MATH G9905 RESEARCH SEMINAR IN NUMBER THEORY (FALL 2014)

LECTURE 11 (NOVEMBER 25, 2014) MICHAEL WOODBURY

AN ADELIC KUZNETSOV TRACE FORMULA FOR GL(4)

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Abstract. An important tool in analytic number theory for GL(2)-type questions is Kuznetsov's trace formula. Recently, in work of Blomer and of Goldfeld/Kontorovich, generalizations of this to GL(3) have been given which are useful for number theoretic applications. In my talks I will discuss joint work with Dorian Goldfeld in which we further generalize the said GL(3) results to GL(4). I will discuss some of the new features and complications which arise for GL(4) as well as applications to low lying zeros of L-functions and a vertical Sato–Tate theorem.

1. GL(2)

To set things up, we will start by talking about GL(2) in classical language. Let $\{\varphi_j\}$ be a basis of Hecke Maass forms of full level (for simplicity). Each φ_j is an eigenfunction for the Laplacian operator

$$\Delta\varphi_j = \lambda_j\varphi_j$$

and the Hecke operator

$$T_p\varphi_j = a_j(p)\varphi_j$$

with a suitable normalization. For $z = x + iy \in \mathbb{H}$,

$$\varphi_j(z) = \sum_{n\neq 0} e^{2\pi inx}\sqrt{y}K_{it_j}(2\pi|n|y)a_j(n).$$

where $\lambda_j = \frac{1}{4} + t_j^2$.

Kuznetsov proved for "nice" test functions h,

$$\sum_{j=1}^{\infty} \frac{a_j(m)\overline{a_j(n)}}{\cosh(\pi t_j)}h(t_j) + \mathcal{E} = \frac{\delta_{n,m}}{\pi^2}\int_{-\infty}^{\infty} r\tanh(\pi r)h(r)\,dr + \sum_{c=1}^{\infty}\frac{s(n,m,c)}{c}\times J_h\left(\frac{4\pi\sqrt{nm}}{c}\right)$$

where $\mathcal{E}$ is the Eisenstein contribution, $s(n,m,c)$ is the Kloosterman sum and J_h is the Kloosterman integral.

Applications of this include:

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- Kuznetsov used this to give bounds for the Kloosterman sums

$$\sum_{c\leq X} \frac{s(n,m,c)}{c} \ll X^{\frac{1}{6}+\epsilon}.$$

- Important ingredient in proving GL(2)-type subconvexity results.
- Can be used to study $\{\varphi_j(n)\}$.
- Low lying zeros.

2. Applications (for GL(n))

We introduce the vertical Sato–Tate theorem. Define the Sato–Tate measure

$$d\mu_\infty^{(2)} = \begin{cases} \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}} & \text{if } |x|\leq 2, \\ 0 & \text{otherwise,} \end{cases}$$

and the Plancherel measure

$$d\mu_p^{(2)} = \begin{cases} \frac{(p+1)\sqrt{4-x^2}}{2\pi((p^{1/2}+p^{-1/2})^2-x^2)} & \text{if } |x|\leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Note that as $p\to\infty$, we have $d\mu_p^{(2)} \to d\mu_\infty^{(2)}$.

Theorem 1 (Sarnak, 1987)**.** *The sequence of eigenvalues $a_1(p), a_2(p), a_3(p), \cdots$ is equidistributed with respect to $d\mu_p^{(2)}$.*

Theorem 2 (Bruggeman)**.** *Let $h_T(\nu) = e^{\frac{-1/4-\nu^2}{T}}$, and $f:\mathbb{R}\to\mathbb{R}$ be continuous. Then*

$$\frac{\sum_{j=0}^{\infty} f(a_j(p))h_T(t_j)}{\sum_{j=0}^{\infty} h_T(t_j)} \to \int f\, d\mu_p^{(2)}$$

as $T\to\infty$.

Both of these results used the Selberg trace formula. In addition, Bruggeman proved that using the Kuznetsov trace formula, one can get a formula for the weighted sum

$$\frac{\sum_{j=0}^{\infty} \frac{f(a_j(p))h_T(t_j)}{\mathcal{L}_j}}{\sum_{j=0}^{\infty} \frac{h_T(t_j)}{\mathcal{L}_j}} \to \int f\, d\mu_\infty^{(2)},$$

where $\mathcal{L}_j = L(1, \operatorname{Ad}\varphi_j)$.

