MATH G9905 RESEARCH SEMINAR IN NUMBER THEORY (FALL 2014)

LECTURE 5 (OCTOBER 7, 2014) XIAOQING LI

LOWER BOUNDS FOR RANKIN–SELBERG TYPE INTEGRALS

NOTES TAKEN BY PAK-HIN LEE

Abstract. In this talk, we will derive an almost sharp lower bound for a Rankin–Selberg type integral involving truncated Eisenstein series on $\mathrm{GL}(2n)$. Some orthogonality relations in the Fourier expansion of automorphic functions on $\mathrm{GL}(n)$ will be discovered and results from sieve methods will be applied. This is a joint work with Goldfeld.

1. The modified truncated Eisenstein series

We will start by explaining the relationship between Authur's truncated Eisenstein series $\hat{E_A}$ and the modified truncated Eisenstein series $\hat{E_A^*}$.

Let f be a Maass form for $\mathrm{SL}(n, \mathbb{Z})$, and

$$\phi(nmk) := |\det m_1|^{ns} |\det m_2|^{-ns} f(m_1) f(m_2).$$

Here nmk is the Iwasawa decomposition: n is in the unipotent radical of $P_{n,n}$, $m = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix}$ is in the Levi, and $k \in \mathrm{O}(2n, \mathbb{R})$. The cuspidal Eisenstein series is

$$E(z, s) := \sum_{\gamma \in P_{n,n} \backslash \Gamma} \phi(\gamma z).$$

Arthur's truncated Eisenstein series is

$$\hat{E_A}(z, s) = E(z, s) - \sum_{\substack{\gamma \in P_{n,n} \backslash \mathrm{SL}_{2n} \\ h(\gamma \cdot z) \geq A}} c_P E|_\gamma$$

where $h(z) = \left| \frac{\det m_1}{\det m_2} \right|$ is the height function and $c_P E$ is the constant term of E along P.

We don't know how to compute the Fourier series of Arthur's truncation, so we introduce a modified version.

Definition 1.

$$\hat{E_A^*}(z, s) := E(z, s) - A^{\frac{n}{2}} E\left(z, s - \frac{1}{2}\right) + \frac{\Lambda(2ns - 2n, f \times f)}{\Lambda(1 + 2ns - 2n, f \times f)} E(z, 2 - s)$$

Last updated: October 9, 2014. Please send corrections and comments to phlee@math.columbia.edu.

$$- \sum_{\substack{\gamma\in P_{n,n}\backslash \mathrm{SL}_{2n} \\ h(\gamma\cdot z)\geq A}} h(z)^{ns}\left(1-\frac{A^{\frac{n}{2}}}{h(z)^{\frac{n}{2}}}\right) f(m_1)f(m_2)\Big|_{\gamma}$$

$$- \frac{\Lambda(2ns-2n, f\times f)}{\Lambda(1+2ns-2n, f\times f)} \sum_{\substack{\gamma\in P_{n,n}\backslash \mathrm{SL}_{2n} \\ h(\gamma\cdot z)\geq A}} h(z)^{n(2-s)}\left(1-\frac{A^{\frac{n}{2}}}{h(z)^{\frac{n}{2}}}\right) f(m_1)f(m_2)\Big|_{\gamma}.$$

We can compute the Fourier expansion of $\hat{E}_A^*$ because

$$\int_{(2)} \frac{x^{-w}dw}{w(w+1)} = \begin{cases} \frac{1}{1-x} & \text{if } x>1, \\ 0 & \text{if } x\leq 1. \end{cases}$$

See Lemma 4 below.

Proposition 2. *$h(\gamma\cdot z)\leq h(z)$ for $y_i\geq \frac{\sqrt{3}}{2}$, $1\leq i\leq 2n-1$ and $\gamma\in \mathrm{SL}_{2n}(\mathbb{Z})$.*

Proof. Let $\nu = \left(0,\cdots,0,\frac{1}{n},0,\cdots,0\right)$ be the $(2n-1)$-tuple with $\frac{1}{n}$ at the n-th place. Then

$$I_\nu(z) := \prod_{i=1}^{2n-1}\prod_{j=1}^{2n-1} y_i^{b_{i,j}\nu_j} = h(z).$$

Here

$$b_{i,j} = \begin{cases} ij & \text{if } i+j\leq 2n, \\ (2n-i)(2n-j) & \text{if } i+j\geq 2n. \end{cases}$$

We have

$$h(\gamma\cdot z) = I_\nu(\gamma\cdot z) = \|e_{n+1}\gamma\cdot z\wedge\cdots\wedge e_{2n-1}\gamma\cdot z\wedge e_{2n}\gamma\cdot z\|^{-2}\|e_{n+1}z\wedge\cdots\wedge e_{2n-1}z\wedge e_{2n}z\|^2 h(z).$$

