NON-VANISHING RESULTS OF SPECIAL VALUES OF L-FUNCTIONS

ERIC URBAN

NOTES TAKEN BY PAK-HIN LEE

Abstract. Here are the notes I am taking for Eric Urban's ongoing course on non-vanishing results of special values of L-functions offered at Columbia University in Spring 2015 (MATH G6675: Topics in Number Theory). As the course progresses, these notes will be revised. I recommend that you visit my website from time to time for the most updated version.

Due to my own lack of understanding of the materials, I have inevitably introduced both mathematical and typographical errors in these notes. Please send corrections and comments to phlee@math.columbia.edu.

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1. Lecture 1 (February 5, 2015)

I want to talk about certain non-vanishing results for special values of L-functions. There are a lot of features about this kind of statements. Some of the ones I am interested in are obtained by algebraic and ergodic methods. Using this method we can even get non-vanishing in positive characteristic, which has a lot of applications in arithmetics. Let me give a general picture of the L-functions we are interested in.

We will be studying motivic L-functions. Let M be a pure motive over $\mathbb{Q}$, which will always assumed to be irreducible. We can think of this as several realizations:

- ℓ-adic realization: a $\mathbb{Q}_\ell$-vector space M_ℓ with action of $G_{\mathbb{Q}}$. This Galois action is unramified outside finitely many primes containing ℓ.
- de Rham realization: a $\mathbb{Q}$-vector space M_{dR}.
- Betti realization: a $\mathbb{Q}$-vector space M_{B}.

We have the comparison isomorphisms

$$M_{\mathrm{B}} \otimes \mathbb{Q}_\ell \cong M_\ell$$

and

$$M_{\mathrm{B}} \otimes \mathbb{C} \cong M_{\mathrm{dR}} \otimes \mathbb{C}.$$

The second isomorphism gives us periods, which are defined modulo rational numbers. But if we fix a lattice that is stable under Galois, we can define periods up to ℓ-adic units. These will be the Deligne periods $\Omega_M \subset \mathbb{C}^\times/\mathbb{Q}^\times$.

The assumption that the motive M is pure implies the following. If we take a prime $q \neq \ell$ such that M_ℓ is unramified at q, then we can look at the characteristic polynomial

$$P(\mathrm{Frob}_q, X) = \prod_i (X - \alpha_{q,i})$$

where $\alpha_{q,i} \in \overline{\mathbb{Q}}$ with a fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. For M to be pure, we have

$$|\alpha_{q,i}| = q^{\frac{w}{2}}$$

where w is a fixed integer called the motivic weight of M. We can see that the polynomial P belongs to $E[x]$, where E is the field of coefficients, with degree equal to $\dim_E M_B$. We can also see that P is independent of ℓ, and M_ℓ is called a compatible system of Galois representations.

Once we have this, we can form an L-function

$$L(M, s) = \prod_q P_q(M, q^{-s})$$

with $P_q(M, X) = \det(1 - X \,\mathrm{Frob}_q \,|M_\ell^{I_q})$, called the local factor at q. From the hypothesis that M is pure, this Euler product $L(M, s)$ is convergent if $\mathrm{Re}(s) > 1 + \frac{w}{2}$. Moreover, $L(M, s) \neq 0$ as long as $\mathrm{Re}(s) > 1 + \frac{w}{2}$. This is just a formal argument:

Proof. Take the logarithm of the Euler product. □

Let $M' = M^\vee(w)$. Then M' is pure of the same weight. Conjecturally we have a functional equation of the form

$$L(M', 1 + w - s) = \epsilon(M, s)L(M, s).$$

This epsilon factor is constructed at infinity and primes of ramification. The archimedean factors are expressed in terms of gamma factors, which are expressed in terms of the Hodge numbers of M_{dR}. Using the functional equation, we know the behavior (vanishing or non-vanishing) at $\mathrm{Re}(s) < \frac{w}{2}$. What is unknown and non-trivial is the vanishing or not of $L(M, s)$ for $\mathrm{Re}(s) \in [\frac{w}{2}, \frac{w}{2} + 1]$, which is called the critical strip.

Example 1.1. Taking M to be the trivial motive (1-dimensional space with trivial Galois action), the periods are powers of $2\pi i$ and $L(M, s)$ is the Riemann zeta function $\zeta(s)$. The result $\zeta(s) \neq 0$ for $\mathrm{Re}(s) = 1$ is already non-trivial.

There are two important cases to consider:

(1) the central value $s = \frac{w+1}{2}$ if this is a critical value (in particular it is an integer),
(2) $s = \frac{w}{2} + 1$ if this is an integer.

(1) and (2) cannot happen at the same time.

Let me give a list of examples that we will be considering. This kind of vanishing or non-vanishing results is not known in many situations. We will consider only cases when M is potentially self-dual or conjugate self-dual. “Potentially” is not standard terminology. By “potentially self-dual” I mean there is an abelian extension $F/\mathbb{Q}$ such that $M_\ell' \cong M_\ell$ as G_F-modules, and “potentially conjugate self-dual” means $M_\ell' \cong M_\ell^c$ as G_M-modules, where M is defined over a CM field M_0 and M/M_0 is an abelian extension.

