STRUCTURE AND REPRESENTATION THEORY OF p-ADIC REDUCTIVE GROUPS

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Abstract. These are notes I am taking for Shrenik Shah's ongoing course on p-adic reductive groups offered at Columbia University in Fall 2017 (MATH GR8674: Topics in Number Theory).

WARNING: I am unable to commit to editing these notes outside of lecture time, so they are likely riddled with mistakes and poorly formatted.

Contents

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1. Lecture 1 (September 12, 2017)

1.1. **Overview.** This class is about the structure of reductive groups over non-archimedean fields[1] and their (smooth) representation theory. More precisely, we will cover the following topics:

(1) Basics: Definitions, induction, Hecke algebras, Jacquet modules, supercuspidals;
(2) Structure theory: Iwahori–Hecke algebra, buildings;
(3) Principal series: Intertwining operators, formulae of Macdonald and Casselman–Shalika;
(4) Categorical perspective: Bernstein–Zelevinsky, Bernstein's theory;
(5) Beyond:
  (a) Langlands program (e.g. how the local and global aspects of the theory interact with and inform of each other);
  (b) p-adic deformation of automorphic forms (e.g. Hida families);
  (c) Rankin–Selberg method;
  (d) Eisenstein series;
  (e) Trace formula, "mixed aspects" (e.g. generalization of Ribet's lemma), arithmetic invariant theory (AIT[2]).

Here are a few words on motivation. Langlands program predicts that there is a connection

$$\text{Automorphic world} \leftrightsquigarrow \text{Galois world}$$

$$\begin{array}{lcc} \text{Global} & \Pi = \bigotimes_v \Pi_v & G_{\mathbf{Q}} \text{ acting on } V_{/\mathbf{C}} \\ & & \Big\downarrow \text{restriction} \\ \text{Local} & ? & G_{\mathbf{Q}_p} \text{ acting on } V_{/\mathbf{C}} \end{array}$$

Roughly speaking, the representation theory of p-adic reductive groups corresponds to the study of local automorphic forms. To construct such representations, we will study induction and compact induction. A natural question is whether all (reasonable) representations come from induction from a parabolic subgroup. One amazing feature of the theory for p-adic groups, as opposed to real or complex Lie groups, is the existence of *supercuspidal* representations, namely ones that do not arise from induction. They correspond to a specific type of representations of $G_{\mathbf{Q}_p}$ on the Galois side, but a purely local proof is only known in the case of GL(2); all known proofs of the local Langlands correspondence for GL(n) are global in nature.

1.2. **Smooth representations.** For the representation theory of p-adic groups, the correct analogy is going to be representations of finite groups, rather than the theory for reductive or algebraic groups which makes heavy use of weights. For example, consider the natural 2-dimensional representation[3] of $\mathrm{GL}_2(\mathbf{Q}_p)$ on V_2. This is a prototypical *algebraic* representation, but is very far from being *smooth*.

[1] In one of the Friday lectures we might consider the case of real or complex groups.
[2] also known as Bhargavalogy
[3] Note that we are being intentionally agnostic about the field of definition.

Definition 1.1. A *smooth representation* is an abstract representation (π, V) of G, where V is a vector space over $\mathbf{C}$, such that for all $v \in V$, there exists $K \subseteq G$ open and compact such that K fixes v.

A stabilizer of $v \in V_2$ as above is an upper triangular subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$ satisfying certain conditions on its diagonal, which is closed and not open: open subgroups of $\mathrm{GL}_2(\mathbf{Q}_p)$ look like $\mathrm{GL}_2(\mathbf{Z}_p)$ or $\mathbb{1}_2 + p^k M_2(\mathbf{Z}_p)$. Thus V_2 is not smooth. We will see that most smooth representations are going to be infinite-dimensional.

For now, let us relax that condition that G be the points of an algebraic group over a non-archimedean field.

Definition 1.2. A *locally profinite group* is a group G satisfying any one of the following equivalent conditions:

- $G = \varprojlim_{K \text{ open compact}} G/K$.
- G is totally disconnected and locally compact.

Example 1.3. The basic examples are:

- $\mathbf{Q}_p = \bigcup p^{-k}\mathbf{Z}_p$.
- $\mathbf{Q}_p^\times = p^{\mathbf{Z}} \times \mathbf{Z}_p^\times$.

This illustrates two directions in which p-adic reductive groups differ from each other: $\mathbf{Q}_p$ is unipotent, and $\mathbf{Q}_p^\times$ is reductive (abelian).

A *character* of G is a homomorphism $\psi : G \to \mathbf{C}^\times$ that is continuous, or equivalently that has open kernel (because $\mathbf{C}^\times$ has no small subgroups). Moreover, a *unitary* character is one whose image is contained in S^1.

Example 1.4. Characters of $\mathbf{Q}_p$ are unitary. Characters of $\mathbf{Q}_p^\times$ are products of a character of $\mathbf{Z}$ (pick an element) and one of $\mathbf{Z}_p^\times$ (use local class field theory). This illustrates why we are going to focus on reductive groups rather than unipotent groups: unipotent groups are simultaneously too easy and too difficult.

One condition on smooth representations that will be useful for us is *admissibility*.

Definition 1.5. We say that a smooth representation (π, V) is *admissible* if V^K is finite-dimensional for all open compact K.

Smoothness is equivalent to $V = \bigcup_K V^K$, so admissibility ensures that the representation is "tame".

Smoothness and admissibility are closed under taking subgroups and quotients, so it makes sense to consider *irreducible* (smooth or admissible) representations. Given two representations π_1 and π_2, $\mathrm{Hom}(\pi_1, \pi_2)$ consists of the linear maps that are G-equivariant. We denote

$$\mathrm{Rep}(G) = \text{category of smooth representations},$$

$$\mathrm{Rep}^{\mathrm{adm}}(G) = \text{category of admissible representations}.$$

Recall that averaging is very useful in the representation theory of finite groups, and we will see that this technique carries over to p-adic groups.

Let $K \subset G$ be open compact, and $\widehat{K}$ be the set of irreducible smooth K-representations. Then

$$V = \bigoplus_{\rho \in \widehat{K}} V^\rho,$$

where V^ρ is the sum of all K-subspaces isomorphic to ρ, called the ρ-isotypic component of V.

Proposition 1.6. *Taking K-fixed vectors is exact on smooth representations.*

Proof. Left-exactness is true in extreme generality. To see right-exactness, suppose there is a surjection $W \twoheadrightarrow U$ and $u \in U^K$. Take any preimage $w \mapsto u$. It might not be a K-fixed vector, but the formal integral (really a finite sum)

$$\frac{1}{\mu(K)} \int_K \pi(k) \cdot w \, dk$$

is stable by K. □

Proposition 1.7. *Exactness of $U \to V \to W$ is equivalent to exactness of $U^K \to V^K \to W^K$ for all K.*

Proof. We have already seen one direction of this statement. The proof of the converse is not difficult and omitted. □

For (π, V) and $H \subset G$, define

$$V(H) = \{v - \pi(h)v \mid v \in V, h \in H\}.$$

For open compact K, we have

$$V(K) = \bigoplus_{\rho \neq \mathbb{1}_K} V^\rho.$$

Later on, we will introduce the Jacquet modules to study $V(H)$.

