AUTOMORPHIC L-FUNCTIONS

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Abstract. Here are the notes I took for Salim Ali Altuğ's course on automorphic L-functions offered at Columbia University in Spring 2015 (MATH G6675: Topics in Number Theory). Hopefully these notes will be completed by Spring 2016. I recommend that you visit my website from time to time for the most updated version.

Due to my own lack of understanding of the materials, I have inevitably introduced both mathematical and typographical errors in these notes. Please send corrections and comments to phlee@math.columbia.edu and altug@math.columbia.edu.

Contents

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1. Lecture 1 (January 20, 2015)

1.1. **Introduction.** One central theme in number theory is automorphic L-functions. In the 1960's it was unclear what a general L-function is supposed to be; there were certainly the Riemann zeta function $\zeta(s)$, $L(s,\chi)$ due to Hecke *et al.*, Artin L-functions, and also suggestions from Tamagawa. My goal is to show how automorphic L-functions arose since 1967. The whole theory was born from the Eisenstein series, particularly their constant terms, and was really a coincidence, since Langlands was studying these constant terms just to kill time (more on this later).

I will go over Langlands' book *Euler Products*[1], which was the manuscript written on his lectures at Yale in 1967. In order to talk about this, we need to define a lot of things. One warning is that this book is not easy to read, as is anything that Langlands wrote. I will keep updating the list of references on my website.

This theory was later known as the Langlands–Shahidi method, and was used to prove functoriality of Sym^3 and Sym^4 of cuspidal automorphic representations of $\mathrm{GL}(2)$. One downside of this method is that it is limited; this will be made precise later.

I will try to keep things self-contained. The theory of Eisenstein series involves:

- reductive groups (roots, weights, parabolics, etc.),
- representation theory of local groups (unramified principal series, Satake isomorphism, Langlands' interpretation as L-groups),
- reduction theory (Borel–Harish-Chandra),
- computation.

This topic can be pretty dry if we just want to go over these things, so I will give more examples. There is also one more blackbox, which is the analytic continuation of Eisenstein series. Essentially this is the heart of the matter.

If we can cover all of this, I am open to suggestions.

1.2. **Modular forms.** A reference for this is Serre's *A Course in Arithmetic*.

Historically, modular forms arose from elliptic integrals in the 1800's. These are at the moment irrelevant to L-functions. Elliptic integrals gave rise to elliptic curves and elliptic functions, and these gave rise to modular forms. I am not going to write down elliptic integrals, but they are essentially integrals over elliptic curves. Let me start with elliptic functions, which came from trying to invert elliptic integrals.

An elliptic function is a meromorphic function $f:\mathbb{C}\to\mathbb{C}$ such that there exist $w_1,w_2\in\mathbb{C}$, $w_1/w_2\notin\mathbb{R}$ with $f(w_1+z)=f(w_2+z)=f(z)$. There are two names attached to these functions: Weierstrass (1815–1897) and Jacobi (1804–1851).

- Weierstrass attached to any given any lattice Λ the function

$$\wp_\Lambda(z)=\sum_{w\in\Lambda\setminus\{0\}}\left(\frac{1}{(z-w)^2}-\frac{1}{w^2}\right)+\frac{1}{z^2}$$

  which is meromorphic and periodic.
- Jacobi came before Weierstrass, and literally took the elliptic integral

$$u=\int_0^\phi\frac{dt}{\sqrt{1-k^2\sin^2 t}}$$

[1] Available at `http://publications.ias.edu/sites/default/files/ep-ps.pdf`.

where $0 < k^2 < 1$. This is an elliptic integral of second kind. Jacobi's first theta function (there are a total of twelve) is defined by

$$\mathrm{sn}(u) = \sin(\phi).$$

It is a fact that this is an elliptic function, with

$$\mathrm{sn}(u + 2mK + 2niK') = \mathrm{sn}(u)$$

where

$$K = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} \text{ and } K' = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1 - (1-k^2)\sin^2 t}}$$

are the complete elliptic integrals.

I introduced Jacobi's function only for fun, but Weierstrass' function will be useful. One important property of $\wp_\Lambda(z)$ is

$$(\wp'_\Lambda(z))^2 = 4\wp^3_\Lambda(z) - 60G_2(\Lambda)\wp_\Lambda(z) - 140G_3(\Lambda).$$

This is actually bad notation since in a second we will be using the $z \in \mathbb{H}$ for the variable of G_k (the translation is by writing $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ such that $\omega_1/\omega_2 \in \mathbb{H}$, then Λ corresponds to $z = \omega_1/\omega_2$ and vice versa), but we will change this in a second. Note that $(\wp_\Lambda(z), \wp'_\Lambda(z))$ gives an explicit point on the elliptic curve

$$y^2 = 4x^3 - 60G_2x - 140G_3.$$

Definition 1.1 (Eisenstein series)**.** For $k \in \mathbb{Z}$, the holomorphic Eisenstein series of weight $2k$ is defined as

$$G_k(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+n)^{2k}}.$$

This converges for $k > 1$. Given a lattice Λ, we get a point $z \in \mathbb{H}$. By $G_k(\Lambda)$ we mean G_k evaluated at z.

Here is a fun fact: if we set

$$u = \int_y^\infty \frac{ds}{\sqrt{4s^3 - 60G_2s - 140G_3}},$$

then $y = \wp(u)$.

One property of G_k is that

$$G_k(z+1) = G_k(z),$$

so we can write a Fourier series expansion

$$G_k(z) = \sum_n a_n q^n$$

where $q = e^{2\pi i z}$. The whole class is essentially about calculating a_0 for general Eisenstein series. For holomorphic Eisenstein series, the constant term a_0 is actually a constant. When we introduce non-holomorphic Eisenstein series, the constant term will depend on y: the general Fourier expansion looks like

$$\sum_n a_n f_n(y) e^{2\pi i n x}.$$

Here is a word on the broader picture. In case you have seen this before, the whole theory digresses into two branches: arithmetic and analytic. These holomorphic Eisenstein series are arithmetic objects. But they do not appear in $\mathcal{L}^2(\Gamma\backslash G)$ and have no place in the spectral theory, which originated in the 1950's.

Let us carry out an *ad hoc* calculation of the Fourier expansion. Recall that

$$\pi \cot \pi z = \frac{1}{z} + \sum_{m=1}^{\infty} \left(\frac{1}{z+m} + \frac{1}{z-m} \right).$$

Since

$$\cot x = \frac{\cos(x)}{\sin(x)} = i\frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}},$$

we get

$$\cot \pi z = i\left(1 + \frac{2}{q-1}\right) = i\left(1 - 2\sum_{n=0}^{\infty} q^n\right).$$

On the other hand, note that

$$\begin{aligned}\frac{d^{2k-1}}{dz^{2k-1}}\pi \cot \pi z &= \frac{(-1)^{2k-1}(2k-1)!}{z^{2k}} + (-1)^{2k-1}(2k-1)!\sum \left(\frac{1}{(z+m)^{2k}} + \frac{1}{(z-m)^{2k}} \right) \\ &= (-1)^{2k-1}(2k-1)! \sum_{m\in\mathbb{Z}} \frac{1}{(z+m)^{2k}}.\end{aligned}$$

By the q-expansion, we have

$$\frac{(-2\pi i)^{2k}}{(2k-1)!}\sum_{n=0}^{\infty} n^{2k-1}q^n = \sum_m \frac{1}{(m+z)^{2k}}.$$

This is so *ad hoc* it appears to come out of nowhere.

