MATH G9905 RESEARCH SEMINAR IN NUMBER THEORY (FALL 2014)

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LOWER BOUNDS FOR L-FUNCTIONS AND EISENSTEIN SERIES

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Abstract. The classical Poussin's method deriving zero free regions for the Riemann zeta function can be generalized to all automorphic L-functions. However this method doesn't work for general Rankin–Selberg L-functions. In this talk we will introduce the method of using Eisenstein series deriving zero free regions for general L-functions due to several people. Especially we will analyze Sarnak's derivation of the standard zero free region for the Riemann zeta function through theory of GL(2) Eisenstein series.

Today we will apply the Maass–Selberg relations. Hadamard showed that the prime number theorem is equivalent to the statement that $\zeta(s) \neq 0$ for $\operatorname{Re} s = 1$. In 1899, Poussin generalized this method to get a zero free region:

$$\zeta(s) \neq 0 \text{ for } \sigma > 1 - \frac{c}{\log(|t|+1)},$$

where $s = \sigma + it$. This is called the standard zero free region.

His method was to construct an auxiliary L-function $D(s)$ satisfying:

(1) It has positive coefficients.
(2) It has a pole at 1 to order k.
(3) If $L(\sigma + it) = 0$ and $D(s)$ vanishes at σ to order $> k$, then we have a zero free region. (If the order is exactly k, we only get non-vanishing on the line $\operatorname{Re}(s) = 1$.)

Example 1. For $\zeta(s)$, we can take

$$D(s) = \zeta^3(s)\zeta^2(s+it_0)\zeta^2(s-it_0)\zeta(s+2it_0)\zeta(s-2it_0).$$

This has a pole at 1 of order 3. $\zeta(\sigma + it_0) = 0$ implies $D(\sigma) = 0$ of order 4.

Remark. For the Dirichlet L-function $L(s,\chi)$, this method works for the t-aspect and fixed χ. If χ is real, this method fails because of the possible existence of real zeros close to 1. This is the famous Landau–Seigel zero, which is a famous open problem in number theory.

Given a number field K and a cusp form π_K on $\mathrm{GL}_m(\mathbb{A}_K)$, we can associate a standard L-function $L(s,\pi)$, which has analytic continuation to the whole complex plane and satisfies a functional equation under $s \mapsto 1-s$. We can take

$$D(s) := \zeta(s)L^2(s,\pi\times\tilde{\pi})L^2(s+it_0,\pi)L^2(s-it_0,\tilde{\pi})L(s+2it_0,\pi\times\pi)L(s-2it_0,\tilde{\pi}\times\tilde{\pi})$$

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where $\tilde{\pi}$ is the contragredient of π. We can verify that $D(s)$ satisfies the conditions above.

Poussin's method also works for the Rankin–Selberg L-functions $L(s, \pi \times \pi')$ if one of π and π' is self-dual. This is due to Moreno and Sarnak. In general, $L(s, \pi \times \pi')$ doesn't vanish for $\sigma > 1 - \dfrac{c}{Q_\pi Q_{\pi'}(|t|+1)^A}$ (Brumley).

In 1976, Jacquet and Shalika introduced the Eisenstein series method for GL_n and proved the non-vanishing $L(1+it, \pi) \neq 0$. Sarnak made their approach effective by using spectral theory on the upper half plane. He considered general Fuchsian groups in $\mathrm{SL}(2, \mathbb{Z})$ which are not necessarily arithmetic. Assume that this group Γ has only one cusp at ∞ with stabilizer

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \right\}.$$

The Eisenstein series is defined as

$$E_\Gamma(z, s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s$$

which is absolutely convergent for $\mathrm{Re}(s) > 1$. Let

$$\tau_{m,\Gamma}(c) := \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma / \Gamma_\infty} e\left(\frac{ma}{d}\right).$$

The m-th Fourier coefficient of $E_\Gamma(z, s)$ is

$$\begin{cases} y^s + \left(\displaystyle\sum_{c>0} \frac{\tau_{0,\Gamma}(c)}{c^{2s}} \right) \dfrac{\pi^{\frac{1}{2}} \Gamma(s - \frac{1}{2})}{\Gamma(s)} y^{1-s} & \text{if } m = 0, \\ D_m(s) \dfrac{2\pi^s |m|^{s-\frac{1}{2}} y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|m|y)}{\Gamma(s)} & \text{if } m \neq 0. \end{cases}$$

Here $D_m(s) = \displaystyle\sum_{c>0} \frac{\tau_{m,\Gamma}(c)}{c^{2s}}$.

We introduce the Poincaré series

$$P_m(z, s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\mathrm{Im}\, \gamma z)^s e(m \gamma z).$$

Its inner product with the Eisenstein series is

$$\langle E_\Gamma(\cdot, s), P_m(\cdot, s+1) \rangle = ab^s \frac{\Gamma(2s)}{\Gamma(s)\Gamma(s+1)} D_m(s)$$

for some constants a and b. This was computed by Goldfeld and Sarnak in the 80's.

Consider the sum

$$\begin{aligned} S(x) &:= \sum_{c \leq x} \left(1 - \frac{c}{x}\right) \tau_{m,\Gamma}(c) \\ &= \frac{1}{2\pi i} \int_{(2)} D_m(s) x^{2s} \frac{ds}{s(2s+1)}. \end{aligned}$$

If $E_\Gamma(z,s)$ has no pole in $[\frac{1}{2},1)$, then we can show $S(x)=o(x)$. This statement is equivalent to the prime number theorem for general groups: if $\Gamma=\mathrm{SL}(2,\mathbb{Z})$, then

$$S(x)=\sum_{c\le x}\left(1-\frac{c}{x}\right)\mu(c)$$

and $S(x)=o(x)$ is equivalent to the PNT. A zero free region is equivalent to $E_\Gamma(s,z)$ having no pole $<\frac{1}{2}$. But this is not true for general groups.