As we mentioned last time, $\|e_{n+1}M\wedge\cdots\wedge e_{2n}M\|^2$ is the sum of squares of all the $n\times n$ minors of the last n rows of M. Using the Cauchy–Binet formula, we can see that this is $\leq h(z)$. □

Corollary 3. *For $A\asymp t^{n^{10}}$, $\frac{\sqrt{3}}{2}\leq y_i\leq (t^{1+\epsilon})^{n(2n-1)}$, $1\leq i\leq 2n-1$, we have*

$$\hat{E}_A^*(z,s) = \hat{E}_A(z,s) - A^{\frac{n}{2}}\hat{E}_A\left(z,s-\frac{1}{2}\right) + c_{s-\frac{1}{2}}\hat{E}_A(z,2-s).$$

In other words, if we choose A to be very big, the truncated terms in the modified truncated Eisenstein series are all 0.

Lemma 4.

$$\hat{E}_A^*(z,s) = E(z,s) - A^{\frac{n}{2}}E\left(z,s-\frac{1}{2}\right) + \frac{\Lambda(2ns-2n, f\times f)}{\Lambda(1+2ns-2n, f\times f)}E(z,2-s)$$
$$- \frac{1}{2\pi i}\int_{(2)} \frac{A^{-\frac{n}{2}w}}{w(w+1)}E\left(z,s+\frac{w}{2}\right)dw$$
$$- \frac{\Lambda(2ns-2n, f\times f)}{\Lambda(1+2ns-2n, f\times f)}\cdot\frac{1}{2\pi i}\int_{(2)} \frac{A^{-\frac{n}{2}w}}{w(w+1)}E\left(z,2-s+\frac{w}{2}\right)dw.$$

2. Orthogonality of Fourier coefficients

Recall the integral

$$I = |L(1+2int, f\times f)|^2 \cdot \int_{P_{2n-1,1}(\mathbb{Z})\backslash \eta^{2n}} \left| \int_0^\infty \hat{E_A^*}(z, 1+it) g\left(\frac{A}{\beta}\right)\frac{dA}{A}\right|^2 \psi\left(\frac{\det z}{\delta}\right) d^\times z.$$

Here $\eta^{2n} = \mathrm{GL}(2n,\mathbb{R})/\,\mathrm{O}(2n,\mathbb{R})\mathbb{R}^\times$, $g, \psi \in \mathbb{C}_c^\infty([1,2])$, $\beta = t^{n^{10}}$ and $\delta = \beta^{-1}(\log\log t)^{n^2}$.

Last time we derived an almost sharp upper bound

$$I \ll \beta^n \delta^{-1} \log^2 t |L(1+it, f\times f)|$$

using the Fourier expansion of $\hat{E_A^*}$ and the Maass–Selberg relations. Today we will prove a sharp lower bound, which will only use the Fourier expanion.

Proposition 5. $I \gg \dfrac{\beta^n\delta^{-1}}{\log t}$.

Recall the Fourier expansion of a general automorphic function on $\mathrm{GL}(n)$.

Lemma 6. *Suppose F is an automorphic function for* $\mathrm{SL}(k,\mathbb{Z})$*,* $k \geq 4$*. Then*

$$F(z) = \int_0^1 \cdots \int_0^1 F\left(\begin{pmatrix} 1 & 0 & \cdots & 0 & u_{1k} \\ & 1 & \cdots & 0 & u_{2k} \\ & & \ddots & \vdots & \vdots \\ & & & 1 & u_{k-1,k} \\ & & & & 1 \end{pmatrix} z\right) d^\times u$$

$$+ \sum_{1\leq l\leq k-2} \sum_{m_{k-1}\neq 0} \cdots \sum_{m_{k-l}\neq 0} \sum_{\gamma \in \tilde{P}_{k-1,l}\backslash \mathrm{SL}_{k-1}} \int_0^1 \cdots \int_0^1$$

$$F\left(\begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & u_{1,k-1} & u_{1k} \\ & 1 & \cdots & 0 & \cdots & u_{2,k-1} & u_{2k} \\ & & & \ddots & & \vdots & \vdots \\ & & & & & 1 & u_{k-1,k} \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} \gamma_{k-1} & \\ & 1 \end{pmatrix} z\right)$$

$$\cdot\, e(-m_{k-1}u_{k-1,k} - \cdots - m_{k-l}u_{k-l,k-l+1}) d^\times u$$

$$+ \sum_{m_{k-1}\neq 0} \cdots \sum_{m_1\neq 0} \sum_{\gamma\in \mathrm{U}_{k-1}\backslash \mathrm{SL}_{k-1}} \int_0^1 \cdots \int_0^1$$

$$F\left(\begin{pmatrix} 1 & u_{12} & \cdots & u_{1k} \\ & 1 & \cdots & u_{2k} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z\right)$$

$$\cdot\, e(-m_{k-1}u_{k-1,k} - \cdots - m_1 u_{12}) d^\times u.$$

The last term is called the nondegenerate term $\mathrm{ND}(F)$. The other terms are the degenereate term $\mathrm{D}(F)$. We need the orthogonality of these Fourier coefficients.