Let us go back to G_k.

$$\begin{aligned}G_k(z) &= \sum_{m,n}{}' \frac{1}{(mz+n)^{2k}} \\ &= 2\zeta(2k) + \sum_{m\neq 0,n} \frac{1}{(mz+n)^{2k}} \\ &= 2\zeta(2k) + 2\sum_{m=1}^{\infty}\sum_{n} \frac{1}{(mz+n)^{2k}} \\ &= 2\zeta(2k) + 2\sum_{m=1}^{\infty} \frac{(-2\pi i)^{2k}}{(2k-1)!}\sum_{a=0}^{\infty} a^{2k-1}q^{am} \\ &= 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{\alpha=1}^{\infty} \sigma_{2k-1}(\alpha)q^{\alpha}\end{aligned}$$

where $\sigma_t(\alpha) = \sum_{d|\alpha} \alpha^t$ is the classical divisor sum function.

Here are some reasons for writing this down. Firstly, the special values of the zeta function appear in the constant terms. For non-holomorphic Eisenstein series, we do get a zeta function as the constant term, rather than just special values. Secondly,

$$\sigma_{2k-1}(\alpha)q^{\alpha} = \sigma_{2k-1}(\alpha)e^{\pi i\alpha x}e^{-\pi\alpha y},$$

where $\sigma_{2k-1}(\alpha)$ is the divisor function, $e^{\pi i\alpha x}$ is a function on the unit circle, and $e^{-\pi\alpha y}$ is one of the solutions to $\frac{d^2 f}{dy^2} = f$ (up to a constant). Classically over the Euclidean plane, $\frac{d^2 f}{dy^2} = f$ has two solutions: e^y and e^{-y}. We see e^{-y} in the Eisenstein series. This is a common theme: the Fourier coefficients of general Eisenstein series will satisfy certain differential equations. Only some of those solutions will appear.

This is the state of matters at the beginning of the 20th century. Things were case-by-case at this point.

Definition 1.2 (Modular function)**.** A modular function of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$ is a function $f : \mathbb{H} \to \mathbb{C}$ such that

- $f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = (cz+d)^{2k} f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.
- f is meromorphic.

Definition 1.3 (Modular form)**.** A modular form is a modular function that is holomorphic everywhere. (By holomorphic at ∞, we mean that the q-expansion should not have any negative powers.)

Definition 1.4 (Cusp form)**.** A cups form is a modular form with zeroth coefficient in q-expansion 0.

Example 1.5.

- G_k is a modular form.
- $\Delta(z) = (60G_2(z))^3 - 27(140G_3(z))^2$ is a modular form of weight 12.
- $j(z) = \frac{1728(60G_2(z))^3}{\Delta(z)}$ is a modular function of weight 0, but not a modular form. It has a simple pole with residue 1 at ∞.

Moving on, there were Siegel modular forms, Jacobi theta functions, Hecke's generalization of Riemann's work for higher number fields, Hilbert's (and Blumenthal's) generalization of the notion of modular forms over upper half spaces, and many other names over 1900–1940. And then came Maass (1949), a student of Hecke who worked on non-holomorphic modular forms. He studied the $\mathcal{L}^2$ theory, which was the introduction of spectral theory to the study of modular forms. Of course, there was the Petersson inner product before Maass, but that was limited to cusp forms only.

The Petersson inner product is defined as

$$\langle f, g\rangle_{\mathrm{SL}_2(\mathbb{Z})} = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} f(z)\overline{g(z)}y^k \frac{dx\,dy}{y^2}$$

for f and g of weight k. In order for this to be well-defined, at least one of f or g has to be cuspidal. I did not go through the spectral theory for holomorphic forms, but keep in mind that the Eisenstein series and cusp forms span all the holomorphic modular forms. When we multiply f and $\overline{g}$, all the cross terms except for the zeroth term are integrable, since q grows as e^{-y}.

The spectral theory involves a Hilbert space and a measure. Next time it will be clear why we do this. Let us work with the specific case of $\mathcal{L}^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}, \mu)$, where

$$d\mu = \frac{dx\,dy}{y^2}$$

is the volume element of the classical hyperbolic metric. (For now we are defining these spaces for weight 0. We note that one can extend these to all K-types for a maximal compact subgroup K, i.e. for all weights k.)

In the case of Eisenstein series, we could write down all the Fourier coefficients. But for cusp forms, I did not write any down because they are difficult. The important point is that the cuspidal part of $\mathcal{L}^2$ is the same as the cuspidal part of automorphic forms.

Next time I will talk more about the classical setting, introducing the Laplacian Δ, spectral theory, the non-holomorphic Eisenstein series and their constant terms. Then we will pass to the adelic setting and groups.

2. Lecture 2 (January 22, 2015)

2.1. **Last time.** Last time was an introduction to modular forms. We went back to the 1800's and talked about holomorphic modular forms, which are roughly speaking holomorphic functions on the upper half plane that are modular with respect to the modular group. In some sense these are just "periodic" functions. We defined the Eisenstein series

$$G_k(z) := \sum_{(c,d)\neq 0} \frac{1}{(cz+d)^{2k}}$$

and computed their Fourier expansions

$$G_k(z) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

where $q = e^{2\pi i z} = e^{2\pi i x}e^{-2\pi y}$.

Then Maass came and introduced the $\mathcal{L}^2$ theory; Maass forms came up in 1949. This marked the introduction of spectral theory to the subject, initiated by Maass and Selberg. Spectral theory means we have a Hilbert space with a measure and we try to diagonalize certain operators.

2.2. **Spectral theory.** Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and consider the Hilbert space $\mathcal{L}^2(\Gamma\backslash\mathbb{H}, d\mu)$ where $d\mu = \frac{dx\,dy}{y^2}$ is the invariant measure on $\mathbb{H}$ and the inner product is given by

$$\langle f, g\rangle = \int_{\Gamma\backslash\mathbb{H}} f\cdot\overline{g}\,d\mu.$$

We will keep doing things classically for another lecture before moving to the adelic setting. We will see that the formula for $d\mu$ essentially comes from the Iwasawa decomposition.

Let Δ be the Laplacian, which in these coordinates is normalized as

$$\Delta = \Delta_0 = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Remark. This is the theory for weight 0 functions (i.e., K-invariant on the right).

Let us analyze Δ on $\mathbb{H}$ a bit. Eventually we will see that holomorphic modular forms (more precisely y^k times a holomorphic form of weight $2k$) and Maass forms are eigenfunctions for Δ (and its extensions for all weights k), and it is important to first study its eigenvalues. This gives motivation for the Selberg's eigenvalue conjecture and the Ramanujan conjecture.

Spectral theory is nice if we have a self-adjoint operator. For infinite-dimensional spaces, we need to be a bit careful because differentiation is in general unbounded. Basically, the idea of self-adjointness is not straightforward, and we need to look at a dense subspace. Some properties of Δ are:

(1) symmetric (i.e., $\langle \Delta f, g\rangle = \langle f, \Delta g\rangle$),
(2) non-negative (i.e., $\langle \Delta f, f\rangle \geq 0$).

Proof. These are very easy, and essentially just integration by parts. Let $f, g \in C_0^\infty(\mathbb{H})$. Then

$$\langle \Delta f, g\rangle = \int_{\mathbb{H}} \Delta f \cdot \overline{g}\, \frac{dx\, dy}{y^2} = \int_{\mathbb{H}} \nabla f \cdot \overline{\nabla g}\, dx\, dy$$

where $\nabla f = (f_x, f_y)$. □

This is a straightforward application of integration by parts, but immediately shows the following consequences. Remember we are trying to understand the eigenvalues, λ, of Δ. At the beginning, there is no reason λ has any restrictions, but because of (1) and (2) above we have:

- Eigenvalues are real.
- Eigenvalues are non-negative.

