MATH G9905 RESEARCH SEMINAR IN NUMBER THEORY (FALL 2014)

LECTURE 2 (SEPTEMBER 16, 2014) DORIAN GOLDFELD

MAASS–SELBERG RELATIONS FOR EISENSTEIN SERIES ON $\mathrm{GL}(n)$

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Abstract. I will introduce Arthur's truncation operator and the inner product formula for truncated Eisenstein series (Maass–Selberg relation) for $\mathrm{GL}(n)$ with $n > 2$.

Today we will generalize the Maass–Selberg relations to $\mathrm{GL}(n)$. Let me first talk about the Eisenstein series for $\mathrm{GL}(n)$. Consider the basic set-up: $G = \mathrm{GL}(2n)$, $\Gamma = \mathrm{GL}(2n, \mathbb{Z})$, $K = \mathrm{O}(2n, \mathbb{R})$. To keep things simple, we will look at the action of $\mathrm{GL}(2n, \mathbb{Z})$. If we want to work adelically, we consider $\mathrm{GL}(2n, \mathbb{A}_k)$ and $\mathrm{GL}(2n, k)$ instead. We are focusing on $2n$ because we will be applying this to Rankin–Selberg. We are also interested in the parabolic $P = P_{n,n} = \begin{pmatrix} \mathrm{GL}(n) & * \\ 0 & \mathrm{GL}(n) \end{pmatrix}$, the unipotent radical $N^P = \begin{pmatrix} I_n & * \\ 0 & I_n \end{pmatrix}$, and the Levi $M^P = \begin{pmatrix} \mathrm{GL}(n) & 0 \\ 0 & \mathrm{GL}(n) \end{pmatrix}$. More generally, we have

$$P_{r_1,\cdots,r_k} = \begin{pmatrix} \mathrm{GL}(r_1) & * & * \\ 0 & \ddots & * \\ 0 & 0 & \mathrm{GL}(r_k) \end{pmatrix}, N^P = \begin{pmatrix} I_{r_1} & * & * \\ 0 & \ddots & * \\ 0 & 0 & I_{r_k} \end{pmatrix}, M^P = \begin{pmatrix} \mathrm{GL}(r_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathrm{GL}(r_k) \end{pmatrix}.$$

Let $F : N^P \to \mathbb{C}$ be a smooth function such that $F(ng) = F(g)$ for all $n \in N^P(\mathbb{Z})$. The constant term along P of F is defined to be

$$c_P F(g) = \int_{N^P(\mathbb{Z})\backslash N^P(\mathbb{R})} F(ng)dn.$$

We are interested in the constant terms along a parabolic of Eisenstein series.

Definition 1. Let $\phi(pg) = \phi(g)$ for all $p \in P(\mathbb{Z})$. Then

$$E^P(\phi, g) := \sum_{\gamma \in P(\mathbb{Z})\backslash G(\mathbb{Z})} \phi(\gamma g).$$

Let me show how the constant term can be computed in general. The constant term can be broken into pieces using the Bruhat decomposition. Let P, Q be two parabolics for G. Then

$$G = \bigcup_{w \in W} PwQ = \bigcup_{w \in (W \cap P)\backslash W/(W \cap Q)} PwQ$$

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where W is the Weyl group.

Consider the Eisenstein series $E^Q(\phi, g)$. The constant term along P is

$$\begin{aligned} c_P E^Q(\phi, g) &= \int_{N_{\mathbb{Z}}^P \backslash N_{\mathbb{R}}^P} E^Q(\phi, ng) dn \\ &= \int_{N_{\mathbb{Z}}^P \backslash N_{\mathbb{R}}^P} \sum_{\gamma \in Q_{\mathbb{Z}} \backslash G_{\mathbb{Z}} / P_{\mathbb{Z}}} \sum_{\beta \in Q_{\mathbb{Z}} \backslash Q_{\mathbb{Z}} \gamma P_{\mathbb{Z}}} \phi(\gamma \beta n g) dn \\ &= \sum_{w \in (W \cap Q) \backslash W / (W \cap P)} \int_{N_{\mathbb{Z}}^P \backslash N_{\mathbb{R}}^P} \sum_{\beta \in ((w^{-1} Q_{\mathbb{Z}} w) \cap P_{\mathbb{Z}}) \backslash P_{\mathbb{Z}}} \phi(w \beta n g) dn. \end{aligned}$$

Assume that $Q_{\mathbb{Z}} \cap P_{\mathbb{Z}} = P_{\mathbb{Z}}$. There are two interesting cases.

