MATH G9905 RESEARCH SEMINAR IN NUMBER THEORY (FALL 2014)

LECTURE 4 (SEPTEMBER 30, 2014) XIAOQING LI

UPPER BOUNDS FOR A RANKIN–SELBERG TYPE INTEGRAL

NOTES TAKEN BY PAK-HIN LEE

Abstract. In this talk, we will give an almost sharp upper bound for a Rankin–Selberg type integral involving the Arthur truncated Eisenstein series on $\mathrm{GL}(n)$. Langlands–Shahidi constant formula, Maass–Selberg relations and other tools on higher rank groups will be used. This is a joint work with Goldfeld.

We will generalize Sarnak's method to derive a logarithmic zero free region for Rankin–Selberg L-functions on $\mathrm{GL}(n)$. His idea is to consider the integral

$$|\zeta(1+2it)|^2 \int_\eta^\infty \int_0^1 \left| \hat{E_A}\left(z, \frac{1}{2}+it\right)\right|^2 d^\times z \ll |\zeta(1+2it)|.$$

How can we generalize this integral to $\mathrm{GL}(2n)$? A natural generalization is

$$|L(1+it, f\times f)|^2 \int_{\eta_1}^\infty \int_{\eta_2}^\infty \cdots \int_{\eta_{2n-1}}^\infty \int_0^1 \cdots \int_0^1 \left| \hat{E_A}\left(z, \frac{1}{2}+it\right)\right|^2 d^\times z$$

but this doesn't work because the Fourier expansion of $\hat{E_A}$ involves sums of the form $\sum_{\gamma \in \mathrm{U}_{2n-1} \setminus \mathrm{SL}_{2n-1}}$. A better solution is to replace $\int_0^1 \cdots \int_0^1 |\hat{E_A}|^2$ by a certain Rankin–Selberg integral.

Let $G = \mathrm{GL}(2n, \mathbb{R})$, $\Gamma = \mathrm{SL}(2n, \mathbb{Z})$ and $K = \mathrm{O}(2n, \mathbb{R})$. Let $P_{n,n}(\mathbb{Z}) = \left\{ \begin{pmatrix} n\times n & * \\ 0 & n\times n \end{pmatrix} \right\} \subset \Gamma$ be the maximal parabolic, $N^P = \left\{ \begin{pmatrix} I_{n\times n} & * \\ 0 & I_{n\times n} \end{pmatrix} \right\}$ be the unipotent radical, and $M^P = \left\{ \begin{pmatrix} n\times n & 0 \\ 0 & n\times n \end{pmatrix} \right\}$ be the standard Levi.

The cuspidal Eisenstein series is

$$E(z,s) := \sum_{\gamma \in P_{n,n}\setminus \Gamma} \left(\frac{|\det m_1|}{|\det m_2|}\right)^{ns} f(m_1) f(m_2) \Big|_\gamma$$

where m_1 and m_2 come from the Iwasawa decomposition of $z = nmk$ with $n \in N^P$, $m = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \in M^P$ and $k \in K$, and f is a fixed Hecke–Maass form for $\mathrm{GL}(n, \mathbb{Z})$.

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Arthur's truncated Eisenstein series is

$$\hat{E_A}(z,s) := E(z,s) - \sum_{\substack{\gamma \in P_{n,n}\backslash\Gamma \\ h(z)|_\gamma \geq A}} c_P E(z,s)|_\gamma.$$

Here $h(z) = \left|\frac{\det m_1}{\det m_2}\right|$, and $c_P E$ is the constant term of E along $P_{n,n}$.

There is no obvious connection between the Fourier expansions of $\hat{E_A}$ and E. The idea is to smooth out the sharp truncation, so we introduce a smooth version of Arthur's truncated Eisenstein series

$$\begin{aligned}\hat{E^*_A}(z,s) := E(z,s) - A^{\frac{n}{2}} E\left(z, s-\frac{1}{2}\right) + \frac{\Lambda(2ns-2n, f\times f)}{\Lambda(1+2ns-2n, f\times f)} E(z, 2-s) \\ - \sum_{\substack{\gamma\in P\backslash\Gamma \\ h(\gamma\cdot z)\geq A}} h(z)^{ns}\left(1-\frac{A^{\frac{n}{2}}}{h(z)^{\frac{n}{2}}}\right) f(m_1)f(m_2)\Bigg|_\gamma \\ - \frac{\Lambda(2ns-2n, f\times f)}{\Lambda(1+2ns-2n, f\times f)} \sum_{\substack{\gamma\in P\backslash\Gamma \\ h(\gamma\cdot z)\geq A}} h(z)^{n(2-s)}\left(1-\frac{A^{\frac{n}{2}}}{h(z)^{\frac{n}{2}}}\right) f(m_1)f(m_2)\Bigg|_\gamma .\end{aligned}$$