In fact more is true.

Proposition 2.1. *If λ is an eigenvalue of Δ on $\mathbb{H}$, then $\lambda \geq \frac{1}{4}$.*

This $\frac{1}{4}$ will appear everywhere and is an interesting object.

Proof. Without loss of generality assume F is real-valued and $\Delta F = \lambda F$; otherwise we can just separate the real and imaginary parts of F (this is to avoid complex conjugations). Then integration by parts gives

$$\begin{aligned}\langle \Delta F, F\rangle &= -\int \left(y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)F\right)\cdot F\,\frac{dx\,dy}{y^2} \\ &= \int (F_x^2 + F_y^2)\, dx\, dy \geq \int F_y^2\, dx\, dy \qquad (1)\end{aligned}$$

and

$$\int F^2\, \frac{dy}{y^2} = 2\int F\cdot F_y\, \frac{dy}{y}.$$

The second relation implies

$$\int F^2\, \frac{dx\,dy}{y^2} = 2\iint F\cdot F_y\, \frac{dx\,dy}{y} \leq \left(4\int F_y^2\, dx\, dy\right)^{\frac{1}{2}} ||F||$$

by the Cauchy–Schwarz inequality, so

$$||F||^2 \leq 4\int F_y^2\, dx\, dy. \qquad (2)$$

By (1) and (2), we conclude

$$\langle \Delta F, F\rangle \geq \frac{1}{4}\langle F, F\rangle. \qquad \square$$

This is the story for $\mathbb{H}$. What usually happens is that for a big space without any discrete group action, it is easy to determine eigenvalues: here we have the half line $[\frac{1}{4}, \infty)$, which is a continuous spectrum. Once we take the quotient $\Gamma\backslash\mathbb{H}$, we will get discrete eigenvalues, which correspond to Maass forms. $\Gamma\backslash\mathbb{H}$ is like the global (real) picture. We can also localize and consider $\mathrm{PGL}_2(\mathbb{Q}_p)/\mathrm{PGL}_2(\mathbb{Z}_p)$, which is like a tree, and we can try to analyze the spectrum of Δ_p. This is a pure spectrum and everything is continuous, but once we take quotients we will get something discrete. Selberg's eigenvalue conjecture states that the same bound $\lambda \geq \frac{1}{4}$ holds even after we pass to the quotient $\Gamma\backslash\mathbb{H}$.

Now let us move to the eigenfunctions $\Delta f = \lambda f$ over $\mathbb{H}$. Recall that

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Step 1: Look at eigenfunctions which only depend on y. So we are looking for functions

$$F(x, y) = \phi(y)$$

such that

$$\Delta F = -y^2\frac{\partial^2}{\partial y^2}\phi = \lambda\phi.$$

There are some obvious eigenfunctions: $\phi(y) = y^s$ and y^{1-s} (linearly independent), where $s(1-s) = \lambda$ and $s \neq \frac{1}{2}$. If $s = \frac{1}{2}$, then we can consider the solutions $\{y^s, y^s\log(y)\}$.

Let us parametrize $\lambda = s(1-s)$. Then $\lambda \in \mathbb{R}$ and $\lambda \geq \frac{1}{4}$. These functions are the building blocks of real-analytic Eisenstein series, and will also come up when we study the intertwining operators.

Step 2: Assume that the eigenfunction is periodic in x. This is because if we want to get eigenfunctions on $\Gamma\backslash\mathbb{H}$, then they should be invariant under the action

$$f\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z\right) = f(z).$$

Then we can write $F(z) = e(x)\phi(2\pi y)$, where $e(x) = e^{2\pi i x}$, and

$$\begin{aligned}
\Delta F &= -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)e(x)\phi(2\pi y) \\
&= -y^2\left(\frac{\partial^2}{\partial x^2}e(x)\right)\phi(2\pi y) - y^2 e(x)\phi''(2\pi y) \\
&= y^2 4\pi^2 e(x)(\phi(2\pi y) - \phi''(2\pi y)).
\end{aligned}$$

In order for $\Delta F = \lambda F$, we want

$$\phi''(2\pi y) + \phi(2\pi y)\left(\frac{\lambda}{(2\pi y)^2} - 1\right) = 0.$$

This is Bessel's differential equation of second type. The upshot is, people have studied these. We have the following facts:

(1) There are two linearly independent solutions: (i) $\sqrt{2\pi^{-1}y}K_{s-\frac{1}{2}}(y)$, (ii) $\sqrt{2\pi y}I_{s-\frac{1}{2}}(y)$.

(2) Their behavior is: $(i) \sim e^{-y}$ and $(ii) \sim e^{y}$ as $y \to \infty$. (To convince oneself informally, consider the equation for $y \to \infty$. Then $\phi'' = \lambda\phi$ has solutions e^y and e^{-y}.)

Remember that we are trying to work on $\mathcal{L}^2(\Gamma\backslash\mathbb{H}, d\mu)$. For harmonic analysis on this, only $K_{s-\frac{1}{2}}$ is relevant. From the above discussion we get the following proposition.

Proposition 2.2. *Any $F \in \mathcal{L}^2(\Gamma\backslash\mathbb{H})$ that is an eigenfunction of Δ with eigenvalue $s(1-s)$ has a Fourier–Whittaker expansion*

$$\sum_{n\in\mathbb{Z}} a_n \sqrt{y} K_{s-\frac{1}{2}}(2\pi|n|y)e(xn)$$

where $a_n \in \mathbb{C}$.

Rough proof. Let $F(z) = F(x+iy)$ be such a function. Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$ we know that the function is periodic in the x-variable and has a Fourier expansion of the form

$$F(z) = \sum_{n\in\mathbb{Z}} a(n,y)e(xn)$$

Since $F(z)$ is an eigenfunction each $a(n,y)$ satisfies the Bessel differential equation, therefore the solutions are expressible as a linear combination of $\sqrt{2\pi^{-1}y}K_{s-\frac{1}{2}}(y)$ and $\sqrt{2\pi y}I_{s-\frac{1}{2}}(y)$. Finally since F is in $\mathcal{L}^2$ only K-Bessel function appears. □

Let us go back to $\Gamma\backslash\mathbb{H}$ and Maass forms. The whole point of the above discussion is that the eigenfunctions of Δ on $\mathbb{H}$ give us building blocks of these functions, and their eigenvalues satisfy certain properties.

We want functions $f : \mathbb{H} \to \mathbb{C}$ which are

- invariant under $\mathrm{SL}_2(\mathbb{Z})$,
- eigenfunctions of Δ,
- of moderate growth (i.e., f should grow at most polynomially at ∞).

Invariance under $\mathrm{SL}_2(\mathbb{Z})$ implies invariance under $x \mapsto x+1$, so F has a Fourier expansion. There are various ways of cooking up such eigenfunctions, but the best way is following your nose: averaging. Start with ϕ, an eigenfunction on $\mathbb{H}$ that is invariant under $x \mapsto x+1$, and make it invariant under the rest of the group by averaging.

Example 2.3. Let $\pm\Gamma_\infty = \left\langle \pm\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \right\rangle = \left\{ \pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\}$ be the stabilizer of y. Define $\tilde{\phi}_s(z) = (\operatorname{Im} z)^s = y^s$, and

$$\tilde{E}(z;s) = \sum_{\pm\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} \tilde{\phi}_s(\gamma z).$$

This is the non-normalized non-holomorphic Eisenstein series.