Fan Zhou generalized these. His proof is unconditional whenever one has an asymptotic character formula, which is known for GL(3).

3. Adelic KTF for GL(n)

Write $\mathbb{A}$ for the adeles of $\mathbb{Q}$, and v will denote a place of $\mathbb{Q}$. We have the Iwasawa decomposition

$$G = \mathrm{GL}_n = UTK$$

where U is the upper triangular unipotent matrices, T is the diagonal matrices, and K is the maximal compact. Denote

$$[G] = \mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A})/Z(\mathbb{Q}).$$

For $a = (a_1, a_2, \cdots, a_{n-1}) \in \mathbb{Q}^{n-1}$, we can define the character on U

$$\psi_{a,\infty}\left(\begin{pmatrix} 1 & x_1 & * & * \\ & \ddots & \ddots & * \\ & & 1 & x_{n-1} \\ & & & 1 \end{pmatrix}\right) = e^{2\pi i(x_1a_1+\cdots+x_{n-1}a_{n-1})}.$$

We set

$$\psi_a = \bigotimes \psi_{a,v} : U(\mathbb{Q})\backslash U(\mathbb{A}) \to \mathbb{C}^\times.$$

We have a unitary representation $H = \bigotimes H_v : T(\mathbb{A}) \to \sigma = \bigotimes_v \sigma_v$, where σ_v is a finite-dimensional representation of K_v. We also have a function $||\cdot||_{\mathrm{tor}}^{\nu} : T(\mathbb{A}) \to \mathbb{C}$ for $\nu = (\nu_1, \cdots, \nu_n) \in \mathbb{C}^n$. Define the Poincaré series

$$P^a(g, \nu) = \sum_{\gamma \in Z(\mathbb{Q})U(\mathbb{Q})\backslash G(\mathbb{Q})} \psi_0(\gamma g)H(\gamma g)||\gamma g||_{\mathrm{tor}}^{\nu}.$$

Note $H(utk) = \sigma(k)^{-1}H(t)$.

The trace formula is in principle very simple: we take two Poincaré series and compute their inner product in two different ways. For the first way, we want to calculate

$$\langle P^a(\cdot,\nu), P^b(\cdot,\nu')\rangle_{[G]} = \int_{[G]} \langle P^a(g,\nu), P^b(g,\nu')\rangle_\sigma \, dg.$$

At almost all places, the integrand is just $P^a\overline{P^b}$. Plugging in and unfolding, we get that this is equal to

$$\sum_{w\in W}\sum_{\tau\in Z(\mathbb{Q})\backslash T(\mathbb{Q})}\prod_v \int_{T_v} \langle \mathrm{Kl}_v(t_v,\nu,a,b,w,\tau), H_v(t_v)\rangle \, dt.$$

This is a Kloosterman integral for each place. The second way uses the spectral expansion

$$P^b(g,\nu) = \sum_j \langle P^b(\cdot,\nu), \varphi_j\rangle\varphi_j + \text{rest of spectrum}.$$

Our goal is to prove a formula of the type

$$\sum_{\lambda_j \le T} \frac{A_j(a)\overline{A_j(b)}}{\mathcal{L}_j} = c\delta_{a,b}T^{\Box} + O(T^{\triangle})$$

where $\triangle < \Box$.

From now on we will consider GL(4). Let

$$p_{T,R}^{\#}(\alpha_1,\alpha_2,\alpha_3) = e^{\frac{\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2}{2T}} \prod_{1\le j\ne k\le 3} \Gamma\left(\frac{2+R+\alpha_j-\alpha_k}{4}\right)$$

where $\alpha_4 = \alpha_1+\alpha_2+\alpha_3$. This is the Lebedev–Whittaker transform of

$$p_{T,R}(y) = \iiint_{\mathrm{Re}(\alpha)=0} p_{T,R}^{\#} \prod \frac{1}{\Gamma\left(\frac{\alpha_j-\alpha_k}{2}\right)} \overline{W_\alpha(y)} \, d\alpha$$

Next time we will bound the inner product using this.