The truncated sum can be rewritten as

$$\sum_{\substack{\gamma\in P\backslash\Gamma \\ h(\gamma\cdot z)\geq A}} h(z)^{ns}\left(1-\frac{A^{\frac{n}{2}}}{h(z)^{\frac{n}{2}}}\right) f(m_1)f(m_2)\Bigg|_\gamma = \frac{1}{2\pi i}\int_{(2)} \frac{A^{-\frac{n}{2}w}}{w(w+1)} E\left(z, s+\frac{w}{2}\right) dw$$

which is smooth. Note that

$$\int_{(2)} \frac{x^{-w}}{w(w+1)} dw = \begin{cases} \frac{1}{1-x} & \text{if } x > 1, \\ 0 & \text{if } x \leq 1. \end{cases}$$

Let $\beta := t^{n^{10}}$, $\delta := \beta^{-1}(\log\log t)^{n^2}$, and $g(x), \psi(x) \in C_c^\infty([1,2])$ be test functions. Define

$$I := |L(1+2int, f\times f)| \cdot \int_{P_{2n-1,1}\backslash\eta^{2n}} \left|\int_0^\infty \hat{E^*_A}(z, 1+it) g\left(\frac{A}{\beta}\right)\frac{dA}{A}\right|^2 \psi\left(\frac{\det z}{\delta}\right) d^\times z$$

where $P_{2n-1,1}$ is the maximal parabolic $\left\{\begin{pmatrix} (2n-1)\times(2n-1) & * \\ 0 & * \end{pmatrix}\right\}$ and $\eta^{2n} = \mathrm{GL}(2n,\mathbb{R})/K\cdot \mathbb{R}^\times$.

Our main theorem is

Theorem 1.

$$I \ll_f \delta^{-\frac{1}{2}}\beta^{\frac{1}{2}+n}(\log t)^2$$

as $t\to\infty$.

This bound is sharp modulo some log terms. Next time we will get a lower bound.

Recall that if ϕ is a Maass form, then the integral

$$\int_{\mathrm{SL}(2n,\mathbb{Z})\backslash\eta^{2n}} |\phi|^2 E_{P_{2n-1,1}}(z,s) d^\times z$$

is roughly equal to the Rankin–Selberg L-function $\Lambda(s, \phi\times\phi)$.

The maximal parabolic Eisenstein series is

$$E_{P_{2n-1,1}}(z,w) := \sum_{\gamma \in P_{2n-1,1}\backslash \mathrm{SL}_{2n}} (\det z)^w|_\gamma.$$

Unfolding gives

$$I = |L(1+2int, f\times f)| \cdot \frac{1}{2\pi i}\int_{(2)} \tilde{\psi}(-w)\delta^{-w} \cdot \int_{\mathrm{SL}_{2n}(\mathbb{Z})\backslash \eta^{2n}} \left| \int_0^\infty \hat{E_A^*}(z,1+it) g\left(\frac{A}{\beta}\right)\frac{dA}{A}\right|^2 E_{P_{2n-1,1}}(z,w) d^\times z dw.$$

Shifting the w-line to $\frac{1}{2}$, we pick up the pole of $E_{P_{2n-1,1}}(z,w)$ at $w=1$ and write

$$I = I_1 + I_2,$$

where

$$I_1 = c|L(1+2int, f\times f)|\tilde{\psi}(-1)\delta^{-1}\int_{\mathrm{SL}(2n,\mathbb{Z})\backslash\eta^{2n}} \left|\int_0^\infty g\left(\frac{A}{\beta}\right)\hat{E_A^*}(z,1+it)\frac{dA}{A}\right|^2 d^\times z$$

for some constant c, and

$$I_2 = |L(1+2int, f\times f)|\cdot\frac{1}{2\pi i}\int_{(\frac{1}{2})}\tilde{\psi}(-w)\delta^{-w}\int_{\mathrm{SL}_{2n}(\mathbb{Z})\backslash\eta^{2n}}\left|\int_0^\infty \cdots\right|^2 E_{P_{2n-1,1}}(z,w) d^\times z dw.$$

We want to bound these two integrals.