- If $w = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ is the trivial Weyl element, then the integral is

$$\int_{N_{\mathbb{Z}}^P \backslash N_{\mathbb{R}}^P} \phi(ng) dn = \mathrm{Vol}(N_{\mathbb{Z}}^P \backslash N_{\mathbb{R}}^P) \phi(g).$$

- If $w_0 = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$ is the long element, then $(w_0^{-1} P_{\mathbb{Z}} w_0) \cap N_{\mathbb{Z}}$ is trivial, so

$$\int_{N_{\mathbb{Z}}^P \backslash N_{\mathbb{R}}^P} \sum_{\gamma \in N_{\mathbb{Z}}^P} \phi(w_0 \gamma n g) dn = \int_{N_{\mathbb{R}}^P} \phi(w_0 n g) dn.$$

We will now get to the Maass–Selberg relations. Again we are in the case $G = \mathrm{GL}(2n)$ and $P = \begin{pmatrix} \mathrm{GL}(n) & * \\ 0 & \mathrm{GL}(n) \end{pmatrix}$. Fix a cusp form f on $\mathrm{GL}(n, \mathbb{R})$. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 1$, we define

$$E_f^P(g, s) = \sum_{\gamma \in P_{\mathbb{Z}} \backslash \mathrm{SL}(2n, \mathbb{Z})} \phi_s(\gamma g).$$

To define ϕ_s, we need to introduce the height function

$$h^P(g) := \left(\frac{|\det(m_1)|}{|\det(m_2)|} \right)^n$$

where $g \in G$ is written as $g = nmk$ with $m = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$ in the Levi. Then

$$\phi_s(g) := h^P(g)^s f(m_1) f(m_2).$$

If $\mathrm{Re}(s)$ is sufficiently large, $E_f^P(g, s)$ converges because f is a cusp form.

More generally, if we decompose the Levi as $\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$ where $m_1 \in \mathrm{GL}(n_1)$ and $m_2 \in \mathrm{GL}(n_2)$, then

$$h^P(g) := \frac{|\det(m_1)|^{n_2}}{|\det(m_2)|^{n_1}}.$$

This satisfies $h^P(zg) = h^P(g)$ for central elements $z = \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix}$, $\lambda \neq 0$.

The associated L-function to f is

$$L(s, f) = \prod_p \prod_{i=1}^n \left(1 - \frac{\alpha_{p,i}}{p^s}\right)^{-1}$$

for some $\alpha_{p,i} \in \mathbb{C}$. The famous Ramanujan conjecture states that $|\alpha_{p,i}| = 1$. We also have the Rankin–Selberg L-function, which is a double product

$$L(s, f \times f) = \prod_p \prod_{i=1}^n \prod_{j=1}^n \left(1 - \frac{\alpha_{p,i}\alpha_{p,j}}{p^s}\right)^{-1}.$$

It is the Rankin–Selberg L-function that will appear in the constant term of Eisenstein series. Langlands computed this and proposed to Shahidi (who was then a grad student) that this can be used to prove the functional equation for the Eisenstein series.