We will start with something self-dual or conjugate self-dual, and consider the twists by finite order characters. For central values, we will always consider the case when M is self-dual or conjugate self-dual.

In the self-dual case, the functional equation is

$$L(M, 1 + w - s) = \epsilon(M, s)L(M, s).$$

We can define $\epsilon(M) = \epsilon(M, \frac{w+1}{2}) = \pm 1$. If $\epsilon(M) = -1$, then $L(M, \frac{1+w}{2})$ vanishes with odd order. If $\epsilon(M) = 1$, then $L(M, \frac{1+w}{2})$ vanishes with an even order, which could be zero in which case we have non-vanishing.

What we will consider is a family of such self-dual or conjugate self-dual motives, and we ask if we have non-generic vanishing in the second case.

I will continue by going into more examples.

Example 1.2.

(1) Dirichlet motives (Dirichlet L-functions): take a Dirichlet character χ and consider

$$L(\chi, s) = \prod_p (1 - \chi(p)p^{-s})^{-1}$$

with the understanding that $\chi(p) = 0$ if p divides the conductor of χ.

(2) Hecke L-functions attached to CM fields: consider χ, a Hecke character of a CM field M of type A_0 (this is Weil's terminology) given by

$$\chi((\alpha)) = \prod_{\sigma: M \hookrightarrow \overline{\mathbb{Q}}} \sigma(\alpha)^{\kappa_\sigma}$$

where $\kappa_\sigma \in \mathbb{Z}$.

(3) Modular forms: let f be a cuspidal modular form of weight k and Nebentypus χ, and consider its L-function $L(f, s)$. The associated motive V_f is of weight $k-1$, and we have $V'_{f,\ell} = V_{f,\ell} \otimes \chi^{-1}$.

(4) Take two modular forms f and g. Consider $M = M_f \otimes M_g$. Then $L(M, s) = L(f, g, s)$ is the Rankin–Selberg L-function attached to f, g. We need to assume weight$(f) >$ weight(g), and moreover g is CM. We have

$$L(f, g, s) = L_K(f, \chi, s)$$

where K is an imaginary quadratic field, and χ is an anti-cyclotomic Hecke character of K.

Let us state the kind of results we have.

In the case of Dirichlet L-functions, we are interested in critical values. Suppose χ is a Dirichlet character. Then $L(\chi, s) \neq 0$ if $\mathrm{Re}(s) \geq 1$ (at $s = 1$ this is non-trivial). For $n \geq 1$,

$$L(\chi, 1-n) = \frac{B_{n,\chi}}{n} \in \mathbb{Q}(\chi)$$

where $B_{n,\chi}$ is the Bernoulli number defined by

$$\sum_{a \mod N} \frac{\chi(a) t e^{at}}{1 - e^t} = \sum_n B_{n,\chi} \frac{t^n}{n!}.$$

By the functional equation, we get something for the positive integers, but we have to divide by some power of π. Actually we don't get any algebraicity result for the positive integers because the gamma function has a pole. If χ is odd, $B_{n,\chi}$ is zero for n odd.

If χ is the quadratic character attached to $\mathbb{Q}(\sqrt{-p})/\mathbb{Q}$, we have a formula

$$\frac{\sqrt{p}}{\pi} L(1, \chi) = h(\mathbb{Q}(\sqrt{-p}))$$

for the class number of $\mathbb{Q}(\sqrt{-p})$.

In my original introduction I only mentioned about non-vanishing, but let me add this. When $s = \frac{w}{2} + 1$ or $\frac{w+1}{2}$ is critical, we can look at the algebraic part

$$L^{\mathrm{alg}}(M, s_0) := \frac{L(M, s_0)}{\Omega^{\mathrm{Deligne}}_{M,\ell}}.$$

Deligne conjectured that this is an algebraic number, and if the ℓ-adic unit is chosen appropriately this will even belong to $\overline{\mathbb{Z}_\ell}$. Then we can look at the reduction modulo ℓ.

Going back to the quadratic character above, the left hand side is $L^{\mathrm{alg}}(1, \chi)$. The arithmetic meaning of vanishing modulo ℓ of this special value is the divisibility by ℓ of a class number.

I only wrote the case of quadratic extensions, but we can do this for all Dirichlet characters. In particular, an interesting case is when χ varies in the set of Dirichlet characters of conductor p^n, for a fixed prime number p. The arithmetic meaning is that we are looking at the class number $h(K_n)$, where $K_n = \mathbb{Q}(\zeta_{p^n})$. This is becoming Iwasawa theory. There are two interesting cases:

(1) If $\ell \neq p$, then we can look at $\mathrm{ord}_\ell(h(K_n))$ and we ask the behavior as n varies. This is basically looking at $\mathrm{ord}_\ell(L^{\mathrm{alg}}(\chi, 1))$ modulo ℓ when χ is of conductor p^n. For

almost all ℓ, this is 0. The question is: what is the behavior of $\mathrm{ord}_\ell(L^{\mathrm{alg}}(\chi,1))$ and $\mathrm{ord}_\ell(h(K_n))$ when $n \to +\infty$?