If (π, V) is any representation of G (not necessarily smooth), then

$$V^\infty = \bigcup_K V^K \subset V$$

is stable under G, and in fact a smooth representation of G. This construction appears in the p-adic Langlands program, where G acts on certain Banach spaces. For our purpose, we will see an application next week.

Exercise. If $0 \to U \to V \to W \to 0$ is exact, then $0 \to U^\infty \to V^\infty \to W^\infty$ is left-exact. Why not right-exact?

Let $H \subset G$ be a subgroup and (σ, V) be any representation of H. Then we can consider the (abstract) induced representation

$$I_H^G(\sigma) = \{f : G \to V \mid f(hg) = \sigma(h) f(g)\}.$$

Even if σ is smooth, $I_H^G(\sigma)$ might not be. This suggests that we might at certain times need to briefly leave the category of smooth representations.

2. Lecture 2 (September 14, 2017)

2.1. **Induced representations.** Induced representations provide a way of constructing representations, especially infinite-dimensional ones in the case of smooth representations. It is natural to see how far we get by considering induced representations, and then study what we miss. We will see that the two procedures of induced representations and compactly induced representations are going to be "orthogonal", with the former being as far as possible from supercuspidals, and the latter related to finite groups and producing supercuspidals.

Let G be a locally profinite group and $H \subseteq G$ be a closed subgroup. Our main example will be $G = \mathrm{GL}_n(\mathbf{Q}_p)$ with varying H. Let (σ, W) be a representation of H. Then (abstract) *induction* is defined as

$$I_H^G(\sigma) = \{f : G \to W \mid f(hg) = \sigma(h)f(g) \text{ for all } h \in H\},$$

and *smooth induction* is

$$\mathrm{Ind}_H^G(\sigma) = I_H^G(\sigma)^\infty,$$

i.e., the space of functions $f : G \to W$ satisfying:

(1) $f(hg) = \sigma(h)f(g)$ for all $h \in H$;

(2) there exists an open compact K such that $f(gk) = f(g)$ for all $k \in K$.

The action of G on these is given by $(g \cdot f)(x) = f(xg)$.

There is a canonical H-homomorphism $\alpha : \mathrm{Ind}_H^G \sigma \to W$ given by $f \mapsto f(1)$.

Proposition 2.1 (Frobenius reciprocity)**.** *If (π, V) is a smooth representation of G, then there is a canonical isomorphism*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) \xrightarrow{\sim} \mathrm{Hom}_H(\pi, \sigma),$$
$$\phi \mapsto \alpha \circ \phi,$$

which is functorial in π and σ.

Proof. We construct an inverse to the map above. For any H-homomorphism $f : V \to W$, we define $f_* : V \to \mathrm{Ind}_H^G \sigma$ by $f_*(v)(g) = f(\pi(g)v)$. □

Fact. $\mathrm{Ind}_H^G(\sigma)$ is additive and exact.

Proof. Additivity is clear. By factoring through I_H^G, it suffices to check right-exactness. Suppose (σ, W) and (τ, U) are H-representations with a surjection $W \twoheadrightarrow U$. Let $\phi \in \mathrm{Ind}_H^G(\tau)$, and fix K such that $\phi(gk) = \phi(g)$ for all $k \in K$.

The support of ϕ can be decomposed as a disjoint union: $\mathrm{Supp}\,\phi = \coprod_g HgK$. Then $\phi(g)$ is stable by $\tau(H \cap gKg^{-1})$, where $H \cap gKg^{-1}$ is a compact subgroup of H. Thus we have a surjection $W^{H\cap gKg^{-1}} \twoheadrightarrow U^{H\cap gKg^{-1}}$, so $\phi(g)$ has a preimage w_g.

For each double coset representative g, we define

$$\Phi(hgk) = \sigma(h)w_g$$

for all $h \in H$ and $k \in K$, and set Φ to be 0 outside $\mathrm{Supp}\,\phi$. It is easy to see that $\Phi \in \mathrm{Ind}_H^G(\sigma)$ maps to ϕ. □

Compact induction $\text{c-Ind}_H^G(\sigma)$ is the space of functions $f : G \to W$ satisfying, in addition to (1) and (2):

(3) $\mathrm{Supp}(f) \subset HC$ for some compact subset C.

It is an exercise in Bushnell–Henniart that (1) and (3) together imply (2), but we will not prove this. If H is cocompact in G (e.g. a parabolic), then compact induction agrees with smooth induction.

For compact induction, we will typically have H open compact.

If H is open compact and (σ, W) is a representation of H, we define a canonical H-homomorphism

$$\alpha_c : W \to \text{c-Ind}_H^G \sigma$$
$$w \mapsto f_w,$$

where $\operatorname{Supp} f_w \subseteq H$ and $f_w(h) = \sigma(h)w$ for all $h \in H$.

Remark. Fix $\mathcal{W}$ a basis of W, and $\mathcal{G}$ a set of representations for G/H. Then

$$\{g f_w \mid g \in \mathcal{G}, w \in \mathcal{W}\}$$

is a basis of $\text{c-Ind}_H^G(\sigma)$.

Proposition 2.2 (Frobenius reciprocity)**.** *If (π, V) is a smooth representation of G, then there is a canonical isomorphism*

$$\operatorname{Hom}_G(\text{c-Ind}_H^G \sigma, \pi) \xrightarrow{\sim} \operatorname{Hom}_H(\sigma, \pi)$$
$$f \mapsto f \circ \alpha_c,$$

which is functorial in π and σ.

Proof. For any H-homomorphism $\phi : W \to V$, there exists a unique G-homomorphism $\phi_* : \text{c-Ind}_H^G \sigma \to V$ defined by $\phi_*(f_w) = \phi(w)$ by the previous remark. □

2.2. **Schur's lemma.** In the rest of this *entire* course, we will make the following

Hypothesis. G/K is *countable* for every (equivalently, for any one) open compact K.

This implies that every smooth irreducible representation has countable dimension, and that Schur's lemma holds:

Proposition 2.3 (Schur's lemma)**.** *If (π, V) is an irreducible smooth representation of G, then* $\operatorname{End}_G(V) = \mathbf{C}$.

Proof. Clearly $\operatorname{End}_G(V)$ is a division algebra, but we are not done yet because this might be of uncountable dimension over $\mathbf{C}$. Since any $v \in V$ generates, $\phi \in \operatorname{End}_G(V)$ is determined by $\phi(v)$. This implies $\operatorname{End}_G(V)$ is has countable dimension, as follows.

Take $\phi \in \operatorname{End}_G(V)$. If ϕ is transcendental over $\mathbf{C}$, then $\mathbf{C}(\phi) \subset \operatorname{End}_G(V)$ contains $\frac{1}{\phi - a}$ for all $a \in \mathbf{C}$ and must have uncountable dimension over $\mathbf{C}$, a contradiction. □

Corollary 2.4.

