MATH G9905 RESEARCH SEMINAR IN NUMBER THEORY (FALL 2014)

LECTURE 10 (NOVEMBER 18, 2014) YOUNGJU CHOIE

SCHUBERT EISENSTEIN SERIES

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Abstract. We define Schubert Eisenstein series as sums like usual Eisenstein series but with the summation restricted to elements of a particular Schubert cell, indexed by an element of the Weyl group. They are generally not fully automorphic. We will develop some results and methods for GL_3 that may be suggestive about the general case. The six Schubert Eisenstein series are shown to have meromorphic continuation and some functional equations. This is a joint work with D. Bump.

Today I will introduce Schubert Eisenstein series, which are not automorphic but have nice arithmetic properties.

1. Schubert Variety

Let G be a split reductive algebraic group defined over a global field $\mathbb{F}$, and B be a Borel subgroup. We know $B\backslash G$ is a flag variety. Since G is a reductive group, we can decompose

$$G = \bigcup_{w \in W} BwB$$

where $W = N_G(T)/T$ is the Weyl group, $B = TU$, and T is a maximal torus.

Let Y_w be the image of BwB in $B\backslash G$. Take X_w to be the Zariski closure of Y_w, i.e.,

$$X_w := \overline{Y_w} = \bigcup_{u \le w} Y_u,$$

where $\le$ is the Bruhat order. X_w is called the Schubert variety.

Example 1. If $G = \mathrm{GL}_3$, then $W = \{\mathrm{id}, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$, where for example

$$s_1 = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}$$

satisfies $s_1^2 = 1$ and corresponds to the simple root $\alpha_1 = (1, -1, 0)$.

For example, $X_{s_1s_2} = B\backslash B \cup B\backslash Bs_1s_2B$.

Last updated: February 28, 2015. Please send corrections and comments to phlee@math.columbia.edu.

2. Bott–Samelson Variety

Let $w \in W$, which we write as $s_{i_1} s_{i_2} \cdots s_{i_k}$ where s_{i_j} are simple reflections. Denote by α_{i_j} the simple root corresponding to s_{i_j}. Let $\mathfrak{w} = (s_{i_1}, \cdots, s_{i_k})$. Let P_{i_j} be the parabolic subgroup spanned by B and s_{i_j}, i.e., $P_{i_j} = B \cup B s_{i_j} B$. We have $B \backslash P_{i_j} \simeq \mathbb{P}^1$.

Consider the left action of B^k on $P_{i_1} \times P_{i_2} \times \cdots P_{i_k}$ given by

$$(b_1, \cdots, b_k)(p_{i_1}, p_{i_2}, \cdots, p_{i_k}) := (b_1 p_{i_1} b_2^{-1}, b_2 p_{i_2} b_3^{-1}, \cdots, b_k p_{i_k}).$$

The quotient $Z_{\mathfrak{w}} := B^k \backslash P_{i_1} \times \cdots \times P_{i_k}$ is called the Bott–Samelson variety. We have the following facts.

(1) There exists a morphism $\varphi_{\mathfrak{w}} : Z_{\mathfrak{w}} \to X_w$ induced from the "multiplication' map

$$(p_{i_1}, \cdots, p_{i_k}) \mapsto p_{i_1} p_{i_2} \cdots p_{i_k}.$$

(2) $\varphi_{\mathfrak{w}}$ is a surjective birational map.
(3) $Z_{\mathfrak{w}}$ is always nonsingular while X_w may be singular. Therefore, this gives a resolution of singularities of X_w.
(4) $\varphi_{\mathfrak{w}}$ may not be an isomorphism.
(5) We have:

Lemma 2. *When $\varphi_{\mathfrak{w}} : Z_{\mathfrak{w}} \to X_w$ is an isomorphism, every element in X_w can be represented uniquely as a product of $\iota_{\alpha_{i_1}}(\gamma_1)\iota_{\alpha_{i_2}}(\gamma_2) \cdots \iota_{\alpha_{i_k}}(\gamma_k)$, where $\iota_{\alpha_{i_j}} : \mathrm{SL}_2 \hookrightarrow G$ is an embedding such that the image is in the Levi subgroup of P_{i_j} (Chevalley embedding), and $\gamma_j \in B_{\mathrm{SL}_2} \backslash \mathrm{SL}_2$.*

Example 3. Let $G = \mathrm{GL}_3$. Then $X_{s_1 s_2} \ni A = \iota_{\alpha_1}(\gamma_1)\iota_{\alpha_2}(\gamma_2) = \begin{pmatrix} \gamma_1 & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & \gamma_2 \end{pmatrix}$.

