^\vee$ via $H^1_{\mathrm{dR}}(\mathbb{C}^\times) \times H_1(\mathbb{C}^\times) \to \mathbb{C}$ given by

$$\langle \omega, \gamma \rangle = \int_\gamma \omega.$$

Note

$$\int_{S^1} \frac{dz}{z} = 2\pi i.$$

The complex conjugation F_∞ acts on $H_1(\mathbb{C}^\times)$ via $S^1 \to -S^1$.

We want to generalize this to $\zeta(n) = L(\mathbb{Z}_\ell(n), 0)$. Let

$$H_1(\mathbb{C}^\times(n)) := (2\pi i)^n H_1(\mathbb{C}^\times)$$

with action of F_∞ given by $(-1)^n$, and

$$H^1_{\mathrm{dR}}(\mathbb{C}^\times(n)) := H^1_{\mathrm{dR}}(\mathbb{C}).$$

If n is even, F_∞ is trivial on $H_1(\mathbb{C}^\times(n))$; if n is odd, F_∞ is -1 on it.

We are ready to explain the $2\pi i$ in $\zeta(n)$ for n positive even:

$$(2\pi i)^n = \det\left(H_1(\mathbb{C}^\times(n))^{\vee, F_\infty=1} \xrightarrow{\sim} H^1_{\mathrm{dR}}(\mathbb{C}^\times(n))\right).$$

For $\zeta(n)$ with n negative odd, we have

$$1 = \det\left(H_1(\mathbb{C}^\times(n))^{\vee, F_\infty=1} \to H^1_{\mathrm{dR}}(\mathbb{C}^\times(n))\right)$$

because $H_1(\mathbb{C}^\times(n))^{\vee, F_\infty=1} = 0$.

To generalize this to higher-dimensional L-functions,

Definition 2. A motive M is the data of

- M_{B} a vector space over $\mathbb{Q}$ with F_∞ involution and a Hodge decomposition $M_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{p,q\in\mathbb{Z}} H^{p,q}$ such that $F_\infty(H^{p,q}) = H^{q,p}$,
- M_{dR} a vector space over $\mathbb{Q}$ with filtration $\mathrm{Fil}^i M_{\mathrm{dR}}$ on $M_{\mathrm{dR}} \otimes \mathbb{C}$,
- for all prime ℓ, M_ℓ a Galois representation of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over a $\mathbb{Q}_\ell$-vector space,

which are compatible in the following sense:

- $M_{\mathrm{B}} \otimes \mathbb{C} \simeq M_{\mathrm{dR}} \otimes \mathbb{C}$ such that $\bigoplus_{p'>p} H^{p',q} \to \mathrm{Fil}^p M_{\mathrm{dR}}$;
- $M_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C} \simeq M_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell}$ for a fixed $\mathbb{C} \simeq \overline{\mathbb{Q}_\ell}$;
- the M_ℓ's form a compatible system: for all $q \nmid \ell\ell'$,
$$\det(1 - T\,\mathrm{Fr}_q\,|M_\ell^{I_q}) = \det(1 - T\,\mathrm{Fr}_q\,|M_{\ell'}^{I_q}) \in \mathbb{Z}_\ell[T]$$
where the geometric Frobenius Fr_q is given by $\mathrm{Fr}_q^{-1}(x) \equiv x^q \pmod{q}$.

This is quite abstract. Let X be an algebraic, projective, smooth variety over $\mathbb{Q}$. We define

$$M_{\mathrm{B}} = H_i(X(\mathbb{C}))$$

with Hodge structure,

$$M_{\mathrm{dR}} = H^i_{\mathrm{dR}}(X)$$

and

$$M_\ell = H^i_{\text{ét}}(X_{\overline{\mathbb{Q}_\ell}}, \mathbb{Q}_\ell).$$

Example 3. Let E be the elliptic curve defined by $y^2 = x^3 + ax + b$, where $a, b \in \mathbb{Q}$. Then $E(\mathbb{C}) \simeq \mathbb{C}/(\mathbb{Z} + \gamma\mathbb{Z})$ for some $\gamma \in \mathbb{C}\backslash\mathbb{R}$. This is a torus. We have

$$H_1(E(\mathbb{C})) \simeq 1\mathbb{Z} \oplus \gamma\mathbb{Z}$$

and

$$H^1(E) = \mathbb{Z}\frac{dx}{y} \oplus \mathbb{Z}x\frac{dx}{y}.$$

Then

$$H_1(E(\mathbb{C})) \otimes \mathbb{C} \simeq H^{1,0} \oplus H^{0,1} = \mathbb{C}(1+\gamma) \oplus \mathbb{C}(1+\overline{\gamma}).$$

The filtration is

$$\mathrm{Fil}^1(H^1_{\mathrm{dR}}) = H^0(E, \Omega^1) = \mathbb{Z}\frac{dx}{y}.$$

Finally, we need the ℓ-adic realization. The ℓ^n-torsion is

$$E[\ell^n] \simeq (\mathbb{Z}/\ell^n\mathbb{Z})^2$$

and the Tate module is

$$V_\ell E := \varprojlim_n E[\ell^n] \simeq (\mathbb{Z}_\ell)^2$$

with Galois action, since the ℓ^n-torsion points have $\overline{\mathbb{Q}}$-coefficients. Set $T_\ell E = V_\ell E \otimes \mathbb{Q}_\ell$.

