# MATH G9905 RESEARCH SEMINAR IN NUMBER THEORY (FALL 2014)

## LECTURE 1 (SEPTEMBER 9, 2014)
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## MAASS–SELBERG RELATIONS FOR EISENSTEIN SERIES ON GL(2)

NOTES TAKEN BY PAK-HIN LEE

### 1. Introduction

This is a research seminar that will go on for the entire year. This is aimed for graduate students who are working on a project to present their work, but postdocs and faculty are welcome too. There will occasionally be outside speakers.

I will give two lectures on the Maass–Selberg relations — the first on GL(2) and the second on GL($n$). This is joint work with Xiaoqing Li, who will continue my lectures. In 1899, de la Vallée-Poussin proved that

$$\zeta(s) \neq 0$$

for $\sigma > 1 - \frac{c}{\log(|t|+2)}$, where $s = \sigma + it$ and $c > 0$ is a constant. This is equivalent to

$$\zeta(1+it) > \frac{c}{\log(|t|+2)}.$$

The constant $c$ is effective. This result was improved by his joint work with Hadamard.

In a paper by Jacquet–Shalika, they showed that if $\pi$ is an automorphic cuspidal representation for $\mathrm{GL}(n, \mathbb{A})$, then

$$L(1+it, \pi) \neq 0$$

for $t \in \mathbb{R}$. In the last ten years, people have been trying to make this effective. Now we can prove that

$$L(1+it, \pi) > \frac{c}{\log(|t|+2)}.$$

This is due to Sarnak–Gelbart.

What about the Rankin–Selberg $L$-functions $L(s, \pi \times \pi)$? If $L(s, \pi) = \sum \frac{a(n)}{n^s}$, then on GL(2) we have

$$L(s, \pi \times \pi) = \sum \frac{a(n)^2}{n^s}.$$

(The formula is more complicated for groups other than GL(2), but can be made explicit using Euler products.) We don't know that $\pi \times \pi$ is automorphic unless $\pi$ is on GL(2) (Ramakrishnan). Thus we can't use the theorem of Jacquet–Shalika. Brumley proved that

$$L(1+it, \pi \times \pi) \gg \frac{1}{(2+|t|)^N}.$$

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Recently, Gelbart–Lapid proved that

$$L(1+it,\pi_1\times\pi_2\cdots\times\pi_r)\gg\frac{1}{(2+|t|)^N}$$

where the $\pi_i$'s are on any reductive groups. This result is computable but the bounds are very bad. This is the best of what's known. The proof of this result uses the Maass–Selberg relations.

**Theorem 1** (G.–Li)**.** *Let $\pi$ be a cuspidal automorphic representation on $\mathrm{GL}(n,\mathbb{A}_{\mathbb{Q}})$. Assume $\pi$ is tempered at all places. Then*

$$L(1+it,\pi\times\pi)>\frac{c}{(\log(2+|t|))^{3+\epsilon}}.$$

The proof has not been written up yet. This is the first result on a logarithmic lower bound. One of the main ingredients is the Maass–Selberg relations, which first appeared in Gelbart–Sarnak.

In the relative trace formula, we need to integrate terms of the form $\frac{1}{L(1+it)}$, so it is important to know that $L(1+it)$ is non-zero. This is why we are interested in the lower bounds for $L(1+it)$.

## 2. Maass–Selberg Relation (Simplest Case)

Let $G=\mathrm{GL}(2,\mathbb{R})$, $K=\mathrm{O}(2,\mathbb{R})$, $\Gamma=\mathrm{SL}(2,\mathbb{Z})$, $P=\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\subset G$ be the parabolic subgroup, $N=\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}\subset G$ be the nilpotent radical, and $M=\begin{pmatrix}* & 0\\ 0 & *\end{pmatrix}\subset G$ be the Levi subgroup. Let $\mathcal{F}=\Gamma\backslash G/(K\cdot\mathbb{R}^\times)$.