Note that $\Delta\tilde{E}(z;s) = s(1-s)\tilde{E}(z;s)$. We can rewrite

$$\sum_{\pm\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} \tilde{\phi}_s(\gamma z) = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1 \\ c\geq 0}} \frac{y^s}{|cz+d|^{2s}}.$$

Once we pass to the group-theoretic language, these things will be canonically defined, but for now let us do an *ad hoc* calculation. We have

$$\operatorname{Im}(\gamma z) = \frac{\operatorname{Im}(z)}{|cz+d|^2}$$

and the bijection

$$\Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z}) \longleftrightarrow \{(c,d) \mid \gcd(c,d)=1, c \geq 0\}.$$

This is because if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} a' & b' \\ c & d \end{pmatrix}$, then $ad-bc = a'd-b'c = 1$ and $(a-a')d+(b'-b)c = 0$. These calculations are easy to do for GL(2).

I will do the following normalization

$$\phi_s(z) = (\operatorname{Im} z)^{s+\frac{1}{2}},$$

so that

$$\Delta E(z;s) = \left(\frac{1}{4} - s^2\right) E(z;s).$$

Note that the above equality is true only for those s so that $E(z,s)$ makes sense, and we then get an eigenfunction with eigenvalue $\frac{1}{4} - s^2$.

Note that for any eigenfunction that is actually in $\mathcal{L}^2$, by the previous calculation, the eigenvalue has to satisfy $\frac{1}{4} - s^2 \in [0,\infty)$, and therefore $s \in i\mathbb{R} \cup [0, \frac{1}{2}]$.

Selberg's eigenvalue conjecture says that for a Maass cusp form, the eigenvalue cannot be in $[0, \frac{1}{2})$. These eigenvalues parametrize representations of GL(2): $i\mathbb{R}$ corresponds to the principal series and $[0, \frac{1}{2})$ corresponds to the complementary series. Thus Selberg says the representations coming from Maass forms have to be in the unitary principal series.

Let me finish by stating some properties of these eigenfunctions, and we will prove them next time.

(1) $E(z;s)$ converges for $\operatorname{Re}(s) > \frac{1}{2}$.
(2) $E(\gamma z;s) = E(z;s)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.
(3) $\Delta E(z;s) = (\frac{1}{4} - s^2)E(z;s)$.
(4) $E(z;s)$ has analytic continuation and a functional equation relating $s \leftrightarrow -s$.
(5) $E(z;s)$ has the Fourier expansion

$$E(z;s) = \sum_{n\neq 0} a_n \sqrt{y} K_s(2\pi|n|y)e(nx) + a_0(y),$$

where

$$a_n = \frac{4|n|^s \sigma_{-2s}(|n|)}{\zeta(2s+1)}$$

and

$$a_0 = y^{s+\frac{1}{2}} + \frac{\xi(2s)}{\xi(2s+1)} \cdot y^{\frac{1}{2}-s}$$

with $\xi(s)$ the completed Riemann zeta function.

The $-2s$ in the divisor function dictates the eigenvalue. The $\frac{\xi(2s)}{\xi(2s+1)}$ in a_0 is the completed ζ-function. Once we know the analytic continuation of ζ (classically obtained using integral representation, Poisson summation and partial summation formula), we get the analytic continuation of $E(z;s)$.

Spectral theory reverses this argument: we would like to start from the analytic continuation of $E(z;s)$, and deduce from this the analytic continuation of ζ! This works more generally for $\mathrm{GL}(n)$ and Rankin products of L-functions, which appear as constant terms of Eisenstein series. This is the upshot for the whole class. We try to do this for all groups.

Next time we will calculate a_0, and obtain analytic continuation and functional equation from a very soft spectral argument.

3. Lecture 3 (January 29, 2015)

3.1. **Last time.** Let us start with some clarifications from last time. We introduced the $\mathcal{L}^2$ theory: we have the Laplacian Δ acting on $\mathcal{L}^2(\Gamma\backslash\mathbb{H})$. Let me summarize what I meant to say.

Consider Δ on the universal covering $\mathbb{H}$. It has two eigenfunctions: for each s, we have y^s and y^{1-s}. All the other eigenfunctions are generated by these. One can show that it is sufficient to take $s = \frac{1}{2} + it$ to get the whole spectrum.

But we are interested in $\Gamma\backslash\mathbb{H}$, so what about eigenfunctions that are invariant under $T = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, i.e., under $x \mapsto 1 + x$? They have to be of the form

$$F(x) = \sum_{n\in\mathbb{Z}} a_n \sqrt{y} K_{s-\frac{1}{2}}(2\pi|n|y)e(nx)$$

where $K_{s-\frac{1}{2}}$ is the modified Bessel function of second type. This is just basic harmonic analysis on $\mathbb{H}$. We have shown that eigenfunctions have to be of this form, but we haven't constructed any yet.

Consider the Dirichlet problem on any domain D in $\mathbb{H}$: we want to find a function F such that $\Delta F = \lambda F$ and $F = 0$ on the boundary of D. By the formal argument on $\mathbb{H}$ from last time (integral trick), we have $\lambda \in \mathbb{R}_+$. In fact we have the stronger bound $\lambda \geq \frac{1}{4}$.

Question: why does this not work for $\Gamma\backslash\mathbb{H}$? It is only a formal argument after all. In particular, why would eigenvalues on $\Gamma\backslash\mathbb{H}$ not be at least $\frac{1}{4}$? In fact, it does work a bit.

Assume for simplicity $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ (the argument will work for any discrete subgroup). Let us fix a fundamental domain, which is in particular a subset of $\mathbb{H}$. We can try to solve the Dirichlet problem for a domain D inside the fundamental domain. For $\Delta F = \lambda F$, we get $\lambda \geq \frac{1}{4}$. But this is not a function on $\Gamma\backslash\mathbb{H}$. We can try to make this a function on $\Gamma\backslash\mathbb{H}$ by averaging

$$\sum_{\gamma\in\text{stabilizer}\backslash\mathrm{SL}_2(\mathbb{Z})} F(\gamma z).$$

Whenever this guy converges, the eigenvalue will be at least $\frac{1}{4}$, but there could be other eigenfunctions not of this form! For example, Eisenstein series that contribute to $\mathcal{L}^2$ will have eigenvalues greater than $\frac{1}{4}$.

Selberg's conjecture states that $\lambda \geq \frac{1}{4}$ for Γ arithmetic. In representation theory language this is the Ramanujan conjecture over the real place. The bound $\lambda \geq 0$ still holds, but we know $\lambda \geq \frac{1}{4}$ only for $\mathrm{SL}_2(\mathbb{Z})$ and congruence subgroups of small level (up to 7).

3.2. **Real-analytic Eisenstein series.** Following the same game as above, we start with the eigenfunction $y^{s+\frac{1}{2}}$ and take the sum

$$E(z,s) = \sum_{\gamma \in \mathrm{Stab}(y)\backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(\gamma z)^{s+\frac{1}{2}} = \sum_{\pm\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^{s+\frac{1}{2}},$$

where $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| \, n \in \mathbb{Z} \right\}$ is the unipotent radical. We take $s + \frac{1}{2}$ because things will be more symmetric this way. Note that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d} \Rightarrow \mathrm{Im}(\gamma z)^{s+\frac{1}{2}} = \frac{\mathrm{Im}(z)^{s+\frac{1}{2}}}{|cz+d|^{2s+1}}.$$

We have the following facts:

(1) $E(z,s)$ converges for $\mathrm{Re}(s) > \frac{1}{2}$. (This is easy.)
(2) $E(\gamma z, s) = E(z,s)$. (This is clear by construction.)
(3) $\Delta E(z,s) = (\frac{1}{4} - s^2)E(z,s)$.
(4) $E(z,s)$ has an analytic continuation to $\mathbb{C}$ (in the s-variable) with a simple pole at $s = \frac{1}{2}$ with residue $\frac{3}{\pi}$.
(5) $E(z,s)$ has the Fourier expansion

$$E(z,s) = \sum_{n\neq 0} a_n(s)\sqrt{y}K_s(2\pi|n|y)e(nx) + a_0(y,s),$$

where

$$a_n(s) = \frac{4|n|^s\sigma_{-2s}(|n|)}{\zeta(2s+1)}$$

and

$$a_0(y,s) = y^{s+\frac{1}{2}} + \frac{\xi(2s)}{\xi(2s+1)}y^{\frac{1}{2}-s}.$$

Here $\xi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ is the completed zeta function.