(2) If $\ell = p$, Iwasawa theory tells you that

$$e_n = \mathrm{ord}_p(h(K_n)) = \mu p^n + \lambda n + \nu$$

when $n \gg 0$. We have the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[T]]$, and X is the Λ-module such that $X \otimes \Lambda/((1+T)^{p^n} - 1) \cong \mathrm{Cl}(K_n)$. In this case we have the characteristic ideal $f(T) \in \Lambda$, where $X \simeq \prod \Lambda/(f_i)$ and $f = \prod f_i$.

Using the Weierstrass preparation theorem, we can write

$$f(T) = p^\mu \times P(T) \times u(T)$$

where $P(T)$ is a polynomial of degree λ and $u(T) \in \Lambda^\times$. If we replace the class group by the anti-cyclotomic part, $f(T)$ is the product of p-adic L-functions (Mazur–Wiles). In characteristic $p = \ell$, the non-vanishing is equivalent to the vanishing of the μ-invariant.

In case (1) and (2), we have the

Theorem 1.3 (Ferrero–Washington)**.**

(1) *If $\ell \neq p$, then $\mathrm{ord}_\ell(h(K_n))$ is bounded when $n \to +\infty$. In other words, $L^{\mathrm{alg}}(1,\chi) \not\equiv 0 \mod \ell$ for almost all χ of conductor p^n.*

(2) *If $\ell = p$, then $\mu = 0$. In other words, $\mathrm{ord}_p(h(K_n)) = \lambda n + \nu$ when $n \gg 0$.*

I will not explain the proof, but I will state one key lemma which is used to prove this theorem. We will see that ergodic theory is under the picture. If $\beta \in \mathbb{Z}_p$, we denote by $s_n(\beta) \in \mathbb{Z} \cap [1, p^n - 1]$ the number such that $\beta \equiv s_n(\beta) \mod p^n$. and

$$x_n(\beta) = p^{-n} s_n(\beta) \in [0,1).$$

Lemma 1.4 (Key Lemma)**.** *Fix $\gamma_1, \cdots, \gamma_r \in \mathbb{Z}_p$. Assume that $\gamma_1, \cdots, \gamma_r$ are $\mathbb{Q}$-linearly independent. Consider*

$$X_n(\beta) = (x_n(\beta\gamma_1), x_n(\beta\gamma_2), \cdots, x_n(\beta\gamma_r)) \in [0,1)^r.$$

For almost all $\beta \in \mathbb{Z}_p$, as $n \to +\infty$, $X_n(\beta)$ is equidistributed in $[0,1)^r$.

Let me finish by stating an equivalent form of this result. These numbers look like random variables, in a strong way. There is a way to express this result in terms of one-parameter subgroups. As an analogy,

Theorem 1.5 (Kronecker)**.** *Let $\gamma_1, \cdots, \gamma_r \in \mathbb{R}$ be $\mathbb{Q}$-linearly independent. Consider*

$$(t\gamma_1, t\gamma_2, \cdots, t\gamma_r) \in (\mathbb{R}/\mathbb{Z})^r$$

when $t \in \mathbb{R}$. Then the set $\{(t\gamma_1, t\gamma_2, \cdots, t\gamma_r) : t \in \mathbb{R}\}$ is dense in $(\mathbb{R}/\mathbb{Z})^n$. More generally, for arbitrary γ_i, the closure of $\{(t\gamma_1, t\gamma_2, \cdots, t\gamma_r)\}$ in $(\mathbb{R}/\mathbb{Z})^r$ is a sub-torus of dimension $\mathrm{rank}_{\mathbb{Q}}(\sum_i \mathbb{Q}\gamma_i)$.

Next time we will see that there is another proof of Ferrero–Washington by Sinnott, whose proof is more analogous with Kronecker's result. Orbits being sub-torus is something very general that we will see in results for next time.

2. Lecture 2 (February 10, 2015)

Last time I stated the following result of Ferrero–Washington.

Theorem 2.1.