Proposition 2. *Suppose F is an automorphic function for $\mathrm{SL}(k,\mathbb{Z})$, $k\geq 4$. Then F has a Fourier expansion*

$$F(z) = \int_0^1\cdots\int_0^1 F\left(\begin{pmatrix} 1 & 0 & \cdots & 0 & u_{1k} \\ & 1 & \cdots & 0 & u_{2k} \\ & & \ddots & \vdots & \vdots \\ & & & 1 & u_{k-1,k} \\ & & & & 1\end{pmatrix} z\right) d^\times u$$

$$+ \sum_{1\leq l\leq k-2}\sum_{m_{k-1}\neq 0}\cdots\sum_{m_{k-l}\neq 0}\sum_{\gamma\in\tilde{P}_{k-1,l}\backslash\mathrm{SL}_{k-1}}\int_0^1\cdots\int_0^1$$

$$F\left(\begin{pmatrix} 1 & 0 & \cdots & 0 & u_{1,k-l} & \cdots & u_{1,k-1} & u_{1k} \\ & 1 & \cdots & 0 & u_{2,k-l} & \cdots & u_{2,k-1} & u_{2k} \\ & & \ddots & & & & \vdots & \vdots \\ & & & & & & 1 & u_{k-1,k} \\ & & & & & & & 1\end{pmatrix}\begin{pmatrix}\gamma_{k-1} & \\ & 1\end{pmatrix} z\right)$$

$$\cdot\, e(-m_{k-1}u_{k-1,k} - \cdots - m_{k-l}u_{k-l,k-l+1}) d^\times u$$

$$+ \sum_{m_{k-1}\neq 0}\cdots\sum_{m_1\neq 0}\sum_{\gamma\in \mathrm{U}_{k-1}\backslash\mathrm{SL}_{k-1}}\int_0^1\cdots\int_0^1$$

$$F\left(\begin{pmatrix}1 & u_{12} & \cdots & u_{1,k-1} & u_{1k}\\ & 1 & \cdots & u_{2,k-1} & u_{2k}\\ & & \ddots & \vdots & \vdots\\ & & & 1 & u_{k-1,k}\\ & & & & 1\end{pmatrix}\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} z\right)$$
$$\cdot e(-m_{k-1}u_{k-1,k} - \cdots - m_1 u_{12}) d^\times u.$$

Here

$$\tilde{P}_{k-1,l} = \left\{\begin{pmatrix}\mathrm{SL}_{k-1-l} & * & * & \cdots & *\\ & 1 & * & \cdots & *\\ & & 1 & \cdots & *\\ & & & \ddots & *\\ & & & & 1\end{pmatrix}\right\}.$$

The last term is called the nondegenerate term $\mathrm{ND}(F)$. For details, see Goldfeld and Ichino–Yamana.

Proposition 3 (Langlands–Shahidi)**.** *$E(z,s)$ has constant term*

$$c_P := \left|\frac{\det m_1}{\det m_2}\right|^{ns} f(m_1)f(m_2) + c_s\left(\left|\frac{\det m_1}{\det m_2}\right|^{n(1-s)}\right) f(m_1)f(m_2)$$

with

$$c_s := \frac{\Lambda(2ns-n, f\times f)}{\Lambda(1+2ns-n, f\times f)}$$

along $P_{n,n}$. Along other parabolics, the constant terms are 0.

Corollary 4.

$$E(z,s) = \sum_{\gamma\in\tilde{P}_{2n-1,n-1}\backslash\mathrm{SL}_{2n-1}}\left(\left|\frac{\det m_1}{\det m_2}\right|^{ns} + \left|\frac{\det m_1}{\det m_2}\right|^{n(1-s)}\right) c_s f(m_1) f^*(m_2)\Bigg|_{\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}} + \mathrm{ND}(E).$$

Here

$$f^*(m_2) = \sum_{m_{2n-1}\neq 0}\cdots\sum_{m_{n+1}\neq 0}\frac{B(m_{2n-1},\cdots,m_{n+1})}{\prod_{k=1}^{n-1} m_{2n-k}^{\frac{k(n-k)}{2}}} W_J(Mm_2),$$

where B are the Fourier coefficients of f and W_J is the Jacquet–Whittaker function.

Proposition 5. *Let*

$$F(z) := \sum_{\gamma\in\mathrm{U}_{k-1}\backslash\mathrm{SL}_{k-1}}\sum_{m_1\geq 1}\cdots\sum_{m_k\neq 0}\frac{A(m_1,\cdots,m_{k-1})}{\prod_{l=1}^{k-1}|m_l|^{\frac{l(k-l)}{2}}} W_J(Mz)\Bigg|_{\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}}.$$

If $\max_i y_i \geq (t^{1+\epsilon})^{\frac{k(k-1)}{2}}$, $\min_i y_i \geq \frac{\sqrt{3}}{2}$, and the Langlands parameters $(\alpha_1,\cdots,\alpha_{k-1})$ satisfy $|\operatorname{Im}\alpha_i| \ll t$, then $F(z) \ll (\max_i y_i)^{-N}$ for N very big.