Deligne gave a recipe to construct from a motive an L-function. Consider

$$P_q(T) = \det(1 - T\,\mathrm{Fr}_q\,|M_\ell^{I_q})$$

and

$$L(M,s) = \prod_q P_q(q^{-s})^{-1}.$$

Definition 4. M is pure of weight i if

$$M_{\mathrm{B}} \otimes \mathbb{C} = \bigoplus_{\substack{p,q \\ p+q=i}} H^{p,q}.$$

If M is pure of weight i, we can think of it as a "piece" of $H^1(X)$, for a certain variety X.

Conjecture 5. *If M is pure of weight i and M_ℓ is unramified at q for $q \nmid \ell$, then the roots of $P_q(T)$ α_q's are Weil numbers of weight i, i.e. for all $\sigma : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $|\alpha_q|_{\sigma,\mathbb{C}} = q^{\frac{i}{2}}$.*

This conjecture implies that if M is pure, then $L(M,s)$ converges absolutely for $\mathrm{Re}(s) > \frac{i}{2} + 1$.

We now want to extend $L(M,s)$ for all $s \in \mathbb{C}$. Denote $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}$, and $h^{\frac{1}{2},\pm} = \dim_{\mathbb{C}}(H^{\frac{i}{2},\frac{i}{2}})^{F_\infty = \pm 1}$. If i is odd, this is defined to be 0. Define

$$\Lambda_\infty(M,s) = \prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \Gamma_{\mathbb{R}}\left(s - \frac{i}{2}\right)^{h^{\frac{i}{2},+}} \Gamma_{\mathbb{R}}\left(s - \frac{i}{2} + 1\right)^{h^{\frac{i}{2},-}}$$

where the complex gamma function is $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ (and the indices p and q are not necessarily prime!). The L-function satisfies the functional equation

$$\Lambda_\infty(M,s)L(M,s) = \epsilon(s)\Lambda_\infty(M,i+1-s)L(M,i+1-s).$$

Definition 6. We say that $n \in \mathbb{Z}$ is critical for M if neither $\Lambda_\infty(M,n)$ nor $\Lambda_\infty(M,i+1-n)$ has a pole.

Example 7. For an elliptic curve, $s = 1$ is the only critical point.

Proposition 8. *If n is critical for M, then there is an isomorphism*

$$I_\infty : F^{i+1-n} M_{dR} \xrightarrow{\sim} (M_B \otimes \mathbb{C})^{F_\infty = (-1)^{i+n}}$$

Here $F^i M$ is the filtration on de Rham cohomology, coming from the spectral sequence

$$H^p(X,\Omega^q) \Rightarrow H^{p+q}_{dR}(X).$$

Fix $\mathbb{Q}$-bases ω_i and γ_i of the left hand side and right hand side respectively, and define

$$c^+(M,n) = \det(I_\infty)$$

with respect to (ω_i) and (γ_i).

Conjecture 9 (Deligne)**.**

$$\frac{L(M,n)}{c^+(M,n)} \in \mathbb{Q}.$$

Example 10. If $i = n = 0$, then $F^1 H^1_{\mathrm{dR}} \xrightarrow{\sim} M_{\mathrm{B}}^{F_\infty=1}$. For an elliptic curve E, $F^1 H^1_{\mathrm{dR}} E$ is ω_E ($\frac{dx}{y}$ is the invariant differential, which is a global section of Ω^1). Using the modular parametrization $\mathbb{H} \to X_0(N) \xrightarrow{\pi} E$ and modular form $f_E \leftrightarrow E$, set

$$\omega_E = (2\pi i) f_E(z) dz$$

and

$$\gamma = \mathrm{Im}(\pi_*(i\infty - 0)) \in H_1(E(\mathbb{C}))^{F_\infty}.$$

We have

$$L(f,1) = \int_0^\infty 2\pi i f_E(iy) dy.$$

This formula proves the conjecture.

Fix a prime p. Suppose the Deligne conjecture is true. Then what are the p-adic properties of $\frac{L(M,n)}{c^+(M,n)} \in \mathbb{Q}$? For all $m > 0$, $\zeta(-m) \in \mathbb{Q}$. If m becomes a p-adic variable, then does $\zeta(-m)$ make sense?

The references are:

- Deligne, *Valeurs de fonctions L et périodes d'intégrales*[1];
- Schneider, *Introduction to the Beilinson conjectures*[2].

[1] Available at `http://publications.ias.edu/sites/default/files/33_Valeursde.pdf`.

[2] Available at `http://wwwmath.uni-muenster.de/u/pschnei/publ/beilinson-volume/Schneider.pdf`.