$L(s, f)$ and $L(s, f \times f)$ have functional equations, due to Godement–Jacquet and Jacquet–Piatetski-Shapiro–Shahidi respectively. Define

$$\Lambda(s, f) = \pi^{-\frac{ns}{2}} \prod_{i=1}^n \Gamma\left(\frac{s + \alpha_i}{2}\right) L(s, f)$$

where $\alpha_i \in \mathbb{C}$ are called the Langlands parameter. Conjecturally $\alpha_i \in i\mathbb{R}$. This is now understood to be the Ramanujan conjecture at infinity. The functional equation is

$$\Lambda(s, f) = \epsilon(f)\Lambda(1 - s, \tilde{f})$$

where $|\epsilon(f)| = 1$ and $\tilde{f}(g) := f(w_0{}^tg^{-1}w_0)$ where w_0 is the long element. In terms of representation this is the contragredient. For the Rankin–Selberg product,

$$\Lambda(s, f \times f) := \pi^{-\frac{n^2 s}{2}} \prod_{i=1}^n \prod_{j=1}^n \Gamma\left(\frac{s + \alpha_i + \alpha_j}{2}\right) L(s, f \times f),$$

which satisfies the functional equation

$$\Lambda(s, f \times f) = \epsilon(f \times f)\Lambda(1 - s, \tilde{f} \times \tilde{f})$$

where $|\epsilon(f \times f)| = 1$.

Theorem 2 (Langlands)**.** *$E_f^P(g, s)$ has meromorphic continuation with functional equation*

$$E_f^P(g, s) = c_s E_{\tilde{f}}^P(g, 1 - s).$$

The functional equation is more complicated if P is not self-associate, i.e. not of the form $\begin{pmatrix} \mathrm{GL}(n) & * \\ 0 & \mathrm{GL}(n) \end{pmatrix}$. The idea of the proof is to first show that the constant term c_s has a functional equation.

Theorem 3 (Langlands–Shahidi)**.**

$$c_s = \frac{\Lambda(2ns - n, f \times f)}{\Lambda(1 + 2ns - n, f \times f)}$$

and

$$c_P E_f^P(g,s) = \phi_s(g) + c_s \phi_{1-s}(g).$$

This is a rather long technical computation, and I will not give a proof. This is a general phenomenon that holds for reductive groups.

As we saw last time, the Eisenstein series are not in L^2. We can truncate the constant term and take an inner product. Selberg simply subtracted away the constant term. Arthur introduced a truncation that remains automorphic.

Definition 4 (Arthur's truncated constant term for $T > 0$)**.**

$$c_P^T F(g) = \begin{cases} c_P F(g) & \text{if } h^P(g) \geq T, \\ 0 & \text{if } 0 \leq h^P(g) < T. \end{cases}$$

For $g = nmk$ and $m = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$, $h(g) = \frac{|\det m_1|^{n_2}}{|\det m_2|^{n_1}}$ is the height function.

Definition 5 (Truncation operator Λ^T on Eisenstein series)**.**

$$\Lambda^T E_f^P(g,s) = E_f^P(g,s) - \sum_{\gamma \in P_{\mathbb{Z}} \backslash \mathrm{SL}(2n,\mathbb{Z})} c_P^T E_f^P(\gamma g, s).$$

If P is not a maximal parabolic, the truncation is more complicated.

Remark. $\Lambda^T E_f^P$ is automorphic for $\mathrm{SL}(2n,\mathbb{Z})$, i.e.

$$\Lambda^T E_f^P(\gamma g, s) = \Lambda^T E_f^P(g,s)$$

for $\gamma \in \mathrm{SL}(2n,\mathbb{Z})$.

Now we want to talk about the Siegel sets. Recall the Iwasawa decomposition: if $z \in G/(K \cdot \mathbb{R})^\times$, then $g = y \cdot x$ where

$$y = \begin{pmatrix} y_1 \cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix} \text{ and } x = \begin{pmatrix} 1 & & & \\ & \ddots & x_{ij} & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

with $y_i > 0$ and $x_{ij} \in \mathbb{R}$.