The important thing is there is a ratio of L-functions in a_0 and an L-function in a_n. At the end of the day, we want to do the following. In this very special case one can calculate the Fourier expansion. Then one gets the analytic continuation and functional equation of $E(z,s)$ by the known properties of ζ. BUT we don't want to do this by writing the Fourier expansion. Instead we would like to deduce properties about these L-functions from the analytic properties of $E(z,s)$. For example, if we know (4) and (5) by some other means, then we know $\frac{\xi(2s)}{\xi(2s+1)}$ has analytic continuation. For more general groups, we get a product of quotients of various L-functions in the constant term, but we can still extract information about the individual L-functions by an induction argument.

Although the first Fourier coefficient seems to contain strictly more information than the constant term, it is harder to compute and may not even make sense for general groups. The Shalika–Casselman formula was not available yet. Here is a bit of historical perspective. We are at about 1965. Shalika was quite senior but Casselman was not yet. Then Shahidi came in around 1975.

3.3. **A (classical) computation.** The constant term is given by

$$a_0(y,s) = \int_0^1 E(x+iy,s)\,dx.$$

Before we start computing this, let us remark that there is a one-to-one correspondence

$$\pm\Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z}) \longleftrightarrow \{(c,d) \mid \gcd(c,d)=1, c \geq 0\},$$

Exercise. Prove this bijection.

So,

$$\begin{aligned}\int_0^1 E(z,s)\,dx &= \int_0^1 \sum_{\substack{(c,d)=1\\ c\geq 0}} \frac{y^{s+\frac{1}{2}}}{|cz+d|^{2s+1}}\,dx \\ &= y^{s+\frac{1}{2}} + \sum_{c=1}^{\infty} \sum_{\substack{d\in\mathbb{Z}\setminus\{0\}\\ (c,d)=1}} \int_0^1 \frac{y^{s+\frac{1}{2}}}{|cz+d|^{2s+1}}\,dx + \int_0^1 \frac{y^{s+\frac{1}{2}}}{|x+iy|^{2s+1}}\,dx.\end{aligned}$$

The sum from $c=2$ to ∞ contributes

$$\begin{aligned}&y^{s+\frac{1}{2}} \sum_{c=2}^{\infty} \sum_{\substack{\alpha=1\\ (\alpha,c)=1}}^{c-1} \sum_{k\in\mathbb{Z}} \int_0^1 \frac{dx}{|cx+\alpha+kc+ciy|^{2s+1}} \\ =&y^{s+\frac{1}{2}} \sum_{c=2}^{\infty} \frac{1}{c^{2s+1}} \sum_{\alpha}\sum_{k\in\mathbb{Z}} \int_0^1 \frac{dx}{|x+\frac{\alpha}{c}+k+iy|^{2s+1}} \\ =&y^{s+\frac{1}{2}} \sum_{c=2}^{\infty} \frac{1}{c^{2s+1}} \sum_{\alpha}\sum_{k\in\mathbb{Z}} \int_k^{k+1} \frac{dx}{|x+\frac{\alpha}{c}+iy|^{2s+1}} \\ =&y^{s+\frac{1}{2}} \sum_{c=2}^{\infty} \frac{1}{c^{2s+1}} \sum_{\alpha} \int_{-\infty}^{\infty} \frac{dx}{|x+\frac{\alpha}{c}+iy|^{2s+1}} \\ =&y^{s+\frac{1}{2}} \sum_{c=2}^{\infty} \frac{1}{c^{2s+1}} \sum_{\alpha} \int_{-\infty}^{\infty} \frac{dx}{|x+iy|^{2s+1}} \\ =&y^{s+\frac{1}{2}} \sum_{c=2}^{\infty} \frac{\varphi(c)}{c^{2s+1}} \int_{-\infty}^{\infty} \frac{dx}{|x+iy|^{2s+1}}.\end{aligned}$$

We made a big fuss about this calculation, because this is the unfolding that will appear naturally later. The integral

$$y^{s+\frac{1}{2}} \int_{\mathbb{R}} \frac{dx}{|x+iy|^{2s+1}}$$

is a specialization of the beta function. More precisely, it is equal to

$$= y^{\frac{1}{2}-s} \int \frac{dx}{(x^2+1)^{s+\frac{1}{2}}} = y^{\frac{1}{2}-s} \frac{\Gamma(\frac{1}{2})\Gamma(s)}{\Gamma(s+\frac{1}{2})}.$$

Combining everything (including the $c = 1$ term), we get

$$a_0(y, s) = y^{s+\frac{1}{2}} + y^{\frac{1}{2}-s} \sum_{c=1}^{\infty} \frac{\varphi(c)}{c^{2s+1}} \frac{\Gamma(\frac{1}{2})\Gamma(s)}{\Gamma(s+\frac{1}{2})}.$$

Note by multiplicativity

$$\sum_{c=1}^{\infty} \frac{\varphi(c)}{c^{2s+1}} = \prod_p \left(\sum_{k=0}^{\infty} \frac{\varphi(p^k)}{p^{k(2s+1)}} \right) = \prod_p \frac{1 - \frac{1}{p^{2s+1}}}{1 - \frac{1}{p^{2s}}} = \frac{\zeta(2s)}{\zeta(2s+1)}$$

and so

$$a_0(y, s) = y^{\frac{1}{2}+s} + \frac{\xi(2s)}{\xi(2s+1)} y^{\frac{1}{2}-s}.$$

3.4. **Why is the $\mathcal{L}^2$ theory helpful?** Remember we are trying to give a historical context too. Before 1965, there was no spectral theory and no Hilbert space in the theory of automorphic forms. Everything was complex-analytic and people were just using the Petersson inner product. I am not sure if people proved the analytic continuation and functional equation of Eisenstein series, but I can now give a very soft analytic continuation result. We know that:

- $E(z, s)$ converges for $\mathrm{Re}(s) > \frac{1}{2}$.
- $E(z, s) \sim y^{s+\frac{1}{2}} + (*)y^{\frac{1}{2}-s}$ (where $(*)$ is constant in the y-variable). All the other terms have exponential decay so are negligible.

Claim. $E(z, s)$ is the unique eigenfunction of Δ with eigenvalue $\frac{1}{4} - s^2$, $\mathrm{Re}(s) > \frac{1}{2}$, and asymptotic growth $\sim y^{s+\frac{1}{2}} + O(y^{-\epsilon})$ for some $\epsilon > 0$.

This is not hard to prove. Analytic continuation is often proved in two ways: by studying special integral transforms and their kernels (which is how the theta and zeta functions arise classically), or by using some uniqueness statements. For example, Jacquet–Langlands uses (in the local case) the uniqueness of Kirillov models, unlike Tate's thesis which uses integrals.