Proposition 7.

$$\int_{P_{k-1,1}\backslash\eta^k} \overline{\mathrm{D}}(F)\cdot \mathrm{ND}(F) d^\times z = 0$$

Proof. Since

$$\bigcup_{\gamma\in \mathrm{U}_{k-1}\backslash \mathrm{SL}_{k-1}} \begin{pmatrix}\gamma & \\ & 1\end{pmatrix}\cdot P_{k-1,1}\backslash\eta^k \cong \mathrm{U}_k(z)\backslash\eta^k,$$

the integral unfolds as

$$\begin{aligned}
&\int_{P_{k-1,1}\backslash\eta^k} \overline{\mathrm{D}}(F)\cdot \mathrm{ND}(F) d^\times z\\
&=\sum_{m_1\neq 0}\cdots\sum_{m_{k-1}\neq 0}\int_{\mathrm{U}_k(\mathbb{Z})\backslash\eta^k} \overline{\mathrm{D}}(F)(z)\int_0^1\cdots\int_0^1\\
&\quad F\left(\begin{pmatrix}1 & u_{12} & \cdots & u_{1k}\\ & 1 & \cdots & u_{2k}\\ & & \ddots & \vdots\\ & & & 1\end{pmatrix} z\right) e(-m_1u_{12}-\cdots-m_{k-1}u_{k-1,k}) d^\times u d^\times z\\
&=\sum_{m_1}\cdots\sum_{m_{k-1}}\int_{y_1=0}^\infty\cdots\int_{y_{2n-1}=0}^\infty\int_{x_{12}=0}^1\cdots\int_{x_{k-1,k}=0}^1 \overline{\mathrm{D}}(F)(z) e(m_1x_{12}+\cdots+m_{k-1}x_{k-1,k})d^\times x\\
&\quad\cdot\int_0^1\cdots\int_0^1 F\left(\begin{pmatrix}1 & u_{12} & \cdots & u_{1k}\\ & 1 & \cdots & u_{2k}\\ & & \ddots & \vdots\\ & & & 1\end{pmatrix} y\right) e(-m_1u_{12}-\cdots-m_{k-1}u_{k-1,k}) d^\times u d^\times y.
\end{aligned}$$

The first integral

$$\int_0^1\cdots\int_0^1 \overline{\mathrm{D}}(F)(z) e(m_1x_{12}+\cdots+m_{k-1}x_{k-1,k})d^\times x$$

can be shown to be 0 by induction. □

Applying this to the automorphic function

$$\int_0^\infty g\left(\frac{A}{\beta}\right)\hat{E_A^*}(z,s)\frac{dA}{A},$$

we get

$$\begin{aligned}
I \gg& |L(1+2nit, f\times f)|^2\int_{y_1=0}^\infty\cdots\int_{y_{2n-1}=0}^\infty\sum_{N\le p\le 2N}\\
&\left|\int_0^1\cdots\int_0^1\int_0^\infty \hat{E_A^*}\left(\begin{pmatrix}1 & \cdots & u_{1,2n}\\ & \ddots & \vdots\\ & & 1\end{pmatrix} y, 1+it\right) g\left(\frac{A}{\beta}\right)\frac{dA}{A}\right.\\
&\left.\quad\cdot e(-pu_{2n-1,2n}-u_{2n-2,2n-1}-\cdots-u_{12})d^\times u\right|^2 \psi\left(\frac{\det z}{\delta}\right)d^\times y.
\end{aligned}$$

The u-integral picks up the nondegenerate $(1, \cdots, 1, p)$ coefficients of $\hat{E_A^*}$, which are related to Jacquet's Whittaker function, so we need Stade's formula.