(1) *The center of G acts by scalars on irreducible smooth representations.*
(2) *If G is abelian, then all irreducible smooth representations are 1-dimensional.*

2.3. **Duality.** Let (π, V) be a representation of G. The *linear dual* is $(\tilde{\pi}, V^*)$, where $V^* = \operatorname{Hom}_{\mathbf{C}}(V, \mathbf{C})$ and the action is defined by

$$\langle \tilde{\pi}(g) v^*, v \rangle = \langle v^*, \pi(g^{-1}) v \rangle.$$

The *contragredient* of V is defined as $\widetilde{V} = (V^*)^\infty$.

Proposition 2.5. *Restriction to V^K induces an isomorphism*

$$\widetilde{V}^K \simeq (V^K)^*.$$

Proof. $\tilde{v} \in \widetilde{V}$ is fixed by K if and only if $\langle \tilde{v}, V(K) \rangle = 0$. □

If V^K is infinite-dimensional, then $\widetilde{V}^K$ has uncountable dimension. Thus we want to work with admissible representations whenever duality is involved. More precisely, we have the

Proposition 2.6. *The natural map $V \to \widetilde{\widetilde{V}}$ is an isomorphism if and only if V is admissible.*

Theorem 2.7. *Let (π, V) be admissible. Then (π, V) is irreducible if and only if $(\tilde{\pi}, \widetilde{V})$ is irreducible.*

Exercise. Read Chapter 1, Section 3 of Bushnell–Henniart, especially the last 3 pages.

Theorem 2.8. *Fix a positive semi-invariant measure $\dot{\mu}$ on $H\backslash G$. Let (σ, W) be an H-representation. Then*

$$(\text{c-Ind}_H^G \sigma)^\sim \cong \text{Ind}_H^G(\delta_{H\backslash G} \otimes \tilde{\sigma}).$$

Remark.

- H is not necessarily open.
- This is useful even when H is cocompact.

2.4. **Principal series.** Principal series are the primary objects that we will study for a big chunk of the course. Recall that we would like a complete understanding of supercuspidals and induction.

Let G be split over $\mathbf{Q}_p$ and B be a choice of Borel, with $B = T \cdot N$. Let $\alpha : T(\mathbf{Q}_p)/T(\mathbf{Z}_p) \to \mathbf{C}^\times$ be a character (recall that $T(\mathbf{Q}_p)/T(\mathbf{Z}_p) \simeq \mathbf{Z}^k$ non-canonically, so α is equivalent to naming k complex numbers). Consider the *principal series representation*

$$\text{Ind}_B^G \alpha \delta_B^{1/2}.$$

It is not always the case that this is irreducible.

The reasons for understanding principal series include the following:

- All automorphic representations look mostly like this locally.
- Eisenstein series is based on studying degenerate cases of principal series, and almost everything we can say about L-functions goes via Eisenstein series.

3. Lecture 3 (September 19, 2017)

3.1. **Duality.** Facts about Haar measure will be left as reading, but we will make precise issues involving duality.

Denote $\delta_{H\backslash G} = \delta_H^{-1} \delta_G\big|_H : H \to \mathbf{R}_+^\times$, where δ_G and δ_H are the modulus characters for G and H respectively. (Usually, G will have $\delta_G = 1$, i.e., G is unimodular.)

The measure $\mu_{H\backslash G}$ is defined so that the integral $\int_{H\backslash G} f(g)\, d\mu_{H\backslash G}$ has a value in $\mathbf{C}$ for all $f \in C_c^\infty(H\backslash G, \delta_{H\backslash G})$, the space of functions $f : G \to \mathbf{C}$ satisfying:

1. $f(hg) = \delta_{H\backslash G}(h) f(g)$;
2. f is compactly supported modulo H and smooth.

Note that there is a natural G-action on $C_c^\infty(H\backslash G, \delta_{H\backslash G})$ by right translation (making it $\text{c-Ind}_H^G \delta_{H\backslash G}$ but we shall not need this interpretation). Such a $\mu_{H\backslash G}$ is unique up to $\mathbf{R}_+^\times$ if we insist that it be:

1. right G-invariant;

(2) positive.

In Bushnell–Henniart, this is called a *semi-invariant measure*.

Theorem 3.1. *Let (σ, W) be a smooth H-module. Then*

$$(\text{c-Ind}_H^G \sigma)^\sim \simeq \text{Ind}_H^G \delta_{H\backslash G} \otimes \tilde{\sigma}.$$

Proof. The map is given as follows. Let $\widetilde{W}$ be the underlying space of $\delta_{H\backslash G} \otimes \tilde{\sigma}$, with the pairing $\widetilde{W} \times W \to \mathbf{C}$ given by $(\tilde{w}, w) \mapsto \langle \tilde{w}, w \rangle$. For $\phi \in \text{c-Ind}_H^G \sigma$ and $\Phi \in \text{Ind}_H^G \delta_{H\backslash G} \otimes \tilde{\sigma}$, consider $f : G \to \mathbf{C}$ given by $g \mapsto \langle \Phi(g), \phi(g) \rangle$. One checks that

$$(\Phi, \phi) \mapsto \int_{H\backslash G} f(g)\, d\mu_{H\backslash G}$$

defines a duality pairing between $\text{Ind}_H^G \delta_{H\backslash G} \otimes \tilde{\sigma}$ and $\text{c-Ind}_H^G \sigma$. □

We will renormalize this for the principal series representation.

3.2. **Unramified principal series.** Let $F/\mathbf{Q}_p$ be a finite extension, and G be a reductive group over F that is *quasi-split*, i.e., there exists a Borel defined over F; we will only consider principal series for quasi-split reductive groups. Fix a choice of Borel B over F, and maximal torus $T \subset B$ over F. Assume $T(F)$ has a canonical maximal compact $K_T = T(\mathcal{O}_F)$ (recall that $T/K_T \simeq \mathbf{Z}^k$). A character $\alpha : T(F)/T(\mathcal{O}_F) \to \mathbf{C}^\times$ is called an *unramified*[4] character of $T(F)$.

Example 3.2. Consider $G = \text{Res}_{\mathbf{Q}_p}^{\mathbf{Q}_{p^2}} \mathbf{G}_m$, where $\mathbf{Q}_{p^2}$ denotes the unique unramified quadratic extension of $\mathbf{Q}_p$. Then the maximal torus is $T = G$ itself, with maximal compact $K_T = \mathcal{O}_{\mathbf{Q}_{p^2}}^\times \subset \mathbf{Q}_{p^2}^\times$. Then $T/K_T \simeq \mathbf{Z}$. Unramified characters of T, i.e., characters of T that factor through K_T, correspond to naming a complex number.