Given a function $\phi: G/(K\cdot\mathbb{R}^\times)\to\mathbb{C}$ satisfying

$$\phi(ng)=\phi(g)$$

for all $n\in N$ and $g\in G$, we can construct the Eisenstein series

$$E(g,\phi):=\sum_{\gamma\in(P\cap\Gamma)\backslash\Gamma}\phi(\gamma g).$$

The classical Eisenstein series is when we take $\phi_s(g)=y^s$ for $g=\begin{pmatrix}y & x\\ 0 & 1\end{pmatrix}$, $y>0$, $x\in\mathbb{R}$. Then

$$\begin{aligned}E(g,s)=E(g,\phi_s)&=\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}}\frac{y^s}{|cz+d|^{2s}}\\&=y^s+c_sy^{1-s}+\text{higher terms in }(*)e^{2\pi ilx}\ (l\neq 0)\end{aligned}$$

where

$$c_s=\frac{\zeta^*(2s-1)}{\zeta^*(2s)}$$

and

$$\zeta^*(s)=\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

**Definition 2** (Constant term along a parabolic $P$)**.** Let $\phi : G/(K \cdot \mathbb{R}^\times) \to \mathbb{C}$. Then the constant term along $P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ is $c_P\phi$ where

$$c_P\phi(g) := \int_0^1 \phi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) du.$$

On $\mathrm{GL}(n)$, there are more parabolics and the constant terms get more complicated.

For example, we have

$$c_P E(g,s) = y^s + c_s y^{1-s}.$$

It is well-known that the Eisenstein series is not square-integrable:

$$\iint_{\Gamma\backslash G/(K\cdot\mathbb{R}^\times)} \left| E\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, s\right)\right|^2 \frac{dxdy}{y^2} = \infty.$$

This is due to the presence of the constant term: either $\int |y^s|^2$ or $\int |y^{1-s}|^2$ is not integrable.

The Petersson inner product is

$$\langle f_1, f_2 \rangle := \iint_{\mathcal{F}} f_1(g)\overline{f_2(g)} d^\times g.$$

Maass and Selberg had the idea of cutting off the constant term, so that we can integrate the Eisenstein series. They proposed

$$E^T(g,s) = \begin{cases} E(g,s) & \text{if } 0 \le y \le T, \\ E(g,s) - c_P E(g,s) & \text{if } y > T. \end{cases}$$

One problem is that this modified Eisenstein series is not continuous and not automorphic. But if we simply restrict to the fundamental domain $\mathcal{F}$, $\langle E^T, E^T \rangle$ is well-defined. The original proof is by Green's theorem, but it isn't the right way to do it. Arthur proposed another way to do the truncation.

## 3. Arthur's Truncation of Eisenstein series

Let $T > 0$. As usual we write $g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$ for $g \in G/(K \cdot \mathbb{R}^\times)$.

**Definition 3** (Truncated constant term)**.** For $\phi : G/(K \cdot \mathbb{R}^\times) \to \mathbb{C}$, we put

$$c_P^T\phi(g) := \begin{cases} 0 & \text{if } 0 \le y \le T, \\ c_P\phi(g) & \text{if } y > T. \end{cases}$$

**Definition 4** (Truncation operator)**.**

$$\Lambda^T\phi(g) := \phi(g) - \sum_{\gamma \in \Gamma\cap P\backslash\Gamma} c_P^T(\gamma g).$$

*Remark.* If $\phi$ is automorphic, i.e. $\phi(\gamma g) = \phi(g)$, then $\Lambda^T\phi(g)$ is also automorphic.

Let me prove some properties of this truncation operator.