(1) *(Ferrero–Washington)* $\ell = p$. *We have the vanishing of the* μ*-invariant of the Kubota–Leopoldt* p*-adic* L*-functions.*

(2) *(Washington)* $\ell \neq p$. *Fix a Dirichlet character* λ *of conductor* $N > 1$*, and consider* $L(0, \lambda\chi) \in \overline{\mathbb{F}_p}$ *when* χ *runs in the set of Dirichlet characters of conductor a power of* ℓ *such that* χ *factors through* $\mathbb{Z}_\ell^\times / \mu_{\ell-1}$*, i.e.,*

$$\begin{array}{ccc} \mathbb{Z}_\ell^\times & \xrightarrow{\chi} & \mu_{\ell^\infty} \\ \downarrow & \nearrow & \\ \mathbb{Z}_\ell^\times/\mu_{\ell-1} & & \end{array}$$

where the image of χ *is an* ℓ*-power root of unity. Then* $L(0, \lambda\chi) \neq 0 \in \overline{\mathbb{F}_p}$ *except for finitely many such* χ*.*

I said that one of the arguments in the proof involves some ergodic results. Recall that we defined $x_n(\beta) \in [0, 1)$ for $\beta \in \mathbb{Z}_p$.

Lemma 2.2 (Key Lemma)**.** *Let* $\gamma_1, \cdots, \gamma_r \in \mathbb{Z}_p$ *be* $\mathbb{Q}$*-linearly independent. Then the image of*

$$x_n(\beta) = (x_n(\gamma_1\beta), \cdots, x_n(\gamma_n\beta)) \in [0, 1)^r$$

when $n \to \infty$ *and* $\beta \in \mathbb{Z}_p$ *is dense in* $[0, 1)^r$*.*

I will not use this result, but will make an analogy with the

Theorem 2.3 (Kronecker)**.** *Let* $\gamma_1, \cdots, \gamma_r \in \mathbb{R}$ *be* $\mathbb{Q}$*-linearly independent. Then the image*

$$(t\gamma_1, \cdots, t\gamma_r) \in (\mathbb{R}/\mathbb{Z})^r$$

for $t \in \mathbb{Q}$ *is dense.*

Here we are looking at orbits inside a torus, which is very general.

Another proof of Ferrero–Washington is by taking another analogy of Kronecker's result. Recall that for $a \in \mathbb{Z}_p$, the power series

$$T^a = (1 + (T-1))^a = \sum_r \binom{a}{n} (T-1)^n \mod p \in \mathbb{F}_p[[T-1]]$$

makes sense.

Lemma 2.4 (Sinnott)**.** *Let* $\gamma_1, \cdots, \gamma_r \in \mathbb{Z}_p$ *be* $\mathbb{Q}$*-linearly independent. Then the power series* $T^{\gamma_1}, \cdots, T^{\gamma_r}$ *are algebraically independent in* $\mathbb{F}_p[[T-1]]$*, i.e., the map* $\mathbb{F}_p[X_1, \cdots, X_r] \to \mathbb{F}_p[[T-1]]$ *given by* $X_i \mapsto T^{\gamma_i}$ *is injective.*

We can see this as saying that the induced image of the one-parameter formal torus inside $\mathbb{G}_{\mathrm{m}}^r$ is dense.

Our goal now is to sketch Sinnott's proof of Washington's theorem for $\ell \neq p$. Recall that we want to look at the L-values as generalized Bernoulli numbers

$$\sum_a \frac{\lambda(a) t e^{at}}{e^{Nt}-1} = \sum B_{n,\lambda} \frac{t^n}{n!}. \tag{1}$$

We have

$$L(1-n,\lambda) = -\frac{B_{n,\chi}}{n},$$

and in particular

$$L(0,\lambda) = -B_{1,\lambda}.$$

From (1) we see that

$$B_{1,\lambda} = \sum_a \frac{\lambda(a) a}{N}.$$

For a general $\lambda \neq 1$, we introduce the following power series

$$\Phi_\lambda(t) = \sum_n \lambda(n) t^n,$$

where, of course, it is understood that if n is not prime to the level of λ, then $\lambda(n) = 0$. So we can write this as

$$\Phi_\lambda(t) = \sum_{(a,N)=1} \sum_k \lambda(a) t^{a+kN} = \frac{\sum \lambda(a) t^a}{1-t^N} \in \mathcal{O}_{\mathbb{G}_\mathrm{m},1},$$

which is the local ring of $\mathbb{G}_\mathrm{m}$ over $\overline{\mathbb{F}_p}$ (i.e., of $\operatorname{Spec} \overline{\mathbb{F}_p}[t,t^{-1}]$) at 1. (*A priori* we only know $\Phi_\lambda(t) \in \overline{\mathbb{F}_p}[[t]]$.) Since $\sum_{a \bmod N} \lambda(a) = 0$, we can rewrite

$$\Phi_\lambda(t) = \frac{\sum \lambda(a)(t^a - 1)}{1-t^N}$$

and hence

$$\Phi_\lambda(1) = \frac{-\sum \lambda(a) a}{N} = -B_{1,\lambda} = L(0,\lambda).$$

We can see the rational function $\Phi_\lambda \in \overline{\mathbb{F}_p}[[t]]$ as a function on $\mu_{\ell^\infty} = \varinjlim \mu_{\ell^n}$. Given any such function Φ, we can define the Hecke operator U_ℓ by

$$(\Phi | U_\ell)(t) = \frac{1}{\ell} \sum_{\zeta^\ell = 1} \Phi(t\zeta).$$

Then we have

$$\Phi_\lambda | U_\ell = \lambda(\ell) \Phi_\lambda.$$

When we have such an eigenfunction Φ of U_ℓ, we can define a measure on $\mathbb{Z}_\ell$ as follows.