In order to get a better zero free region, we use the Maass–Selberg relations. Let $\Gamma=\mathrm{SL}(2,\mathbb{Z})$. Consider the integral

$$I:=\int_\eta^\infty\int_0^1|\zeta(1+2it)|^2\left|E_A\left(z,\frac{1}{2}+it\right)\right|^2 d^\times z$$

where $\eta=t^{-\delta}$ for $\delta>0$, and

$$E_A(z,s)=E(z,s)-\sum_{\Gamma_\infty\backslash\Gamma}c_AE|_\gamma$$

with

$$c_AE=\begin{cases}0 & \text{if } y\le A,\\ y^s+\phi(s)y^{1-s} & \text{if } y>A\end{cases}$$

and

$$\phi(s)=\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\zeta(2s)}.$$

We will bound this integral.

Lemma 2. $I\ll\frac{1}{\eta}|\zeta(1+2it)|((\log t)^2+2\log A)$.

Lemma 3. $I\gg\frac{1}{\eta}\frac{1}{\log t}$. *Here* $A=\frac{1}{\eta}$.

Comparing these we get

Theorem 4. $\zeta(1+2it)\gg\frac{1}{\log^3|t|}$.

Proof of Lemma 2. The upper bound follows from the Maass–Selberg relations. We have

$$I=|\zeta(1+2it)|^2\int_{\Gamma\backslash\mathfrak{h}}N(z,\eta)\left|E_A\left(z,\frac{1}{2}+it\right)\right|^2 d^\times z$$

where the counting function is

$$N(z,\eta)=\#\{\gamma\in\Gamma_\infty\backslash\Gamma:\mathrm{Im}(\gamma z)>\eta\}.$$

We have the bound $N(z,\eta)\ll\frac{1}{\eta}$. And then we use the Maass–Selberg relations

$$\int_{\Gamma\backslash\mathfrak{h}}\left|E_A\left(z,\frac{1}{2}+it\right)\right|^2 d^\times z=2\log A-\frac{\phi'}{\phi}\left(\frac{1}{2}+it\right)+\frac{\overline{\phi}(\frac{1}{2}+it)A^{2it}+\phi(\frac{1}{2}+it)A^{-2it}}{2it}.\quad\square$$

The lower bound is trickier.

Proof of Lemma 3. Recall

$$I = |\zeta(1+2it)|^2 \int_\eta^\infty \int_0^1 \left| E_A\left(z, \frac{1}{2}+it\right) \right|^2 d^\times z.$$

We use the truncated Eisenstein series. The proof is broken into 3 steps.

(1) We write

$$E_A(z,s) = \sum_{n\in\mathbb{Z}} a_n(y,s)e(nx).$$

By Parseval's identity,

$$\int_0^1 |E_A(z,s)|^2 dx = \sum_{n\in\mathbb{Z}} |a_n(y,s)|^2.$$

(2) For $y > \frac{1}{A}$, $E_A(z,s)$ and $E(z,s)$ have the same nondegenerate Fourier coefficients. This means $\int_0^1 E_A(z,s)e(nx)dx \neq 0$ for $n \neq 0$, and

$$a_n(y,s) = \frac{2\pi^s\sqrt{y}}{\Gamma(s)\zeta(2s)}\sigma_{1-2s}(n)|n|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|y).$$

(3) By positivity,

$$\begin{aligned} I &\gg \sum_n |\sigma_{-2it}(n)|^2 \int_{\eta n}^\infty \left| \frac{K_{it}(2\pi y)}{\Gamma(\frac{1}{2}+it)} \right|^2 \frac{dy}{y} \\ &\gg \frac{1}{t} \sum_{n\leq \frac{t}{4\eta}} |\sigma_{-2it}(n)|^2. \end{aligned}$$

An analytical method won't give us anything, because the generating series is

$$\sum_n \frac{|\sigma_{it_0}(n)|^2}{n^s} = \frac{\zeta^2(s)\zeta(s+it_0)\zeta(s-it_0)}{\zeta(2s)}$$

and our goal is to bound ζ!

Sarnak came up with the idea of using sieve theory. For prime p,

$$|\sigma_{2it}(p)| = |p^{2it}+1| \geq |\cos(2t\log p)+1|.$$

If $2t\log p$ is close to an odd multiple of π, this is bad. We use sieve theory to get rid of such primes and prove

$$\sigma_{2it}(p) \gg 1$$

for a positive proportion of primes p. So we get

$$I \gg \frac{1}{t}\sum_{n\leq\frac{t}{4\eta}} |\sigma_{-2it}(n)|^2 \gg \frac{1}{\eta}\frac{1}{\log t}.$$

□

Sarnak used the Maass–Selberg relations. This method is generalized to a broad class of L-functions. Gelbart–Lapid–Sarnak proved that

$$L(\mathrm{Sym}^9, 1+it) \gg \frac{1}{(|t|+1)^A}$$

even though we don't know much analytical properties of this L-function. Later we will try to get a logarithmic standard zero free region by generalizing this approach. What is the analogue of the integral I on $\mathrm{GL}(n)$?