The *unramified principal series* is defined as the induction[5]

$$i_B^G(\alpha) := \{f : G \to \mathbf{C} \text{ smooth} \mid f(bg) = \alpha(b)\delta_B^{\frac{1}{2}}(b)f(g)\},$$

where $\alpha : B(F) \to \mathbf{C}^\times$ is defined by lifting through $B(F) \to B(F)/N(F) = T(F)$. Note the normalization factor of $\delta_B^{\frac{1}{2}}(b)$. Then we have

- i_B^G is smooth and admissible.
- $\tilde{i}_B^G(\alpha) = i_B^G(\tilde{\alpha})$.

More generally we can define $i_P^G(\sigma)$ for any parabolic P.

Proposition 3.3. *$i_P^G\sigma$ is* unitary[6] *if σ is unitary.*

Definition 3.4. Let $\alpha : T(F)/T(\mathcal{O}_F) \to \mathbf{C}^\times$ be an unramified character.

- α is *unitary* if it lands in S^1.

[4]Why? Because of class field theory.

[5]In the notation from before, $i_B^G(\alpha) = \text{Ind}_B^G(\alpha\delta_B^{\frac{1}{2}})$.

[6]A character is unitary if it lands in S^1. In general, a representation is unitary if it has an invariant form; we will study this in more detail later.

- α is *regular* if it has no stabilizer under $W_G = N_G(T)/C_G(T)$, the rational Weyl group of G, where $N_G(T)$ and $C_G(T)$ are the normalizer and centralizer of T in G respectively, and W_G acts on characters via $\alpha^w(t) = \alpha(wtw^{-1})$.

As a preview of what we will talk about later in the course, we state the

Proposition 3.5. *$i_B^G(\alpha)$ is irreducible if α is unitary and regular.*

This is used in the theory of intertwining operators. Other applications of principal series include the following.

3.2.1. *Macdonald–Casselman formula.* Suppose G is split, with an integral model $K = G(\mathcal{O}_F)$ chosen "reasonably" (a hyperspecial maximal compact).

Fact. If $i_B^G(\alpha)^K \neq 0$, then $i_B^G(\alpha)$ contains some nice irreducible representation π_α, called a *K-spherical* (or spherical) representation.

We are interested in this because every automorphic representation looks like a spherical representation at almost all places. The Macdonald–Casselman formula, due to Macdonald and later reproved by Casselman, describes the qualitative behavior of π_α. More precisely, let $\phi \in \pi_\alpha$ be a K-fixed vector, with a corresponding K-fixed $\tilde{\phi} \in \tilde{\pi}_\alpha$. Consider $f_\alpha(g) = \langle \tilde{\phi}, \pi(g)\phi \rangle$ with $f_\alpha(1) = 1$, which is bi-K-invariant and hence determined by values on the torus (by the Cartan decomposition), or really a piece T^+ of the torus. The Macdonald–Casselman formula gives an explicit description of the function f_α.

3.2.2. *Casselman–Shalika formula.* For certain π's, there is a map

$$\pi \to C^\infty(N\backslash G, \chi) = \{f : G \to \mathbf{C} \text{ smooth} \mid f(ng) = \chi(n)f(g)\},$$

where $B = T \cdot N$ with maximal unipotent N, and $\chi : N \to \mathbf{C}^\times$ is a generic character. For irreducible admissible representations π that appear in $C^\infty(N\backslash G, \chi)$ (and not all do!), they do so with multiplicity one. The Casselman–Shalika formula gives an explicit description of such a model of π, known as a *Whittaker model.*

3.3. **Hecke algebras.** The main use of Hecke algebras in general is to cut down infinite-dimensional representation spaces into finite-dimensional pieces. They are useful in abstract to leverage noncommutative ring (or rng) theory, and useful concretely if a presentation is known. Today we will focus on the abstract application.

Let G be unimodular; this is the only situation where Hecke algebras are studied. The *Hecke algebra* is $\mathcal{H}(G) = C_c^\infty(G)$, the space of compactly supported smooth (i.e., locally constant) functions on G, with convolution product

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g)\, dx.$$

Note that $\mathcal{H}_G$ is a noncommutative ring without an identity. It turns out that the representation theory of G is the same of the representation theory of $\mathcal{H}(G)$, but both objects are complicated in their own ways.

For $K \subset G$ open, define

$$e_K := \frac{1}{\mu(K)} \cdot \mathrm{char}_K(\cdot)$$

which is an idempotent in $\mathcal{H}(G)$.

Theorem 3.6.

(1) $e_K * e_K = e_K$.
(2) $e_K * f = f$ *if and only if* $f(kg) = f(g)$ *for all* $k \in K$.
(3) $\mathcal{H}(G,K) := e_K * \mathcal{H}(G) * e_K$ *is the space of bi-*K*-invariant functions, which is a subalgebra with* e_K *as unit element.*

Proof. (1) is left as an exercise. (3) follows from (2) easily. To prove (2),

$$(e_K * f)(kg) = \int_G e_k(x) f(x^{-1}kg)\, dx = \int_G e_K(kx) f(x^{-1}g)\, dx = (e_K * f)(g)$$

from which the conclusion follows easily. □

Thus $\mathcal{H}(G) = \bigcup_K \mathcal{H}(G,K)$ is a union of commutative subalgebras whose representations are all finite-dimensional, as we will see later.

We say that M is a *smooth* $\mathcal{H}(G)$*-module* if the structure map

$$* : \mathcal{H}(G) \otimes_{\mathcal{H}(G)} M \to M$$

is surjective, i.e, $\mathcal{H}(G) * M = M$.

Proposition 3.7. M *is smooth if and only if for all* $m \in M$*, there exists* e_K *such that* $e_K * m = m$.

Proof. If $m = e_K * f * e_K * m'$, then $e_K * m = m$. □

Thus for M smooth, $M = \bigcup_K M^K$, where $M^K = \{m \mid e_K * m = m\}$.

4. Lecture 4 (September 21, 2017)

Today we will continue talking about Hecke modules, and then do some calculations for GL_2.

4.1. **Hecke modules.** Recall that a $\mathbf{C}$-vector space M is an $\mathcal{H}(G)$-module if there is a map $M \times \mathcal{H}(G) \to M$ (one could also use $\otimes$), and $\mathrm{Hom}_{\mathcal{H}(G)}(M_1, M_2)$ has the usual definition. We say that M is smooth if $\mathcal{H}(G) \cdot M = M$, or equivalently if for all $m \in M$ there exists an open compact K such that $e_K * m = m$ for the idempotent $e_K = \frac{1}{\mu(K)} \cdot \mathrm{char}(K)$.

Let us now make precise the similarity with smoothness for G-representations. Suppose (π, V) is a smooth representation of G. For $f \in \mathcal{H}(G)$ and $v \in V$, define

$$\pi(f)v = \int_G f(g)\pi(g)v\, dg$$

(which is really a countable sum) and check that convolution products behave well:

$$\pi(f_1 * f_2)v = \pi(f_1)\pi(f_2)v.$$

This equips V with the structure of an $\mathcal{H}(G)$-module. It is easy to see that this construction is functorial from the category of smooth G-representations into the category of smooth $\mathcal{H}(G)$-modules. To go in the opposite direction, we have the

Proposition 4.1. *If* M *is a smooth* $\mathcal{H}(G)$*-module, then there exists a unique homomorphism* $\pi : G \to \mathrm{Aut}_{\mathbf{C}}(M)$ *such that* (π, M) *is smooth and* $\pi(f)m = f * m$ *for all* $f \in \mathcal{H}(G)$.