Fix an isomorphism $\mathbb{Z}_\ell \cong \mathbb{Z}_\ell(1)$, which basically means that for all n, we fix a primitive ℓ^n-th root of unity ζ_n such that $\zeta_{n+1}^\ell = \zeta_n$. Thus a function on μ_{ℓ^∞} can be seen as a function on $\mathbb{Z}_\ell$. Let $\mathcal{C}(\mathbb{Z}/\ell^n\mathbb{Z}, A)$ be the space of functions on $\mathbb{Z}/\ell^n\mathbb{Z}$ taking values in A, which will be assumed to be an $\overline{\mathbb{F}_p}$-algebra. Then a measure on $\mathbb{Z}/\ell^n\mathbb{Z}$ is simply a map

$$\mathcal{C}(\mathbb{Z}/\ell^n\mathbb{Z}, A) \to B$$

for any A-module B, so can be seen as

$$\theta_n = \sum_{x \in \mathbb{Z}/\ell^n\mathbb{Z}} \alpha_x \delta_x \in B \cdot (\mathbb{Z}/\ell^n\mathbb{Z})$$

where $\alpha_x \in B$ and δ_x is the Dirac measure at x.

Now we want to construct a measure on $\mathbb{Z}_\ell$. This will be a map in $\mathrm{Hom}(\mathcal{C}(\mathbb{Z}_\ell, A), B)$, where $\mathcal{C}(\mathbb{Z}_\ell, A)$ is the space of continuous functions on $\mathbb{Z}_\ell$ taking values in A. The construction of a B-valued measure on $\mathbb{Z}_\ell$ is given by the data of θ_n for all n such that $\theta_{n+1} \mapsto \theta_n$ under the map $\pi_n : B \cdot (\mathbb{Z}/\ell^{n+1}\mathbb{Z}) \to B \cdot (\mathbb{Z}/\ell^n\mathbb{Z})$.

If Φ is a function satisfying $\Phi | U_\ell = a\Phi$, we can construct such a compatible sequence. If we define

$$\theta_{\Phi,n} = \sum_{x \in \mathbb{Z}/\ell^n\mathbb{Z}} \Phi(\zeta_n^x)\delta_x,$$

then

$$\pi_n(\theta_{\Phi,n+1}) = \ell a \theta_{\Phi,n}.$$

Hence if we choose

$$\hat{\theta}_n = \frac{1}{(a\ell)^n}\theta_{\Phi,n},$$

we get a measure on $\mathbb{Z}_\ell$. We denote by $d\Phi$ this measure, and $d\Phi(t)$ the measure given by

$$\theta_{\Phi,n}(t) = \sum_{x \in \mathbb{Z}/\ell^n\mathbb{Z}} \Phi(\zeta_n^x t)\delta_x.$$

Of course $d\Phi(1) = d\Phi$.

If f is a continuous function on $\mathbb{Z}_\ell$ taking values in $\overline{\mathbb{F}_p}$, then we have that

$$\int f \, d\Phi = \frac{1}{(a\ell)^n} \sum_{x \in \mathbb{Z}/\ell^n\mathbb{Z}} \Phi(\zeta_n^x) f(x)$$

where n is such that f factors through $\mathbb{Z}_\ell \to \mathbb{Z}/\ell^n\mathbb{Z}$ (such an n exists since f is continuous).

Let $\chi : (\mathbb{Z}/\ell^n\mathbb{Z})^\times \to \overline{\mathbb{Z}}^\times \to \overline{\mathbb{F}_p}$ be the reduction mod p of a Dirichlet character. This can be seen as a function on $\mathbb{Z}_\ell$ with support contained in $\mathbb{Z}_\ell^\times$. Since $\Phi_\lambda(t) = \sum \lambda(m)t^m \in \overline{\mathbb{F}_p}[[t]]$, we have

$$\begin{aligned}
\int \chi \, d\Phi_\lambda(t) &= \frac{1}{(\lambda(\ell)\ell)^n} \sum_{x \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times} \Phi_\lambda(\zeta_n^x t)\chi(x) \\
&= \frac{1}{(\lambda(\ell)\ell)^n} \sum_{x,m} \chi(x)\lambda(m)\zeta_n^{xm} t^m \\
&= \frac{1}{(\lambda(\ell)\ell)^n} \sum_{m} \left(\sum_{x} \chi(x)\zeta_n^{xm} \right) \lambda(m) t^m \\
&= \frac{1}{(\lambda(\ell)\ell)^n} \sum_{(m,\ell)=1} \left(\sum_{x} \chi(x)\zeta_n^{xm} \right) \lambda(m) t^m \\
&= \frac{1}{(\lambda(\ell)\ell)^n} \sum_{(m,\ell)=1} \chi(m)^{-1} \left(\sum_{x} \chi(xm)\zeta_n^{xm} \right) \lambda(m) t^m
\end{aligned}$$

$$= \frac{G(\chi)}{(\lambda(\ell)\ell)^n} \sum_{(m,\ell)=1} \chi(m)^{-1}\lambda(m)t^m$$

where $G(\chi)$ is the Gauss sum. Therefore,

$$\int \chi \, d\Phi_\lambda(t) = \frac{G(\chi)}{(\lambda(\ell)\ell)^n} \Phi_{\lambda\chi^{-1}}(t).$$