Lemma 8 (Stade)**.**

$$\int_0^\infty \cdots \int_0^\infty |W(y,\beta)|^2 |\det y|^w d^\times y$$
$$= \frac{\pi^{aw} 2^b}{\Gamma\left(\frac{mw}{2}\right)} \left| \prod_{j=1}^{m-1} \prod_{j \le k \le m-1} \pi^{-\frac{1}{2}-\frac{1}{2}\left(\beta_{m-k}-\beta_{m-k+j}\right)} \Gamma\left(\frac{1+\beta_{m-k}-\beta_{m-k+j}}{2}\right) \right|^{-2}$$
$$\cdot \prod_{j=1}^m \prod_{k=1}^m \Gamma\left(\frac{w+\beta_j-\beta_k}{2}\right).$$

Here $(\beta_1, \cdots, \beta_m)$ are the Langlands parameters of $W(y,\beta)$.

Fact 9. *The $(1, \cdots, 1, p)$ Fourier coefficient of $E\left(z, \frac{1}{2}+it\right)$ is*

$$\frac{\lambda(p)\sigma_{nit}(p)}{p^{\frac{2n-1}{2}}} W(my, \alpha') \frac{1}{L(1+2int, f \times f)}$$

where $\sigma_s(n) := \sum_{d|n} d^s$ and $\lambda(p)$ is the p-th Hecke eigenvalue of f.

Now we want to get a lower bound for

$$\sum_{N \le p \le 2N} |\lambda(p)\sigma_{nit}(p)|^2.$$

3. Lower bounds for Hecke eigenvalues

Lemma 10. *For full density of primes $N \le p \le 2N$,*

$$\sigma_{nit}(p) \gg \frac{1}{\log\log t}.$$

Here $N \ge t^2$.

In his GL(2) paper, Sarnak used $\sigma_{nit}(p) \gg 1$ for a positive proportion of primes.

Proof. We have

$$|\sigma_{nit}(p)| = |p^{2int}+1| \ge |\cos 2nt \log p + 1|.$$

If $2nt \log p$ is close to $(2m+1)\pi$, then the prime is bad. The condition

$$|2nt \log p - (2m+1)\pi| \le \frac{1}{\sqrt{\log\log t}}$$

holds for p in short intervals only. Counting primes in short intervals by sieve theory, we see that the bad primes only contribute to lower order terms. □

What about $\lambda(p)$? If it gets too small, we have no hope.

Lemma 11. *For tempered Hecke–Maass f for $\mathrm{SL}_n(\mathbb{Z})$, there exist at least $\frac{1}{100n^2}\frac{N}{\log N}$ primes in $[N, 2N]$ such that the p-th Hecke eigenvalue satisfies*

$$|\lambda(p)| \gg \frac{1}{\log\log t}.$$

This is the only place where we need f to be tempered, i.e. to satisfy the Ramanujan conjecture.

Proof. By Liu–Wang–Ye,

$$\sum_{N\le n\le 2N} \Lambda(n)|\lambda(n)|^2 \sim N$$

where $\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, \\ 0 & \text{otherwise.} \end{cases}$ Here $|\lambda(p)| \le n$ by the Ramanujan conjecture.

Suppose the conclusion is wrong. Then we split the sum into

$$\begin{aligned} &\sum_{\text{good } p} \Lambda(p)|\lambda(p)|^2 + \sum_{\text{bad } p} \Lambda(p)|\lambda(p)|^2 \\ &\le n^2 \frac{N}{50n^2 \log N} + \sum_{N\le P\le 2N} \Lambda(p) \frac{1}{\log^2 \log t} \\ &\le \frac{N}{50 \log N} + \frac{N}{\log^2 \log t} \end{aligned}$$

which is a contradiction. □

Hence

$$\sum_{N\le p\le 2N} |\lambda(p)\sigma_{nit}(p)|^2 \gg_f \frac{N}{\log N}.$$

Corollary 12. $I \gg \dfrac{\beta^n \delta^{-1}}{\log t}$.

The p-th Hecke eigenvalue of E is $\lambda(p)\sigma_{nit}(p)$. Recall that

$$\phi(z) := h(z)^{ns} f(m_1) f(m_2)$$

and

$$E(z,s) = \sum \phi(z)|_\gamma.$$

For the Rankin–Selberg L-function $L(s, f \times f)$, the eigenvalue is $\lambda(p)p^s + \overline{\lambda(p)}p^{-s}$, which we can bound by taking out $\lambda(p)$. For $L(s, f_1 \times f_2)$ with two different f_1 and f_2, the upper bound still works but we haven't found a trick to obtain a lower bound yet.

4. Reference

Some background materials on the Maass–Selberg relations are available on Paul Garrett's website, in the following two papers:

- *Simplest Example of Truncation and Maass–Selberg Relations*[1],
- *Truncation and Maass–Selberg Relations*[2].

[1]Available at `http://www-users.math.umn.edu/~garrett/m/v/maass_selberg_trivial.pdf`.
[2]Available at `http://www.math.umn.edu/~garrett/m/v/maass_selberg.pdf`.