Proof. If $e_K * m = m$, define $\pi(g)m = \mu(K)^{-1} f * m$ where $f = \mathrm{char}(gK)$. □

Thus there is an equivalence of categories between smooth G-representations and smooth $\mathcal{H}(G)$-modules, under which the K-fixed vectors correspond to $e_K * M$ which is a $\mathcal{H}(G,K)$-module.

With these two equivalent notions, we tend to use G to study functions and distributions, whereas we can name explicit elements in $\mathcal{H}(G)$ (which is useful, for instance, when we study vectors invariant under a given open compact subgroup).

4.2. **Calculations for** GL_2**.** Let $G = \mathrm{GL}_2(\mathbf{Q}_p)$, $B = \begin{pmatrix} * & * \\ & * \end{pmatrix}$, $T = \begin{pmatrix} * & \\ & * \end{pmatrix}$, $K = \mathrm{GL}_2(\mathbf{Z}_p)$, and $K_T = \begin{pmatrix} * & \\ & * \end{pmatrix} \subset K$.

4.2.1. *Iwasawa decomposition.*

$$B(\mathbf{Q}_p) \cdot \mathrm{GL}_2(\mathbf{Z}_p) = \mathrm{GL}_2(\mathbf{Q}_p).$$

Proof. Let $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Q}_p)$. If $a_{21} = 0$, then we are done.

If[7] $v_p(a_{21}) \geq v_p(a_{22})$, then

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & \\ -\frac{a_{21}}{a_{22}} & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ & * \end{pmatrix}.$$

If $v_p(a_{21}) < v_p(a_{22})$, we can use the Weyl group element to switch them:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{pmatrix}.$$

□

Remark. For GL_n, the same argument works: start with the bottom row and work upwards.

Corollary 4.2. *$B(\mathbf{Q}_p)\backslash G(\mathbf{Q}_p)$ is compact.*

4.2.2. *Cartan decomposition.*

$$K\backslash G/K = \coprod_{a \leq b} K \begin{pmatrix} p^a & \\ & p^b \end{pmatrix} K.$$

Proof. Use a Weyl group element to move the entry with the smallest valuation to the upper left corner, and kill the other elements in the first row and column. Then (in the case of GL_n) move down the diagonal.

Here is a slick way to see that this is a disjoint union. Note that the determinant recovers $a + b$. Shifting a and b by the same integer, we may assume they are both non-negative. Then $g\mathbf{Z}_p^n$ is a lattice preserved under K, so the structure of $\mathbf{Z}_p^n/g\mathbf{Z}_p^n$ as a $\mathbf{Z}_p$-module recovers a and b. □

The Cartan decomposition shows that $\mathcal{H}(G,K)$ has generators $\mathrm{char}\left(K \begin{pmatrix} p^a & \\ & p^b \end{pmatrix} K\right)$, and that G/K is countable.

[7] We adopt the convention that $v_p(p^a) = a$ and $|p^a|_p = p^{-a}$.

4.2.3. *Iwahori subgroup.* Let G be split and smooth over $\mathbf{Z}_p$. Consider the map

$$K = G(\mathbf{Z}_p) \to G(\mathbf{F}_p)$$

The Iwahori subgroup[8] I is defined as the preimage of $B(\mathbf{F}_p)$.

One of the reasons for why we prefer to work with I instead of K is the Iwahori decomposition

$$I = (I \cap N') \times (I \cap T) \times (I \cap N),$$

where N' (resp. N) is the lower (resp. upper) triangular unipotent subgroup[9].

The Iwahori decomposition interacts with the Iwasawa decomposition well. Pulling back the Bruhat decomposition

$$\bigcup_w B(\mathbf{F}_p) w B(\mathbf{F}_p) = G(\mathbf{F}_p)$$

to the $\mathbf{Z}_p$-points, we see that $IwI \cup I = K$. Multiplying B on both sides,

$$BIwI \cup BI = BK \implies BwI \cup BI = G(\mathbf{Q}_p)$$

and hence

$$Bw(I \cap N) \cup B(I \cap N') = G(\mathbf{Q}_p).$$

What about $\mathcal{H}(G, I)$? There is an (extended) affine Weyl group that parametrizes double cosets. It contains $\mathbf{Z}^k \rtimes W_0$ for the finite Weyl group W_0. Later we will see two ways of writing down $\mathcal{H}(G, I)$.

4.2.4. *Satake isomorphism.*

Theorem 4.3 (Satake, Haines–Rostami)**.** *Let G be split and smooth over $\mathbf{Z}_p$, T be a maximal torus, $B = TN$ be a Borel, and $K = G(\mathbf{Z}_p)$. Then there is a $\mathbf{C}$-algebra homomorphism $\xi : \mathcal{H}(G, K) \to \mathcal{H}(T, K_T)$ such that*

$$(\xi f)(t) = \delta_B^{\frac{1}{2}}(t) \int_N f(tn)\, dn,$$

which is injective, and in fact an isomorphism onto the Weyl invariants

$$\mathcal{H}(G, K) \xrightarrow{\sim} \mathcal{H}(T, K_T)^{W_G}.$$

As a consequence, $\mathcal{H}(G, K)$ is abelian. An alternative argument is to consider the anti-involution $g \mapsto {}^t g$, but the Satake isomorphism is more canonical.

4.2.5. *Generators for $\mathcal{H}(G, K)$.* For a maximal torus T, consider the cocharacters $X_*(T) = \mathrm{Hom}(\mathbf{G}_m, T)$ and the Weyl chamber

$$X_*^+(T) = \{\lambda \in X_*(T) \mid \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta\}.$$

Then a set of generators of $\mathcal{H}(G, K)$ is given by $\mathrm{char}(K\lambda(p)K)$ for $\lambda \in X_*^+(T)$. This follows from the Cartan decomposition.

[8]In some sense, the Iwahori is a canonical construction, even more so than maximal compacts.

[9]For general G, there is a notion of (B, N)-pairs that makes the Iwahori decomposition work.

Appendix A. Supplementary Lecture 1 (September 8, 2017): Reductive groups

Today's lecture will serve to introduce some background materials for the course, namely the theory of reductive groups over a field (not necessarily p-adic). The references are:

- Borel, *Linear Algebraic Groups*;
- Humphreys, *Linear Algebraic Groups*;
- Springer, *Linear Algebraic Groups*;
- Springer, *Reductive Groups* in the Corvallis proceedings.

"Linear" is an archaic term that is more commonly known as "affine" in modern language.

Definition A.1. An *algebraic group* is a group object in the category of algebraic varieties (thus multiplication and inverse are regular maps satisfying group axioms).