In particular, if we specialize at $t = 1$, we get

$$\int \chi \, d\Phi_\lambda = \frac{G(\chi)}{(\lambda(\ell)\ell)^n} L(0, \chi^{-1}\lambda).$$

Here we can think of $d\Phi_\lambda$ as a measure on $\mathbb{Z}_\ell^\times$. Recall that we are interested in the set of values $L(0, \chi^{-1}\lambda)$ as χ varies among

$$\begin{array}{ccc} \mathbb{Z}_\ell^\times & \xrightarrow{\ \chi\ } & \mu_{\ell^\infty} \\ \downarrow & \nearrow & \\ \Gamma = \mathbb{Z}_\ell^\times/\mu_{\ell-1} & & \end{array} \tag{2}$$

We want a measure on $\Gamma \cong \mathbb{Z}_\ell$. The question is whether we can find a $\Psi \in \overline{\mathbb{F}_p}[[t]]$ such that

$$\int_\Gamma \chi \, d\Psi(t) = \int_{\mathbb{Z}_\ell^\times} \chi \, d\Phi_\lambda$$

for all χ satisfying (2). For that we use the decomposition $\mathbb{Z}_\ell^\times \cong \mu_{\ell-1} \times \Gamma$ to get

$$\Psi(t) = \sum_{\epsilon \in \mu_{\ell-1}} \Phi_\lambda(t^\epsilon).$$

We want to show $L(0, \chi^{-1}\lambda) \neq 0$ for infinitely many χ. By looking at the values of $\Psi(\zeta)$ for $\zeta \in \mu_{\ell^\infty}$, we would like to show that $\Psi(\zeta) \neq 0$ except for finitely many ζ. If we assume that this is zero for all ζ, then $\Psi(t)$ is also zero. This is where the algebraic independence statement of Sinnott's key lemma comes in.

In summary, we have expressed L-values using rational functions. To prove Washington's theorem, there is an ergodic statement about algebraic independence.

3. Lecture 3 (February 12, 2015)

We were proving Washington's theorem for $\ell \neq p$, but we might only have time to do a weak version of it.

Let N be an integer prime to p, and λ be Dirichlet character of level N with $\lambda(-1) = -1$. Recall the parity condition $\lambda(-1) = (-1)^n$ for the non-vanishing of the special value

$$L(1-n, \lambda) = -\frac{B_{\lambda,n}}{n}.$$

We introduce the function

$$\Phi_\lambda(t) = \sum_{n=1}^{\infty} \lambda(n)t^n = \frac{\sum_{a=1}^{N-1} \lambda(a)t^a}{1 - t^N}$$

and look at this as inside $\overline{\mathbb{F}_p}[[t]]$ or $\overline{\mathbb{F}_p}(t)$.

Fix $\ell \neq p$, which will be assumed odd for simplicity. Generally, given a function $\Phi : \mu_{\ell^\infty} \to \overline{\mathbb{F}_p}$ such that $\Phi|U_\ell = a\Phi$ for some $a \neq 0$, $a \in \overline{\mathbb{F}_p}$, we can associate a measure $d\Phi$ on $\mathbb{Z}_\ell$ once we have fixed an isomorphism $\mathbb{Z}_\ell \cong \mathbb{Z}_\ell(1)$. This is the same as fixing a compatible system of ℓ-power roots of unity: $\zeta_n \in \mu_{\ell^n}$ with $\zeta_{n+1}^\ell = \zeta_n$. The measure is given by

$$\int_{\mathbb{Z}_\ell} f \, d\Phi = \frac{1}{(a\ell)^M} \sum_{x \in \mathbb{Z}/\ell^M\mathbb{Z}} f(x)\Phi(\zeta_M^x)$$

where f factors through $\mathbb{Z}/\ell^M\mathbb{Z}$. In particular, if you take $\Phi = \Phi_\lambda$ in which case $a = \lambda(\ell)$ (which is non-zero since $(\ell, N) = 1$) and χ a Dirichlet character of level ℓ^M, then

$$\int_{\mathbb{Z}_\ell^\times} \chi \, d\Phi_\lambda = \frac{G(\chi)}{(\ell\lambda(\ell))^M} L(0, \chi^{-1}\lambda)$$

where $G(\chi)$ is the Gauss sum, and $L(0, \chi^{-1}\lambda)$ is obtained from $\Phi_{\chi^{-1}\lambda}(t)$.