Remark. When $\varphi_{\mathfrak{w}}$ is not an isomorphism, then every element of X_w can still be written in this form, but the representation will not necessarily be unique.

3. Schubert Eisenstein Series

Let G be defined over $\mathbb{F}$, and $B = TU$ where T is a maximal torus and U is the unipotent subgroup. Write $\mathbb{A}$ as the ring of adeles of $\mathbb{F}$. For every place v of $\mathbb{F}$, $G_v := G(\mathbb{F}_v)$. Let $\chi : T(\mathbb{A})/T(\mathbb{F}) \to \mathbb{C}^\times$ be a quasi-character. Let $(\Pi_v(\chi_v), V_v(\chi_v))$ be a principal series representation, where

$$V_v(\chi_v) = \left\{ f_v : G_v \to \mathbb{C} : f_v(bg) = (\delta^{\frac{1}{2}} \chi_v)(b) f_v(g), f_v \text{ is } K_v\text{-finite} \right\},$$

δ is the modular quasi-character, and $K = \prod_v K_v$ is a maximal compact subgroup.

For simplicity, we let $\chi = \bigotimes_v \chi_v$, where χ_v is unramified at every nonarchimedean place, i.e., $V_v(\chi_v)$ has a nonzero K_v-fixed vector f_v^0, normalized such that $f_v^0(1) = 1$. Let $V(\chi)$ be the space of finite linear combinations of $f = \bigotimes_v f_v$, where $f_v = f_v^0$ for almost all v.

For each $w \in W$ and $f \in V(\chi)$, we define

$$E_w(g) := \sum_{\gamma \in X_w} f(\gamma g).$$

This is called the "Schubert Eisenstein series".

If $w = w_0$ is the longest element, then $E_{w_0}(g) = E(g)$ is the usual Eisenstein series.

4. GL_3 examples

For simplicity, we take χ to be unramified at every place v. Take

$$(\delta^{\frac{1}{2}}\chi)\begin{pmatrix} y_1y_2 & * & * \\ & y_1 & * \\ & & 1 \end{pmatrix} = |y_1|^{2\nu_1+\nu_2}|y_2|^{\nu_1+2\nu_2}$$

for $\nu_1,\nu_2 \in \mathbb{C}$, so that, for any $k_v \in K_v = \mathrm{SL}(3,\mathcal{O}_v)$, we have

$$f_v^0\left(\begin{pmatrix} y_1y_2 & * & * \\ & y_1 & * \\ & & 1 \end{pmatrix} k_v;\nu_1,\nu_2\right) = |y_1|^{2\nu_1+\nu_2}|y_2|^{\nu_1+2\nu_2}.$$

We define

$$E_w(g;\nu_1,\nu_2) = \sum_{\gamma\in X_w(\mathbb{F})} f(\gamma g;\nu_1,\nu_2).$$

Example 4.

(1) If $w = \mathrm{id}$, then
$$E_{\mathrm{id}}(g;\nu_1,\nu_2) = f(g;\nu_1,\nu_2).$$

(2) If $w = s_1$, then
$$E_{s_1}(g;\nu_1.\nu_2) = \sum_{\gamma\in X_{s_1}(\mathbb{F})} f(\gamma g;\nu_1,\nu_2).$$
But we know $Z_{(s_1)} \simeq X_{s_1}$, where every element is of the form $\gamma = \iota_{\alpha_1}(\gamma_1)$ with $\gamma_1 \in B_{\mathrm{SL}_2}\backslash\mathrm{SL}_2$, so the sum can be written as
$$\sum_{\gamma_1\in B_{\mathrm{SL}_2}\backslash\mathrm{SL}_2} f\left(\begin{pmatrix}\gamma_1 & \\ & 1\end{pmatrix} g;\nu_1,\nu_2\right).$$
This is essentially the GL_2-Eisenstein series.