Example A.2 (General linear group and special linear group)**.**

$$\mathrm{GL}_n(R) = \{g \in M_n(R) \mid \det(g) \in R^\times\},$$
$$\mathrm{SL}_n(R) = \{g \in M_n(R) \mid \det(g) = 1\}.$$

An *affine* algebraic group is one that is affine as a variety, or equivalently one that admits an embedding $G \hookrightarrow \mathrm{GL}_n$. This is the reason why these groups were called *linear*. One of the beautiful parts of the theory of affine algebraic groups is their classification, which will reflect how rigid their structures are.

Definition A.3. For an affine algebraic group G, the *radical* $\mathrm{Rad}(G)$ is the identity component of its maximal normal solvable subgroup, and the *unipotent radical* $\mathrm{Rad}_u(G)$ consists of the unipotent[10] elements of the radical.

Example A.4.

- $B \subset \mathrm{GL}_n$ (the subgroup of upper triangular matrices)[11]: $\mathrm{Rad}(B) = B$ (because it is solvable), and $\mathrm{Rad}_u(B)$ consists of upper triangular matrices with 1's along the diagonal.
- GL_n: $\mathrm{Rad}(\mathrm{GL}_n) = \mathbf{G}_m$ (diagonally embedded center), and $\mathrm{Rad}_u(\mathrm{GL}_n) = 1$.
- $\mathbf{G}_m(R) = R^\times$: $\mathrm{Rad}(\mathbf{G}_m) = \mathbf{G}_m$ and $\mathrm{Rad}_u(\mathbf{G}_m) = 1$.
- $\mathbf{G}_a(R) = R$: $\mathrm{Rad}(\mathbf{G}_a) = \mathrm{Rad}_u(\mathbf{G}_a) = \mathbf{G}_a$, as can be seen by considering the embedding $\mathbf{G}_a \hookrightarrow \mathrm{GL}_2$ given by $a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$.

Definition A.5. A group is *simple* if it has no nontrivial connected normal subgroups.

Example A.6. SL_n is simple, but $\mathrm{SL}_n \times \mathrm{SL}_n$ is not.

Definition A.7. A group is *semisimple* if it has trivial radical.

Definition A.8. A group is *reductive* if it has trivial unipotent radical.

Root datum is a more symmetric version of root system, which is useful for studying reductive groups.

[10]For GL_n, an element g is unipotent if $g - I_n$ is nilpotent. For general G, an element g is unipotent if for all (equivalently, for any one) embeddings $G \hookrightarrow \mathrm{GL}_n$, the image of g is unipotent. There is an intrinsic definition, but we shall not need it.

[11]We will soon see that this is a prototypical example of an affine algebraic group that is not reductive.

Definition A.9. A *root datum* is the datum[12] $\Psi = (X, \Phi, X^\vee, \Phi^\vee, \vee, \langle \cdot, \cdot \rangle)$ where:

- $X, X^\vee$ are free $\mathbf{Z}$-modules of finite rank;
- there is a pairing $\langle \cdot, \cdot \rangle : X \times X^\vee \to \mathbf{Z}$ making $X^\vee$ dual to X;
- $\Phi \subset X$ and $\Phi^\vee \subset X^\vee$ are finite subsets called *roots* and *coroots* respectively;
- there is a bijection $\Phi \xrightarrow{\vee} \Phi^\vee$ given by $\alpha \mapsto \alpha^\vee$;

along with reflection maps for every $\alpha \in \Phi$

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha, \quad x \in X,$$
$$s_{\alpha^\vee}(u) = u - \langle \alpha, u \rangle \alpha^\vee, \quad u \in X^\vee,$$

satisfying the conditions

(1) $\langle \alpha, \alpha^\vee \rangle = 2$ for all α;
(2) $s_\alpha(\Phi) \subset \Phi^\vee$ and $s_{\alpha^\vee}(\Phi^\vee) \subset \Phi^\vee$.

The dual of a root datum Ψ is defined as $\Psi^\vee = (X^\vee, \Phi^\vee, X, \Phi)$.

Let $Q \subset X$ be the subgroup generated by Φ, and $X_0 \subset X$ be the orthogonal complement to $\Phi^\vee$ (similarly for $Q^\vee \subset X^\vee$ and $X_0^\vee$). Denote $V = Q \otimes_{\mathbf{Z}} \mathbf{Q}$ and $V_0 = X_0 \otimes_{\mathbf{Z}} \mathbf{Q}$ (similarly for $V^\vee$ and $V_0^\vee$).

If $X_0 = 0$, Ψ is called *semisimple*; if Φ is empty, Ψ is called *toral*.

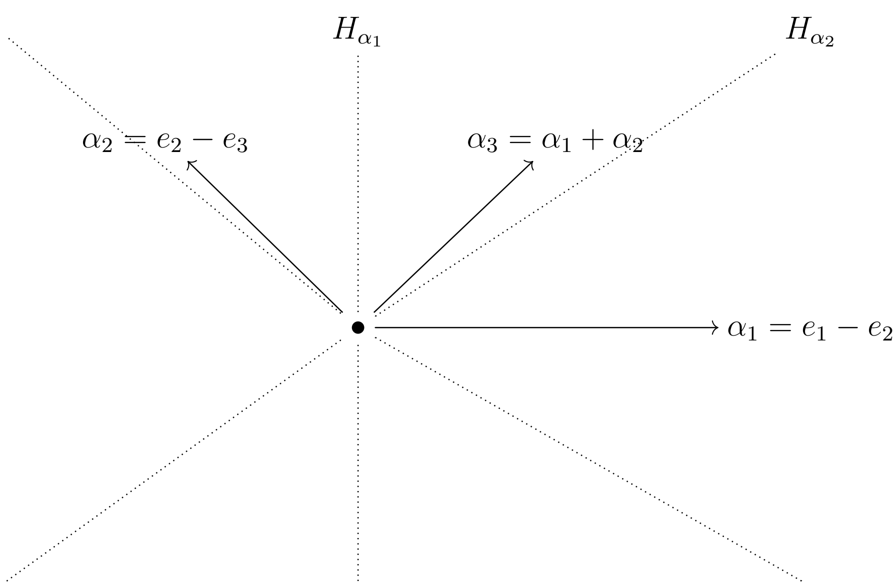
Fact. $Q \cap X_0 = \{0\}$, and $Q + X_0$ is finite index in X.

Each s_α has a fixed hyperplane H_α. Consider $V \otimes \mathbf{R} \backslash \{H_\alpha\}_\alpha$, whose connected components are called *Weyl chambers*.

Example A.10. Consider the root system $A_2 \hookrightarrow \mathbf{R}^3$ (with basis e_1, e_2, e_3) defined by

$$A_2 = \{\pm(e_1 - e_2), \pm(e_2 - e_3), \pm(e_1 - e_3)\},$$

which can be represented as



There are six Weyl Chambers, e.g. the region between H_{α_1} and H_{α_2} in the first quadrant.

[12] $\vee$ and $\langle \cdot, \cdot \rangle$ are often suppressed from the notation.