Proof. Suppose there exists an $F(z)$ satisfying this. Consider the difference $E(z, s) - F(z)$.

(1) $\Delta(E(z, s) - F(z)) = (\frac{1}{4} - s^2)(E(z, s) - F(z))$.
(2) $E(z, s) - F(z) \in \mathcal{L}^2(\Gamma\backslash\mathbb{H})$.

This means $E(z, s) - F(z)$ is an eigenfunction of Δ on $\mathcal{L}^2(\Gamma\backslash\mathbb{H})$, so the eigenvalue has to be ≥ 0 because Δ is non-negative and symmetric on $\mathcal{L}^2(\Gamma\backslash\mathbb{H})$. This is where the $\mathcal{L}^2$ theory is important.

As the final step, the eigenvalue is $\frac{1}{4} - s^2$. The assumption is that $\mathrm{Re}(s) > \frac{1}{2}$, so either $s \in \mathbb{R}$ with $s > \frac{1}{2}$, or s has an imaginary part. The former implies that $\frac{1}{4} - s^2 < 0$ and the latter implies that $\frac{1}{4} - s^2$ is not real. In either case, $\frac{1}{4} - s^2$ cannot be an eigenvalue of Δ in $\mathcal{L}^2(\Gamma\backslash\mathbb{H})$.

But $E(z, s) - F(z)$ actually has eigenvalue $\frac{1}{4} - s^2$, so we conclude $F(z) = E(z, s)$. □

The implication is that for $\mathrm{Re}(s) > \frac{1}{2}$, we have

$$E(z, s) = E(z, -s)a(s)$$

for some function $a(s)$ independent of z. Why?

- $s \mapsto -s$ does not affect the eigenvalue.

- Since
$$E(z,s) \sim y^{s+\frac{1}{2}} + \frac{\xi(2s)}{\xi(2s+1)} y^{\frac{1}{2}-s},$$
we know
$$E(z,-s) \sim y^{\frac{1}{2}-s} + \frac{\xi(-2s)}{\xi(-2s+1)} y^{\frac{1}{2}+s}$$
and so
$$\frac{\xi(-2s+1)}{\xi(-2s)} E(z,-s) = y^{\frac{1}{2}+s} + (*) y^{\frac{1}{2}-s}.$$
Therefore
$$E(z,s) = \frac{\xi(-2s+1)}{\xi(-2s)} E(z,-s).$$

Remark. First of all we needed the fact that $E(z,-s)$ was defined with the same eigenvalue for this argument to work (which the Fourier expansion gives), so in a sense we are cheating, but this does give the correct functional equation for $\mathrm{Re}(s) > \frac{1}{2}$. However in a sense this is useless for the $\mathcal{L}^2$ theory. $E(z,s)$ converges on $\mathrm{Re}(s) > \frac{1}{2}$, but does not belong to $\mathcal{L}^2$ and does not contribute to $\mathcal{L}^2$. By the functional equation we have obtained the same for $\mathrm{Re}(s) < -\frac{1}{2}$. If we can analytically continue the Eisenstein series to the critical line $\mathrm{Re}(s) = 0$, these are the ones that contribute to $\mathcal{L}^2$.

"Contribution" means the following. What is the dual of $(\mathbb{R},+)$, i.e., $\mathrm{Hom}(\mathbb{R},\mathbb{C}^\times)$? For any $a \in \mathbb{C}^\times$, we have $x \mapsto e(ax)$, which gives an isomorphism $\mathbb{C}^\times \cong \mathrm{Hom}(\mathbb{R},\mathbb{C}^\times)$. In order to look at $\mathcal{L}^2(\mathbb{R})$, Fourier theory says
$$f(x) = \int_{\mathbb{R}} \hat{f}(z) e(-zx)\, dz.$$
We see that it is enough to take all the characters that are unitary. Only the ones that are on the unitary axis are "contributing". Note also that none of the harmonics $e(-zx)$ are in $\mathcal{L}^2$, but they form $\mathcal{L}^2$.

Next time we will compute the volume of the fundamental domain, prove the prime number theorem, and prove that there are finitely many imaginary quadratic fields with class number 1. These are kind of random, but as you can guess they share a common theme: they all follow from the computations for $E(z,s)$.

4. Lecture 4 (February 3, 2015)

4.1. **Last time.** We defined the Eisenstein series
$$E(z;s) = \sum_{\pm\Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(\gamma z)^{s+\frac{1}{2}}$$
where $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\}$ is the unipotent radical. We talked about how the $\mathcal{L}^2$ theory helped: using a very soft argument, we proved the

Proposition 4.1. *$E(z;s)$ is the unique eigenfunction of Δ satisfying:*
- $\Delta E(z;s) = (\frac{1}{4} - s^2) E(z;s)$,
- *growth* $y^{s+\frac{1}{2}} + O(y^{-\epsilon})$ *for some* $\epsilon > 0$, *and*

- $\mathrm{Re}(s) > \frac{1}{2}$.

Remark. $\mathrm{Re}(s) > \frac{1}{2}$ is necessary for the convergence of $E(z; s)$.

As a corollary, we get the functional equation.

Corollary 4.2. *$E(z, s) = A(s)E(z, -s)$ where $A(s)$ is meromorphic and $\mathrm{Re}(s) > \frac{1}{2}$.*

Proof. We have

$$E(z, s) = y^{s+\frac{1}{2}} + \frac{\xi(2s)}{\xi(2s+1)} y^{\frac{1}{2}-s} + O(e^{-\frac{y}{2}})$$

where $\xi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ is the completed zeta function. We know

$$\Delta E(z, s) = \left(\frac{1}{4} - s^2\right) E(z, s)$$

and so

$$\Delta E(z, -s) = \left(\frac{1}{4} - s^2\right) E(z, -s).$$

We also have

$$E(z, -s) \sim y^{-s+\frac{1}{2}} + \frac{\xi(-2s)}{\xi(-2s+1)} y^{\frac{1}{2}+s}$$

and hence

$$E(z, -s)\frac{\xi(-2s+1)}{\xi(-2s)} \sim y^{s+\frac{1}{2}} + O(y^{-\epsilon}).$$

By the proposition,

$$E(z, s) = \frac{\xi(-2s+1)}{\xi(-2s)} E(z, -s).$$

□

This result does not really give the analytic continuation of $E(z, s)$. It simply flips the regions $\mathrm{Re}(s) > \frac{1}{2}$ and $\mathrm{Re}(s) < -\frac{1}{2}$. This only gives a philosophical argument for the functional equation. The whole point is to go the other way round:

(1) Analyze Eisenstein series (this is hard!).
(2) Calculate constant terms.
(2.5) Observe that they are L-functions.
(3) Deduce the properties of these L-functions from (1).

4.2. **Analytic continuation.** Today we will go back to the original theory. Let us forget about all the stuff above.

Theorem 4.3 (Analytic continuation of Eisenstein series)**.** *$E(z, s)$, originally defined for $\mathrm{Re}(s) > \frac{1}{2}$, satisfies the following properties:*

- *$E(z, s)$ has analytic continuation to the whole $\mathbb{C}$ (in the s-variable).*
- *$E(z, s) = A(s)E(z, -s)$, where $A(s)$ is meromorphic.*
- *$\Delta E(z, s) = (\frac{1}{4} - s^2)E(z, s)$.*
- *$E(z, s)$ has a simple pole at $s = \frac{1}{2}$ with residue $\frac{3}{\pi}$.*

4.3. **Application: Prime number theorem.** We will prove the following form of the prime number theorem.