(3) If $w = s_1s_2$ (the case $w = s_2s_1$ is similar), then
$$E_{s_1s_2}(g;\nu_1,\nu_2) = \sum_{\gamma\in X_{s_1s_2}(\mathbb{F})} f(\gamma g;\nu_1,\nu_2).$$
Since $Z_{(s_1s_2)} \simeq X_{s_1s_2} \ni \gamma = \iota_{\alpha_1}(\gamma_1)\iota_{\alpha_2}(\gamma_2)$ with $\gamma_1,\gamma_2 \in B_{\mathrm{SL}_2}\backslash\mathrm{SL}_2$, the sum can be written as
$$\sum_{\gamma_1,\gamma_2\in B_{\mathrm{SL}_2}\backslash\mathrm{SL}_2} f(\iota_{\alpha_1}(\gamma_1)\iota_{\alpha_2}(\gamma_2)g;\nu_1,\nu_2) = \sum_{\gamma_2\in B_{\mathrm{SL}_2}\backslash\mathrm{SL}_2} E_{s_1}(\iota_{\alpha_2}(\gamma_2)g;\nu_1,\nu_2).$$

We normalize

$$E_w^*(g;\nu_1,\nu_2) = \zeta^*(3\nu_1)\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2+1)E_w(g;\nu_1,\nu_2)$$

where $\zeta^*(s) = \prod_v \zeta_v(s)$, with

$$\zeta_v(s) = \begin{cases}\Gamma_v(s) & \text{if } v \text{ is archimedean,}\\ (1-q_v^{-s})^{-1} & \text{if } v \text{ is nonarchimedean,}\end{cases}$$

and $q_v = |\mathcal{O}_v/\mathfrak{p}_v|$ where $\mathfrak{p}_v$ is the maximal ideal of $\mathcal{O}_v$.

We have the following facts for $w = w_0$.

(1) $E_{w_0}(g;\nu_1,\nu_2)$ is analytic except poles at $\nu_1,\nu_2,1-\nu_1-\nu_2\in\left\{0,\frac{2}{3}\right\}$.
(2) $E^*_{w_0}(g;\nu_1,\nu_2)=E^*_{w_0}(g;w(\nu_1,\nu_2))$ for all $w\in W$, where

$$w(\nu_1,\nu_2)=\begin{cases}\left(\frac{2}{3}-\nu_1,\nu_1+\nu_2-\frac{1}{3}\right) & \text{if } w=s_1,\\ \left(\nu_1+\nu_2-\frac{1}{3},\frac{2}{3}-\nu_2\right) & \text{if } w=s_2.\end{cases}$$

Proposition 5.

(1) *$E^*_{s_1}(g;\nu_1,\nu_2)$ has a meromorphic continuation for all $\nu_1,\nu_2\in\mathbb{C}$, and similarly for $E^*_{s_2}$.*
(2) *$E^*_{s_1}(g;\nu_1,\nu_2)=E^*_{s_1}(g;s_2(\nu_1,\nu_2))$.*
(3) *$E^*_{s_1s_2}(g;\nu_1,\nu_2)$ has a meromorphic continuation for all $\nu_1,\nu_2\in\mathbb{C}$.*
(4) *$E^*_{s_1s_2}(g;\nu_1,\nu_2)=E^*_{s_1s_2}(g;s_2(\nu_1,\nu_2))$.*