Remark. It is well-known that $G(\chi) \neq 0$ in $\overline{\mathbb{F}_p}$.

We are interested in those values for χ taking values in $\mu_{\ell^\infty} \subset \overline{\mathbb{F}_p}$, i.e., for χ that factors through

$$\begin{array}{ccc} \mathbb{Z}_\ell^\times & \xrightarrow{\chi} & \overline{\mathbb{F}_p} \\ \downarrow & \nearrow & \\ \Gamma = \mathbb{Z}_\ell^\times/\mu_{\ell-1} & & \end{array}$$

where we have the decomposition $\mathbb{Z}_\ell^\times = \mu_{\ell-1} \times \Gamma$, $\Gamma = 1 + \ell\mathbb{Z}_\ell$. (This is where the assumption that ℓ is odd simplifies the situation.)

We want to construct a measure $d\Psi$ on Γ such that

$$\int_\Gamma \chi \, d\Psi = \int_{\mathbb{Z}_\ell^\times} \chi \, d\Phi_\lambda.$$

Note

$$\int_{\mathbb{Z}_\ell^\times} \chi \, d\Phi_\lambda = \sum_{\epsilon \in \mu_{\ell-1}} \int_{\epsilon\cdot\Gamma} \chi \, d\Phi_\lambda,$$

where the integral on the right means

$$\int_{\epsilon\cdot\Gamma} \chi \, d\Phi_\lambda = \int_{\mathbb{Z}_\ell^\times} \chi 1_{\epsilon\cdot\Gamma} \, d\Phi_\lambda = \frac{1}{(\ell a)^M} \sum_{x \in \frac{1+\ell\mathbb{Z}_\ell}{1+\ell^M\mathbb{Z}_\ell}} \chi(\epsilon x)\Phi_\lambda(\zeta_M^{\epsilon x}).$$

Putting all these together, this means

$$\int_{\mathbb{Z}_\ell^\times} \chi \, d\Phi_\lambda = \frac{1}{(\ell a)^M} \sum_{x \in \Gamma/\Gamma^{\ell^M}} \chi(x) \left(\sum_\epsilon \Phi_\lambda(\zeta_M^{x\epsilon}) \right)$$

since we have assumed $\chi|_{\mu_{\ell-1}} = 1$. Therefore we can take $\Psi : \mu_{\ell^\infty} \to \overline{\mathbb{F}_p}$ to be given by

$$\Psi(t) = \sum_\epsilon \Phi_\lambda(t^\epsilon).$$

Remark. If $\epsilon \in \mu_{\ell-1}$ is not ± 1, then $\Phi_\lambda(t^\epsilon)$ is not rational.

We can write

$$\Psi(t) = \sum_{\epsilon \in \mu_{\ell-1}/\{\pm 1\}} \Phi_\lambda(t^\epsilon) + \Phi_\lambda(t^{-\epsilon}).$$

Note that we have $\Phi_\lambda(t^{-1}) = \Phi_\lambda(t)$ because $\lambda(-1) = -1$.

Today we want to show that there exists such a χ such that $L(0, \lambda\chi^{-1}) \neq 0$ in $\overline{\mathbb{F}_p}$. This is a weaker version of the theorem of Washington, which says that this happens for almost all χ. It is not much more work to get the stronger version. To prove this, we assume that $L(0, \lambda\chi^{-1}) = 0$ for all $\chi : \Gamma \to \mu_{\ell^\infty}$. We have seen that

$$0 = L(0, \lambda\chi^{-1}) = \int_\Gamma \chi \, d\Psi = \frac{1}{(\lambda(\ell)\ell)^M} \sum_{x \in \Gamma/\Gamma^{\ell^M}} \chi(x)\Psi(\zeta_M^x).$$

This is true for all characters χ. By the orthogonality relations of characters, this implies $\Psi(\zeta) = 0$ for all $\zeta \in \mu_{\ell^\infty}$. (If we knew that Ψ is rational, it would be easy to derive a contradiction. Now we need to work a little bit more.) From this, we will deduce that $\Phi_\lambda(\zeta^\epsilon) + \Phi_\lambda(\zeta^{-\epsilon}) = 0$ for all ζ and $\epsilon \in \mu_{\ell-1}/\{\pm 1\}$, which will give a contradiction.

Recall that we are studying $\overline{\mathbb{F}_p}$-valued functions on μ_{ℓ^∞}.

Lemma 3.1 (Sinnott). *Take $b_1, \cdots, b_n \in \overline{\mathbb{F}_p}$ not all zero, and $a_1, \cdots, a_n \in \mathbb{Z}_\ell$ pairwise distinct. Consider the function*

$$f(\zeta) = \sum_{i=0}^{n} b_i \zeta^{a_i} \in \overline{\mathbb{F}_p}$$

for $\zeta \in \mu_{\ell^\infty}$. Then f has finitely many zeros.