Theorem 4.4 (Prime number theorem)**.** *$\zeta(s) \neq 0$ on the line $\mathrm{Re}(s) = 1$.*

Proof. Recall

$$E(z,s) = y^{s+\frac{1}{2}} + \frac{\xi(2s)}{\xi(2s+1)} y^{\frac{1}{2}-s} + \frac{4}{\xi(2s+1)} \sum_{n\neq 0} |n|^s \sigma_{-2s}(|n|) K_s(2\pi|n|y) e(nx).$$

Consider $\mathrm{Re}(s) = 0$. By the theorem, $E(z,s)$ has no poles there. Looking at the right hand side, we see that $\xi(2s+1) \neq 0$ for $\mathrm{Re}(s) = 0$. □

This is a fairly straightforward argument, but it gives strictly less information than the original proof of the prime number theorem by de la Vallée-Poussin, which gives a zero-free region of the form

$$\sigma > 1 - \frac{c}{\log(|t|+2)}$$

where $s = \sigma + it$ and $c > 0$ is a constant.

One can analyze the spectral decomposition further to get a zero-free region. This involves looking at the inner products of truncated Eisenstein series. A reference is Sarnak's article on his webpage[2].

Note that once we have the Fourier expansion of $E(z,s)$, the formal argument above will actually prove the functional equation.

4.4. **Application: Gauss class number problem.**

Theorem 4.5 (Deuring, Landau, Gronwall)**.** *There are finitely many imaginary quadratic fields with class number* 1*.*

Side remark. Landau and Gronwall knew that the claim follows from the Riemann hypothesis for $L(s, (\frac{-}{D}))$ for all fundamental discriminants $D < 0$.

Proof. Suppose the Riemann hypothesis is false, so there exists s_0 such that $\zeta(s_0 + \frac{1}{2}) = 0$ with $\mathrm{Re}(s_0) \neq 0$. Let $z_0 \in \mathbb{H}$ be a CM point, i.e., $az_0^2 + bz_0 + c = 0$ for some $a, b, c \in \mathbb{Z}$ with $b^2 - 4ac = D$. Then we have two facts:

- For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, Γ_{z_0} corresponds to the ideal class of $\left(a, \frac{b+\sqrt{D}}{2}\right)$ in $\mathbb{Q}(\sqrt{D})$.
- We can write

$$\begin{aligned} E(z_0, s) &= \sum_{u,v\in\mathbb{Z}}{}' \frac{\mathrm{Im}(z_0)^{s+\frac{1}{2}}}{|uz+v|^{2s+1}} \\ &= \left(\frac{\sqrt{|D|}}{2}\right)^{s+\frac{1}{2}} \sum{}' \frac{1}{|av^2 - buv + cv^2|^{s+\frac{1}{2}}} \\ &= \left(\frac{\sqrt{|D|}}{2}\right)^{s+\frac{1}{2}} \zeta_{z_0}\left(s + \frac{1}{2}\right) \end{aligned}$$

[2]Available at `http://web.math.princeton.edu/sarnak/ShalikaBday2002.pdf`.

where ζ_{z_0} is the zeta function for the ideal class corresponding to z_0. Recall that for any Galois extension $K/\mathbb{Q}$,

$$\zeta_{K/\mathbb{Q}}(s) = \sum_I \frac{1}{N(I)^s} = \sum_{c \in \mathrm{Cl}(K)} \sum_{\alpha \in K} \frac{1}{(\alpha c)^s}$$

and we define

$$\zeta_c(s) = \sum_{\alpha \in K} \frac{1}{(\alpha c)^s}.$$

Summing the above over $z_0 \in \Lambda_D$,

$$\sum_{z \in \Lambda_D} E(z,s) = \left(\frac{|D|^{\frac{1}{2}}}{2}\right)^{s+\frac{1}{2}} \zeta_{\mathbb{Q}(\sqrt{D})}\left(s + \frac{1}{2}\right).$$

Note that

$$\zeta_{\mathbb{Q}(\sqrt{D})}\left(s+\frac{1}{2}\right) = \zeta\left(s+\frac{1}{2}\right) L\left(s+\frac{1}{2}, \left(\frac{-}{D}\right)\right),$$

so evaluating everything at the hypothetical zero gives

$$\sum_{z \in \Lambda_D} E(z, s_0) = 0.$$

In particular, if the class number $h(D) = 1$, then this gives $E(z_D, s_0) = 0$, where $z_D = \frac{1+\sqrt{D}}{2}$.

By the Fourier expansion of $E(z,s)$,

$$E\left(\frac{1+\sqrt{D}}{2}, s_0\right) = \left(\frac{\sqrt{|D|}}{2}\right)^{s_0+\frac{1}{2}} + (*)\left(\frac{\sqrt{|D|}}{2}\right)^{\frac{1}{2}-s_0} + O\left(e^{-\frac{\sqrt{|D|}}{2}}\right).$$

This cannot be all 0 for large $|D|$ if s_0 is not on the line $\mathrm{Re}(s) = 0$. Therefore, this can only happen finitely many times, or the Riemann hypothesis is true in which case we can use the above argument. □

Of course, we were cheating in this proof because we only assumed the falsity of the Riemann hypothesis for ζ. We can indeed push the argument a bit further for the Riemann hypothesis for quadratic L-functions.

4.5. **Application: Volumes of fundamental domains.** Our last application will be quite general: the computation of the volumes of the fundamental domains. This was one of Langlands' first applications.

We want to calculate the volume of $\Gamma\backslash\mathbb{H}$, which is $\frac{\pi}{3}$. The idea is to calculate

$$\int_{\Gamma\backslash\mathbb{H}} E(z,s)\, dz$$

by shifting the contour and picking up the pole at $s = \frac{1}{2}$. But there are two problems:

(1) In general $E(z,s) \notin \mathcal{L}^1(\Gamma\backslash\mathbb{H})$ when it converges.
(2) This integral is 0 when $\mathrm{Re}(s) = 0$.

We are going to modify this idea and somehow make it work.

This is how we are going around. Take $f \in C_0^\infty(\mathbb{R}_{>0})$, and consider the series

$$\theta_f(z) = \sum_{\pm\Gamma_\infty\backslash\Gamma} f(\mathrm{Im}(\gamma z)).$$

Recall that the Mellin transform is given by

$$\tilde{f}(s) = \int_0^\infty f(y) y^s \, \frac{dy}{y}.$$

The Mellin inversion is

$$f(y) = \frac{1}{2\pi i} \int_{-\infty}^\infty \tilde{f}(s) y^{-s} \, ds.$$

The idea is to rewrite f as its own Mellin transform, which is a very general method in analytic number theory. The sum becomes the Eisenstein series

$$\theta_f(z) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=s_0} \tilde{f}(s) E\left(z, -s - \frac{1}{2}\right) \, ds$$

where $s_0 > 1$.