On a compact Riemannian surface, $F(z) \ll |\lambda|^\epsilon$ for the spectral eigenvalue λ. On a non-compact surface, this is not true.

The Jacquet–Whittaker function has rapid decay

$$W_{k,\nu}(z) \ll (\max_i y_i)^{-N}$$

under the above conditions.

Lemma 6 (Bump–Friedberg–Hoffstein)**.**

$$I_\nu(g\cdot z) = \left(\prod_{i=0}^{k-2} \|e_{k-i}gz\wedge\cdots\wedge e_{k-1}gz\|^{-k\nu_{k-i-1}}\right) |\det z|^{\sum_{i=0}^{k-1} i\nu_{k-i}}$$

for $g\in \mathrm{SL}(k,\mathbb{R})$*. Here*

$$I_\nu(z) = \prod_{i=1}^{k-1}\prod_{j=1}^{k-1} y_i^{b_{ij}\nu_j} \ \text{ with } b_{ij} = \begin{cases} ij & \text{if } i+j\le k,\\ (k-i)(k-j) & \text{if } i+j\ge k,\end{cases}$$

and $e_i = (0,\cdots,0,1,0,\cdots,0)$*.*

Note that $\|e_{k-i}M\wedge\cdots\wedge e_kM\|^2$ is the sum of squares of all the $(i+1)\times(i+1)$ minors of the last $i+1$ rows of M.

This bound I_1. To bound I_2 we need to use the maximal parabolic Eisenstein series

$$E_{P_{2n-1,1}}(z,w)$$

which is essentially the Epstein zeta function. The completed Eisenstein series

$$E^*_{P_{2n-1}}(z,w) = \pi^{-nw}\Gamma(nw)\zeta(2nw)E_{P_{2n-1,1}}(z,w)$$

has integral representation

$$\int_1^\infty (\theta_z(u)-1)u^{nw}\frac{du}{u} + \int_1^\infty (\theta_{w_0{}^tz^{-1}w_0}(u)-1)u^{n(1-w)}\frac{du}{u} - \frac{1}{n}\left(\frac{1}{1-w}+\frac{1}{w}\right).$$

Here w_0 is the long element in the Weyl group, and the theta function is

$$\theta_z(u) = \sum_{(a_1,\cdots,a_{2n})\in\mathbb{Z}^{2n}} e^{-\pi(b_1^2+b_2^2+\cdots+b_{2n}^2)u}$$

where

$$\begin{aligned} b_1 &= a_1Y_1,\\ b_2 &= (a_1x_{12}+a_2)Y_2,\\ &\ \vdots\\ b_{2n} &= (a_1x_{1,2n}+a_2x_{2,2n}+\cdots+a_{2n})Y_{2n},\end{aligned}$$

and

$$Y_k = y_1\cdots y_{2n-k}(y_1^{2n-1}\cdots y_{2n-1})^{-\frac{1}{2n}}.$$

Proposition 7. *For* $y_i\ge\frac{\sqrt{3}}{2}$*,* $1\le i\le 2n-1$*,* $\mathrm{Re}\,w=\frac12$*, we have*

$$E_{P_{2n-1,1}}(z,w) \ll \sum_{2\le k\le 2n}\Big((y_1y_2^2\cdots y_{2n-k}^{2n-k})^{\frac12}(y_{2n-k+1}^{k-1}\cdots y_{2n-1})^{\frac12}$$
$$+(y_1y_2^2\cdots y_{2n-k}^{2n-k})^{\frac{k-1}{2n}}(y_{2n-k+1}^{k-1}\cdots y_{2n-1})^{\frac{2n-k+1}{2n}}\Big)\ln y_1\cdots y_{2n-1}.$$

Recall the Maass–Selberg relation

$$\langle \hat{E}_A(\cdot, r), \hat{E}_A(\cdot, s)\rangle = |\langle f, f\rangle|^2 \left(\frac{A^{r+\overline{s}-1}}{r+\overline{s}-1} + \overline{c_s}\frac{A^{r-\overline{s}}}{r-\overline{s}} + c_r\frac{A^{\overline{s}-r}}{\overline{s}-r} + c_r\overline{c_s}\frac{A^{1-r-\overline{s}}}{1-r-\overline{s}}\right).$$

We can apply this to the smooth truncation above, which is just a linear combination of Arthur's truncation.

We need an upper bound for the Rankin–Selberg L-function.

Lemma 8.

$$L(1+2int, f\times f) \ll_{n,f} \log(|t|+1)$$

and

$$L'(1+2int, f\times f) \ll_{n,f} \log^2(|t|+1).$$

These are all the ingredients of the proof.