Proof. Consider the finite field $k \subset \overline{\mathbb{F}_p}$ containing $b_1, \cdots, b_n$. Let N_0 be the integer such that $\ell^{N_0} = \#(\mu_{\ell^\infty} \cap k)$, and N_1 be such that the a_i's are all distinct modulo ℓ^{N_1}. Let $N \geq N_0 + N_1$ and $\zeta \in \mu_{\ell^N}$ be such that $f(\zeta) = 0$ (which is possible if f had infinitely many zeros).

We know that

$$0 = \zeta^{-a_j} f(\zeta) = b_j + \sum_{i \neq j} b_i \zeta^{a_i - a_j}$$

and take the trace $\mathrm{tr}_{k(\zeta)/k}$. Since $\zeta^{a_i - a_j} \notin k$, it has trace 0 and so

$$0 = [k(\zeta) : k] b_j.$$

Therefore we get that $b_j = 0$ for all j, which is a contradiction. □

It is convenient to introduce $\mathcal{F}$, the ring of $\overline{\mathbb{F}_p}$-valued functions on μ_{ℓ^∞}, and $\mathcal{N} \subset \mathcal{F}$, the ideal of functions which are almost equal to 0 (i.e., vanishing except for finitely many values). Let $\mathcal{F}_0 = \mathcal{F}/\mathcal{N}$. Then the lemma says that

$$f(\zeta) = \sum_i b_i \zeta^{a_i}$$

is a unit in $\mathcal{F}_0$ when $a_i \neq a_j$ for all $i \neq j$ and b_i are not all zero.

Corollary 3.2. *Take $a_1, \cdots, a_n \in \mathbb{Z}_\ell$ linearly independent over $\mathbb{Z}$. Then the functions $t^{a_1}, \cdots, t^{a_n}$ (in $\mathcal{F}_0$) are algebraically independent over $\overline{\mathbb{F}_p}$. In other words, the map*

$$\theta : \overline{\mathbb{F}_p}[X_1, \cdots, X_n] \to \mathcal{F}_0$$

sending $X_i \mapsto (\zeta \mapsto \zeta^{a_i})$ is injective.

Proof. Consider this map

$$X_1^{k_1} X_2^{k_2} \cdots X_n^{k_n} \mapsto (\zeta \mapsto \zeta^{a_1 k_1 + a_2 k_2 + \cdots + a_n k_n}).$$

More generally, for any

$$f = \sum \beta_{(k_1, \cdots, k_n)} X_1^{k_1} X_2^{k_2} \cdots X_n^{k_n},$$

we have

$$\theta(f)(\zeta) = \sum \beta_{(k_1, \cdots, k_n)} \zeta^{a_1 k_1 + \cdots + a_n k_n}.$$

Let us call $\alpha(k_1, \cdots, k_n) = a_1 k_1 + \cdots + a_n k_n$. If $(k_1, \cdots, k_n) \neq (k_1', \cdots, k_n')$, then $\alpha(k_1, \cdots, k_n) \neq \alpha(k_1', \cdots, k_n')$ because the a_i's are linearly independent.

If f is in the kernel, then it has infinitely many zeros, so the lemma forces all the $\beta_{(k_1, \cdots, k_n)}$ to be zero. Therefore $f = 0$. □

Proposition 3.3. *Let k be a field, and $Y_1, \cdots, Y_r$ be monomials in $X_1, \cdots, X_n$*

$$Y_i = \prod_j X_j^{c_{ij}}$$

such that $Y_i^a = Y_j^b$ implies either $a = b = 0$ or $i = j$. Let $P_i(z) \in k(z)$ be such that

$$P_1(Y_1) + P_2(Y_2) + \cdots + P_r(Y_r) = 0 \tag{3}$$

then $P_1(z), \cdots, P_r(z) \in k$.

Sketch proof. Let $R = k[X_1, \cdots, X_n, X_1^{-1}, \cdots, X_n^{-1}]$, which is a factorial ring with units given by constants and monomials. The condition implies that if $f(z), g(z) \in k[z]$ are relatively prime, then $f(Y_i)$ and $g(Y_i)$ are also relatively prime.

We can easily reduce to the case $r = 2$ by induction. Writing $P_i = \frac{f_i}{g_i}$, (3) becomes

$$f_1(Y_1) g_2(Y_2) = -f_2(Y_1) g_1(Y_2).$$

We can deduce that $P_i(z) \in k[z, z^{-1}]$. From (3) again, this forces $P_i(z) \in k$. □

Using this proposition and the previous corollary, we will consider for $r = \frac{\ell - 1}{2}$,

$$\Phi(t^{\epsilon_1}) + \cdots + \Phi(t^{\epsilon_r}) = 0$$

and deduce that $\Phi \in \overline{\mathbb{F}_p}$. We will come back to this next time.