Definition A.11. For any fixed Weyl chamber C, we say:

- α is *positive with respect to* C (denoted $\alpha >_C 0$) if $\langle x, \alpha^\vee \rangle > 0$ for all $x \in C$;
- α is *simple with respect to* C if it is not the sum of two positive (with respect to C) roots.

Let Δ be the set of simple roots with respect to C, and call $(X, \Phi, \Delta, X^\vee, \Phi^\vee, \Delta^\vee)$ a *based* root datum.

From now on, suppose the base field k is algebraically closed.

Fact. Given (G, T) where T is a choice of a maximal torus (unique up to conjugacy), it is easy to describe a root datum $\Psi(G, T)$. This induces an equivalence of categories.

To describe the association $(G, T) \rightsquigarrow \Psi(G, T)$, we need an analogue of Lie algebras. Consider the ring of dual number $k[\epsilon] = k[x]/(x^2)$ and define

$$\mathfrak{g} = \mathrm{Lie}(G) = \ker(G(k[\epsilon]) \to G(k)).$$

Then $G(k[\epsilon])$ acts on $\mathfrak{g}$ by conjugation. This gives the *adjoint action* $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$ on the level of algebraic groups. The derivative of the induced map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is denoted $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}_k(\mathfrak{g})$. The Lie bracket is defined as $[x, y] = \mathrm{ad}(x)(y)$.

Given (G, T) where T is a maximal torus, the adjoint action of T on $\mathfrak{g}$ gives a decomposition

$$\mathfrak{g} = \mathrm{Lie}(T) \oplus \sum_{\alpha \in \Phi} U_\alpha,$$

where each U_α is a one-dimensional eigenspace of T.

Now we set

$$X = X^*(T) = \mathrm{Hom}_{\mathrm{AlgGps}}(T, \mathbf{G}_m),$$
$$X^\vee = X_*(T) = \mathrm{Hom}_{\mathrm{AlgGps}}(\mathbf{G}_m, T),$$

and define $X \times X^\vee \to \mathbf{Z}$ by $(x, u) \mapsto m$ where $x \circ u$ is the homomorphism $z \mapsto z^m$ on $\mathbf{G}_m$. The decomposition above naturally gives the roots Φ as the set of nontrivial eigenvalues of T on $\mathfrak{g}$.

To define the coroots $\Phi^\vee$, let $T_\alpha = \ker(\alpha : T \to \mathbf{G}_m)$. Then $Z_G(T_\alpha)$, the derived group of the centralizer of T_α in G, is isomorphic to either SL_2 or PSL_2, onto which $\mathbf{G}_m$ surjects. There exists a unique $\alpha^\vee$ whose image lies in $Z_G(T_\alpha)$ and such that $\langle \alpha, \alpha^\vee \rangle = 2$.

Definition A.12. A *parabolic subgroup* is a closed subgroup $P \subset G$ such that G/P is a projective variety. A *Borel* is an absolutely minimal parabolic subgroup.

Let B be a choice of Borel containing our fixed choice of T. Then $\mathrm{Lie}(B)$ contains half of the U_α corresponding to the positive roots with respect to one of the Weyl chambers. This gives a based root datum $\Psi(G, T, B)$.

There are decomposition theorems that make the study of reductive groups easier.

Theorem A.13 (Bruhat decomposition)**.**

$$G = \coprod_{w \in W} BwB$$

where the longest w *corresponds to an open cell. In fact,* BwB *can be replcaed by* BwU_B *for the unipotent radical* U_B *of* B*.*

We will study other decomposition theorems specifically for real groups and p-adic groups.

Appendix B. Supplementary Lecture 2 (September 15, 2017): Applications and related questions

B.1. **p-adic deformations of automorphic forms.** For motivational purpose, consider modular forms, which are functions $f = \sum a_n q^n$ (where $q = e^{2\pi i\tau}$) on the upper half plane satisfying certain transformation properties. A *p-adic deformation* means varying the coefficients a_n in a way that is p-adically continuous.

Let π be an irreducible cuspidal automorphic representation, i.e., a subrepresentation of $L^2_0(G(\mathbf{Q})\backslash G(\mathbf{A}))$ where G is a reductive group over $\mathbf{Q}$ and $\mathbf{A} = \mathbf{R} \times \mathbf{A}_f = \mathbf{R} \times \prod'_p \mathbf{Q}_p$ is the adeles over $\mathbf{Q}$. Then there is a decomposition $\pi = \pi_\infty \otimes \pi_f$ where π_∞ is a representation of $G(\mathbf{R})$ and π_f is a representation of $G(\mathbf{A}_f)$. This motivates the following question: Is there some finite-dimensional construction that recovers π_f (each π_p)?

Choose an open compact $K_f \subset G(\mathbf{A}_f)$ such that $\pi_f^{K_f} \neq 0$. (Write $K_f = K_p \times K^p$ where K^p is the component at primes away from p.) Then there is a Hecke algebra $\mathcal{H}(G(\mathbf{A}_f), K_f)$ acting on $\pi_f^{K_f}$. We require $K_f = \prod K_\ell$, so that there is decomposition

$$\mathcal{H}(G(\mathbf{A}_f), K_f) = \bigotimes_\ell{}' \mathcal{H}(G(\mathbf{Q}_\ell), K_\ell).$$

The important point is that for almost all ℓ, $\mathcal{H}(G(\mathbf{Q}_\ell), K_\ell)$ is commutative. These objects will be studied next week.

Next, think of the representation as being on a $\overline{\mathbf{Q}_p}$-vector space. In order to get a suitable algebra of operators for *spectral* theory, we need to consider a certain subspace $\mathcal{H}(G(\mathbf{Q}_p), K_p) \supset \mathcal{U}^+$. This requires understanding what K_p and $\mathcal{H}(G(\mathbf{Q}_p), K_p)$ look like:

$$\mathcal{H}(G(\mathbf{Q}_p), K_p) = C_c^\infty(K_p\backslash G/K_p)$$

is the convolution algebra of bi-K_p-invariant functions with compact support.

Let us define a particular choice of K_p. Fix G split over $\mathbf{Z}_p$, a maximal torus T split over $\mathbf{Z}_p$, and a Borel $B = T \cdot N$ over $\mathbf{Z}_p$. Then the *Iwahori subgroup* is

$$\mathrm{Iw}_m := \text{preimage of } B(\mathbf{Z}_p/p^m\mathbf{Z}_p) \text{ under the map } G(\mathbf{Z}_p) \to G(\mathbf{Z}_p/p^m\mathbf{Z}_p).$$

Example B.1. For $G = \mathrm{GL}_n$ and B the upper triangular matrices,

$$\mathrm{Iw}_m = \begin{pmatrix} \mathbf{Z}_p & & \mathbf{Z}_p \\ & \ddots & \\ p^m\mathbf{Z}_p & & \mathbf{Z}_p \end{pmatrix}.$$

Later we will see that the Hecke algebra $\mathcal{H}(\mathrm{GL}_n(\mathbf{Q}_p), \mathrm{Iw}_m)$ is generated by elements of $\mathbf{Z}^n \rtimes W_m$ with relations that are complicated to write down.