**Definition 5** (Siegel set $S_\delta$)**.** Let $\delta > 0$. Then

$$S_\delta := \left\{ x + iy : -\frac{1}{2} \le x \le \frac{1}{2}, y > \delta \right\}.$$

**Proposition 6.** *Let $T \ge 1$. Then $\Lambda^T E(g,s)$ is of rapid decay on any Siegel set $S_\delta$ with $\delta > 0$.*

*Proof.* We know that $E(g,s) - c_P E(g,s)$ is of rapid decay in $S_\delta$. Let $g \in \mathcal{F}$. Then

$$c_P^T E(\gamma g) = \begin{cases} c_P E(g,s) & \text{if } \gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } y > 1, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that $\Lambda^T E(g,s)$ is of rapid decay on $\mathcal{F}$. Since $\Lambda^T E$ is automorphic, it is of rapid decay everywhere. ◻

Let $s, r \in \mathbb{C}$, and $E_s = E(g,s)$. Then $\langle \Lambda^T E_s, \Lambda^T E_r \rangle$ is well-defined.

**Theorem 7** (Maass–Selberg Relation)**.** *Let $r, s \in \mathbb{C}$ with $s \ne \overline{r}$ and $s + \overline{r} \ne 1$. Then*

$$\langle \Lambda^T E_s, \Lambda^T E_r \rangle = \frac{T^{s+\overline{r}-1}}{s+\overline{r}-1} + c_s \frac{T^{\overline{r}-s}}{\overline{r}-s} + c_{\overline{r}} \frac{T^{s-\overline{r}}}{s-\overline{r}} + c_s c_{\overline{r}} \frac{T^{1-s-\overline{r}}}{1-s-\overline{r}}.$$

In the remainder of this lecture I will give Arthur's proof. Next time we will generalize this to higher rank Eisenstein series, but the main idea of the proof is exactly the same.

*Proof.* The first step is $\langle \Lambda^T E_s, \Lambda^T E_r \rangle = \langle \Lambda^T E_s, E_r \rangle$. We need to prove

$$I := \left\langle \Lambda^T E_s, \sum_{\gamma \in \Gamma \cap P \backslash \Gamma} c_P^T E(\gamma g, r) \right\rangle = 0$$

This can be proved by Rankin–Selberg. Unraveling the sum, we get

$$\begin{aligned} I &= \iint_{\mathcal{F}} \Lambda^T E\left( \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, s \right) \overline{\sum_\gamma c_P^T E\left( \gamma \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, r \right)} \frac{dx dy}{y^2} \\ &= \int_0^1 \int_0^\infty \Lambda^T E\left( \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, s \right) \overline{c_P^T E\left( \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, r \right)} \frac{dx dy}{y^2}. \end{aligned}$$

For $0 \le y \le T$, the second term is 0; for $y > T$, the second term is $y^r + c_r y^{1-r}$ but the first term is a power series in $e^{2\pi i l x}$ with $l > 0$. So the integral is 0.

The final step is a computation. Unfolding,

$$\begin{aligned} \langle \Lambda^T E_s, E_r \rangle &= \int_0^1 \int_0^\infty (y^s - c_P^T E(g,s)) \cdot E(g, \overline{r}) \frac{dx dy}{y^2} \\ &= \int_{y=0}^T \int_{x=0}^1 y^s E(g,\overline{r}) \frac{dx dy}{y^2} - \int_{y=T}^\infty \int_{x=0}^1 c_s y^{1-s} E(g,\overline{r}) \frac{dx dy}{y^2} \\ &= \int_0^T y^s (y^{\overline{r}} + c_{\overline{r}} y^{1-\overline{r}}) \frac{dy}{y^2} - c_s \int_T^\infty y^{1-s} (y^{\overline{r}} + c_{\overline{r}} y^{1-\overline{r}}) \frac{dy}{y^2} \end{aligned}$$

which is easy to compute. ◻

If $\phi$ is cusp form on $\mathrm{GL}(n)$, it induces a cuspidal automorphic representation on the Levi of $\mathrm{GL}(2n)$. Xiaoqing Li will explain how to use this to get a lower bound for $L(1+it, \phi \times \phi) \gg \frac{1}{(\log t)^{3+\epsilon}}$.

Reference: Paul Garrett, *Simplest Example of Truncation and Maass–Selberg Relations*[1].

[1]Available on his "Vignettes" webpage at `http://www-users.math.umn.edu/~garrett/m/v/maass_selberg_trivial.pdf`.