5. Fourier Expansion of Eisenstein Series

We write $E(g)=E(g;\nu_1,\nu_2)=E_{w_0}(g;\nu_1,\nu_2)$ following Bump's notations. It is known that

$$E(g)=E_0^0(g)+\sum_{\gamma_1\in B_{\mathrm{SL}_2}\backslash\mathrm{SL}_2}E_{0,1}(\iota_{\alpha_1}(\gamma_1)g)+\sum_{\gamma_1\in U_{\mathrm{SL}_2}\backslash\mathrm{SL}_2}W(\iota_{\alpha_1}(\gamma_1)g)$$

where:

- for c,d,

$$E_c^d(g)=\int_{(\mathbb{A}/\mathbb{F})^2}E\left(\begin{pmatrix}1 & & x_3\\ & 1 & x_1\\ & & 1\end{pmatrix}g\right)\psi(cx_3+dx_1)\,dx_3\,dx_1$$

  and

$$E_{c,d}(g)=\int_{(\mathbb{A}/\mathbb{F})^3}E\left(\begin{pmatrix}1 & x_2 & x_3\\ & 1 & x_1\\ & & 1\end{pmatrix}g\right)\psi(cx_2+dx_1)\,dx_3\,dx_2\,dx_1$$

  where ψ is an additive character of $\mathbb{A}/\mathbb{F}$.
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$$W(g)=\int_{(\mathbb{A}/\mathbb{F})^3}E\left(\begin{pmatrix}1 & x_2 & x_3\\ & 1 & x_1\\ & & 1\end{pmatrix}g\right)\psi(x_2+x_1)\,dx_1\,dx_2\,dx_3$$

  and we denote $W^*(g)$ to be the same integral of E^*.

Theorem 6.

(1) $E^*(g;\nu_1,\nu_2)=H^*(g;\nu_1,\nu_2)+\sum_{\substack{w^2=1\\ w\in\{\mathrm{id},s_1,s_2\}}}\left(E^*_{s_1s_2}(g;w(\nu_1,\nu_2))-E^*_{s_1}(g;w(\nu_1,\nu_2))\right).$
(2) *$H^*(g)$ is entire in ν_1,ν_2.*

6. Applications

- It is known that the GL_3-Eisenstein series $E(g)$ has a pole at $\nu_1 = \nu_2 = 0$.
- Let $\kappa(g)$ be the coefficient of ν_1^{-1} in the Taylor expansion of E at $\nu_1 = \nu_2 = 0$.
- (Bump–Goldfeld) If $F/\mathbb{Q}$ is a cubic field, and $\mathfrak{a}$ is an ideal class in F, one may associate with $\mathfrak{a}$ a compact torus of GL_3. If $L_{\mathfrak{a}}$ is the period of $\kappa(g)$ on this torus, then the Taylor expansion of L-function is
$$L(s, \mathfrak{a}) = \frac{\rho}{s} + L_{\mathfrak{a}} + O(s).$$
Therefore,
$$L(s; \theta) = \sum_{\mathfrak{a} \text{ ideal class}} \theta(\mathfrak{a}) L(s, \mathfrak{a})$$
for θ a character on ideal classes.
- (Connection with Schubert Eisenstein series) Write
$$\zeta^*(s) = \frac{\rho}{s} + \delta + O(s).$$
Then
$$E^{**}_{s_1}(g; \nu_1, \nu_2) = \zeta^*(3\nu_1) E_{s_1}(g; \nu_1, \nu_2) = \frac{\rho}{3\nu_1} + \phi_{s_1}(g; \nu_2) + O(\nu_1).$$

Theorem 7.
$$\kappa(g) = \frac{\rho}{s}\zeta^*(2)\left(\widehat{E}^{**}_{s_2 s_1}(g; 0, 0) + E^{**}_{s_1}(g; 1, 0)\right) + c_0$$
where
$$\widehat{E}^{**}_{s_2 s_1}(g; \nu_1, \nu_2) = E^{**}_{s_2 s_2}(g; \nu_1, \nu_2) - E^{**}_{s_2}(g; \nu_1, \nu_2) - E^{**}_{s_2}(g; s_1(\nu_1, \nu_2))$$
and
$$c_0 = \frac{\rho}{3}(\delta\zeta^*(-1) + \rho(\zeta^*)'(-1)).$$