Define the subalgebras

$$\mathcal{H}(G(\mathbf{Q}_p), \mathrm{Iw}_m) \supseteq \mathcal{U} \supseteq \mathcal{U}^+ \supseteq \mathcal{U}^{++},$$

where:

- $\mathcal{U}$ is the maximal commutative subalgebra of $\mathcal{H}(G(\mathbf{Q}_p), \mathrm{Iw}_m)$, i.e., the subalgebra generated by the characteristic functions $\mathrm{char}(\mathrm{Iw}_m t \mathrm{Iw}_m)$ for diagonal matrices t;
- $\mathcal{U}^+$ is the subalgebra generated by those t's that pair "positively" with positive roots;
- $\mathcal{U}^{++}$ is the subalgebra generated by those t's that pair strictly "positively" with positive roots.

Example B.2. For $G = \mathrm{GL}_2$ and $B = \begin{pmatrix} t_1 & * \\ & t_2 \end{pmatrix}$, the positive root is $\alpha = \frac{t_1}{t_2}$. A diagonal matrix $t = \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}$ pairs positively with α if $\left|\frac{t_1}{t_2}\right|_p \leq 1$, and strictly positively if $\left|\frac{t_1}{t_2}\right|_p < 1$.

$\mathcal{U}^{++}$ acts on $\pi_p^{\mathrm{Iw}_1}$. There is a "machine" that takes an element of $\mathcal{U}^{++}$ and produces a spectral expansion, which is important in the construction of eigenvarieties.

B.2. **Weil–Deligne representations.** The local Langlands conjecture says that

$$\left\{\begin{matrix}\text{Weil–Deligne representations of } W_{\mathbf{Q}_p} \\ \text{(Frobenius-semisimple) of dimension } k\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{irreducible admissible representations} \\ \text{of } \mathrm{GL}_k(\mathbf{Q}_p)\end{matrix}\right\}.$$

Note that irreducibility is not required on the LHS! This should be thought of as a generalization of reciprocity laws. Recall that Artin reciprocity gives a relationship between Galois groups and ideal class groups.

Let $F/\mathbf{Q}$ be a global number field. Then $G_F = \mathrm{Gal}(\overline{F}/F)$ has "canonical" representation theory. Suppose $\rho : G_F \to \mathrm{GL}_n(E)$ where E is any field, and $\mathfrak{p}$ is a prime ideal of F. The theory of Weil–Deligne representations gives an alternative description of $\rho_{\mathfrak{p}} : G_{F_{\mathfrak{p}}} \to \mathrm{GL}_n(E)$ that is equivalent but easier to think about. This will make more apparent the relationship between representation theories of p-adic groups and Galois groups.

Consider the map

$$G_{F_{\mathfrak{p}}} \to \mathrm{Gal}(F_{\mathfrak{p}}^{\mathrm{ur}}/F_{\mathfrak{p}}) = \widehat{\mathrm{Frob}^{\mathbf{Z}}}$$

whose kernel is the inertia subgroup $I_{\mathfrak{p}}$. The *Weil–Deligne group* $W_{F_{\mathfrak{p}}}$ is defined to be the preimage of $\mathrm{Frob}^{\mathbf{Z}}$, so it contains $I_{\mathfrak{p}}$ as an open subgroup. Recall by local class field theory that under $F_{\mathfrak{p}}^{\times} \simeq \mathbf{Z} \times \mathcal{O}_{F_{\mathfrak{p}}}^{\times}$, $\mathcal{O}_{F_{\mathfrak{p}}}^{\times}$ corresponds to the abelianization of $I_{\mathfrak{p}}$.

Definition B.3. A *Weil–Deligne representation* is the data (V, r, N), where:

- V is a $\mathbf{C}$-vector space,
- $r : W_{F_{\mathfrak{p}}} \to \mathrm{GL}(V)$ is a homomorphism with open kernel,
- N is endomorphism of V,

such that $r(\sigma)Nr(\sigma)^{-1} = |\sigma|^{-1}N$, where $|\cdot|^{-1} : W_{F_{\mathfrak{p}}} \to \mathbf{C}^{\times}$ sends $\mathrm{Frob} \mapsto q$, the size of the residue field of $F_{\mathfrak{p}}$.

A reference is Tate's article *Number theoretic background* in the Corvallis proceedings. Here are some operations on Weil–Deligne representations:

(1) Frobenius-semisimplification: $(V, r, N) \overset{\text{Fr-ss}}{\rightsquigarrow} (V, r^{\mathrm{ss}}, N)$ given by

$$r^{\mathrm{ss}}(\sigma^n\tau) = s^n r(\tau),$$

where σ is a fixed lift of $\mathrm{Frob}_{\mathfrak{p}}$ and $r(\sigma) = us$ with u unipotent and s semisimple. This doesn't lose information.

(2) Semisimplification: $(V, r, N) \overset{\text{ss}}{\rightsquigarrow} (V, r^{\mathrm{ss}}, 0)$. This does lose information about (V, r, N).

(3) Given a Frobenius-semisimple $(W, r, 0)$ and an integer $k \geq 1$,

$$\mathrm{Sp}_k(W) = (W^k, r(|\cdot|^{-1})^{k-1} \oplus r(|\cdot|^{-1})^{k-2} \oplus \cdots \oplus r(|\cdot|^{-1}) \oplus r, N)$$

where $N : r(|\cdot|^{-1})^{i-1} \to r(|\cdot|^{-1})^{i}$ sends $W \to W$ identically, and kills $r(|\cdot|^{-1})^{k-1}$.

Fact. If r is irreducible, then $\mathrm{Sp}_k(W)$ is indecomposable. All indecomposable (Frobenius-semisimple) representations arise in this way.

We will see that, roughly speaking, this process of constructing Weil–Deligne representations from irreducible representations of the Weil group corresponds to the construction of smooth representations via parabolic induction.

Here are a few more words on the monodromy operator; a good reference is Taylor–Yoshida. An $\alpha \in \overline{\mathbf{Q}}$ is a *Weil q-number of weight k* if $|\tau(\alpha)|^2 = q^k$ for all $\tau : \overline{\mathbf{Q}} \to \mathbf{C}$. We say that (V, r, N) is:

- *strictly pure of weight* $k \in \mathbf{R}$ if $N = 0$ and the eigenvalues of (any) Frobenius lift are Weil q-numbers of weight k;
- *pure of weight* k if:
  (1) there exists an increasing separated exhaustive filtration Fil_i of V such that the i-th graded $\mathrm{gr}_i = \mathrm{Fil}_i/\mathrm{Fil}_{i-1}$ is strictly pure of weight i (this implies $N(\mathrm{Fil}_i V) \subseteq \mathrm{Fil}_{i-2}V$);
  (2) all weights are in $k + \mathbf{Z}$ and $N^i : \mathrm{gr}_{k+i} \xrightarrow{\sim} \mathrm{gr}_{k-i}$.

If W is strictly pure of weight k, then $\mathrm{Sp}_\ell(W)$ is pure of weight $k - (\ell - 1)$.