Definition 6 (Siegel set)**.** Fix $\delta > 0$. The Siegel set is

$$S_\delta = \left\{ xy : -\frac{1}{2} \leq x_{ij} \leq \frac{1}{2}, y_i > \delta \right\}$$

Let $\mathcal{F}$ be the fundamental domain for $\Gamma \backslash G/(K \cdot \mathbb{R}^\times)$, $\Gamma = \mathrm{SL}(2n,\mathbb{Z})$.

Theorem 7. $\mathcal{F} \subset S_{\frac{\sqrt{3}}{2}}$.

A proof can be found in my book *Automorphic Forms and L-Functions for the Group* $\mathrm{GL}(n,\mathbb{R})$.

It is known that $E_f^P(g,s) - c_P E_f^P(g,s)$ is of rapid decay on S_δ ($\delta > 0$).

Proposition 8. *$\Lambda^T E_f^P$ is of rapid decay on any S_δ for $T \geq 1$.*

This implies that $\Lambda^T E_f^P \in L^2(\Gamma\backslash G/(K \cdot \mathbb{R}^\times))$, and that $\langle \Lambda^T E_f^P, \Lambda^T E_f^P \rangle < \infty$. More generally, we can take $|\langle \Lambda^T E_f^P(*, r), \Lambda^T E_f^P(*, s)\rangle| < \infty$.

Theorem 9 (Maass–Selberg relation)**.** *Assume $r, s \in \mathbb{C}$ with $r \neq \overline{s}$ and $r + \overline{s} - 1 \neq 0$. Then*

$$\langle \Lambda^T E_f^P(*, r), \Lambda^T E_f^P(*, s)\rangle = \langle f, f\rangle \cdot \left(\frac{T^{r+\overline{s}-1}}{r+\overline{s}-1} + c_{\overline{s}} \frac{T^{r-\overline{s}}}{r-\overline{s}} + c_r \frac{T^{\overline{s}-r}}{\overline{s}-r} + c_r c_s \frac{T^{1-r-\overline{s}}}{1-r-\overline{s}} \right).$$

Again we are in the special situation $G = \mathrm{GL}(2n)$, $P = \begin{pmatrix} \mathrm{GL}(n) & * \\ & \mathrm{GL}(n) \end{pmatrix}$ and f a cusp form on $\mathrm{GL}(n)$. This was first proved by Langlands, but he used a different truncation. This is the only way we know to prove the functional equation for Eisenstein series in the most general case.

The original proof by Selberg uses Green's theorem. Using Arthur's truncation, we don't need any differential operators; the proof is just unfolding.

Proof. First we can show that

$$\left\langle \Lambda^T E_f^P(*, r), \sum_{\gamma \in P_{\mathbb{Z}}\backslash\Gamma} c_P^T E_f^P(\gamma \cdot *, s) \right\rangle = 0.$$

This implies

$$\begin{aligned} &\langle \Lambda^T E_f^P(*, r), \Lambda^T E_f^P(*, s)\rangle \\ =&\langle \Lambda^T E_f^P(*, r), E_f^P(*, s)\rangle \\ =&\int_{P_{\mathbb{Z}}\backslash G/(K\cdot\mathbb{R}^\times), h^P(g)\geq T} -c_r\phi_{1-r}(g)E(g,\overline{s})dg + \int_{P_{\mathbb{Z}}\backslash G/(K\cdot\mathbb{R}^\times), h^P(g)<T} \phi_r(g)E(g,\overline{s})dg. \end{aligned}$$

Unraveling the sum, we get a very simple integral. □

Starting next week, Xiaoqing Li will give a series of lectures on the application of Maass–Selberg relations on the lower bounds of Rankin–Selberg L-functions. For cusp forms f on $\mathrm{GL}(n)$, we get

$$L(1+it, f\times f) \gg \frac{1}{(\log|t|)^{3+\epsilon}}$$

for large t. We don't know that $f \times f$ is automorphic. Since we are just working with $f \times f$, we only need the Maass–Selberg relations on the parabolic $P = \begin{pmatrix} \mathrm{GL}(n) & * \\ & \mathrm{GL}(n) \end{pmatrix}$, which is the case we considered today.