One can compute

$$\int_{\Gamma\backslash\mathbb{H}} \theta_f(z) \, dz$$

in two ways: by shifting the contour, and by unfolding. Shifting the contour gives

$$\int_{\Gamma\backslash\mathbb{H}} \theta_f(z) \, dz = \tilde{f}(-1) \cdot \frac{3}{\pi} \mathrm{vol}(\Gamma\backslash\mathbb{H}) + \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=-\frac{1}{2}} \tilde{f}(s) \int_{\Gamma\backslash\mathbb{H}} E\left(z, -s - \frac{1}{2}\right) \, d\mu(z) \, ds,$$

whereas unfolding gives

$$\int_{\Gamma\backslash\mathbb{H}} \theta_f(z) \, d\mu(z) = \int_{\pm\Gamma_\infty\backslash\mathbb{H}} f(\mathrm{Im}(\gamma z)) \, d\mu(z) = \int_0^\infty f(y) \, \frac{dy}{y^2} = \tilde{f}(-1).$$

So we have the following identity

$$\tilde{f}(-1) = \tilde{f}(-1) \frac{3}{\pi} \mathrm{vol}(\Gamma\backslash\mathbb{H}) + \frac{1}{2\pi i} \int_{s \in i\mathbb{R}} \tilde{f}\left(-s - \frac{1}{2}\right) \int_{\Gamma\backslash\mathbb{H}} E(z, s) \, d\mu(z) \, ds.$$

For $\mathrm{Re}(s_0) = 0$, the integral

$$\int_{\Gamma\backslash\mathbb{H}} E(z, s_0) \, d\mu(z)$$

is 0, so we get

$$\mathrm{vol}(\Gamma\backslash\mathbb{H}) = \frac{\pi}{3}.$$

Among the things we did this is the most general. Langlands calculated this for split Chevalley groups over $\mathbb{Q}$. At the end of the day, once we have an Eisenstein series, this will work for more general groups.

Next time we will leave the classical language and move to groups and the adelic language.

5. Lecture 5 (February 5, 2015)

5.1. **Convergence of averaging.** Let me comment on something way earlier. We obtained eigenfunctions φ by considering the Dirichlet problem on any domain D in the upper half plane $\mathbb{H}$. Assume that we can extend φ across the boundary of D. Then since Γ acts discontinuously, for a fixed g, γg only hits D finitely many times. So $\sum \varphi(\gamma g)$ converges.

5.2. **Reminder (last time).** Last time we were calculating the volume of the fundamental domain. We got the following equality

$$\tilde{f}(-1)\left(\frac{3}{\pi}\operatorname{vol} -1\right) = \frac{1}{2\pi i}\int_{\mathrm{Re}(t)=0} I(t)\tilde{f}\left(-t+\frac{1}{2}\right)\,dt,$$

where

$$I(t) = \int_{\Gamma\backslash\mathbb{H}} E(z,t)\,\frac{dx\,dy}{y^2}.$$

We had a lemma last time.

Lemma 5.1. *Suppose $I(t)$ is defined. Then it is identically zero.*

A heuristic reason is that the Eisenstein series is orthogonal to the constant function.

Proof. Consider

$$\frac{1}{2\pi i}\int_{\mathrm{Re}(t)=0} I(t)\tilde{f}\left(-t+\frac{1}{2}\right)\,dt = c\tilde{f}(-1)$$

for all $f \in C_c^\infty(\mathbb{R}_{\geq 0})$. We will show that if this is true, then $I(t)\equiv 0$.

We can shift the eigenvalue by taking $F(y) = y^{\frac{1}{2}}f(y)$. Then $\tilde{F}(s) = \tilde{f}(s+\frac{1}{2})$. So we can consider F in place of f:

$$\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=0}\tilde{F}(-s)I(s)\,ds = c\tilde{F}\left(-\frac{3}{2}\right).$$

Now let $G(y) = yF' + \frac{3}{2}F$. Then $\tilde{G}(s) = (s+\frac{3}{2})\tilde{F}(s)$. Since the above is true for all $\tilde{F}$, we have the same relation

$$\frac{1}{2\pi i}\int \tilde{G}(-s)I(s)\,ds = c\tilde{G}\left(-\frac{3}{2}\right)\equiv 0,$$

so

$$\frac{1}{2\pi i}\int \tilde{F}(-s)\left(-s+\frac{3}{2}\right)I(s)\,ds \equiv 0$$

for all F. This implies $I(s)=0$. ☐

Therefore $\operatorname{vol} = \frac{3}{\pi}$. This is a roundabout argument.

5.3. **Modular forms as Maass forms.** Today we will end our discussion on $\mathbb{H}$ and pass to G. We talked about modular forms and real-analytic Eisenstein series.

Remark. All modular forms (holomorphic of weight k) are Maass forms.

So far we have talked about Maass forms that are invariant, but we need to define them more generally.

Definition 5.2. A Maass form of weight k (for trivial Nebentypus) is $f : \mathbb{H}\to\mathbb{C}$ such that

- $f(\gamma z) = \left(\dfrac{cz+d}{|cz+d|}\right)^k f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$,
- $\Delta_k f(z) = s(1-s)f(z)$, where $\Delta_k = -y^2\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) + iky\dfrac{\partial}{\partial x}$ is the weight k Laplacian, and
- f is of moderate growth.

With this, modular forms become Maass forms.

Proposition 5.3. *If f is a weight k holomorphic form, then $F(z) = y^{\frac{k}{2}} f(z)$ is a Maass form of weight k and eigenvalue $\frac{k}{2}(1-\frac{k}{2})$.*

Proof. Exercise. □

This unifies the picture of modular forms and Maass forms.

5.4. **From $\mathbb{H}$ to group.** We defined these Eisenstein series on the upper half plane, which is a manifold. Recall that the upper half plane is a symmetric space

$$\mathbb{H} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2).$$

Instead of looking at $\mathbb{H}$, we want to look at functions on $\mathrm{SL}_2(\mathbb{R})$ and more generally on $\mathrm{GL}_2(\mathbb{R})$.

First we will go from a modular form f on $\mathbb{H}$ to a function on

$$\mathrm{GL}_2^+(\mathbb{R}) = \{g \in \mathrm{GL}_2(\mathbb{R}) \mid \det(g) > 0\}.$$

Let

$$j(g,z) = \frac{cz+d}{\sqrt{\det(g)}}$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $g \in \mathrm{SL}_2(\mathbb{Z})$, then $\det(g) = 1$. This function satisfies the following properties:

- $j(gh,z) = j(g,hz)j(h,z)$ (cocycle property). (Using this, we can define half-integral weight modular forms over any groups, for example over function fields. If we do the same calculation over anything that is not $\mathbb{R}$, the formula for $j(g,z)$ does not work but the cocycle condition is what generalizes. This turns out to be the Hilbert symbol.)
- $j(gk_\theta, i) = e^{i\theta} j(g,i)$ where $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}(2)$.

If f is a modular form of weight k, we construct

$$\phi_f(g) = j(g,i)^{-k} f(gi).$$

This lifts to the group G and is invariant modulo Γ, i.e., ϕ_f is a function on $\Gamma\backslash G$.

More generally, if f is a Maass form of weight k, we define the cocycle

$$j_{\mathrm{Maass}}(g,z) = \left(\frac{cz+d}{|cz+d|}\right)\frac{1}{\sqrt{\det(g)}}$$

then $\phi_f(g) = j_{\mathrm{Maass}}(g,i)^{-k} f(gi)$ is a function on the group.

5.5. **To adelic group.** Now we want to pass from a function ϕ on $\mathrm{GL}_2(\mathbb{Z})\backslash G(\mathbb{R})$ to Φ on $G(\mathbb{A})$.

Recall strong approximation. Suppose we have a group $G/\mathbb{Q}$ with S a finite set of places. The pair (G,S) satisfies strong approximation if

$$\prod_{v\in S} G(\mathbb{Q}_v) \times G(\mathbb{Q})$$

is dense in $G(\mathbb{A})$. In practice we usually take S to be the archimedean places. This essentially means we can do Chinese remainder theorem for the group G.

It is a fact that $(\mathrm{SL}(2),\{\infty\})$ has strong approximation. More generally, we have the

Theorem 5.4 (Kneser)**.** *If G is simply connected and $\prod_{v\in S} G(\mathbb{Q}_v)$ is not compact, then (G,S) satisfies strong approximation.*

But we will not need such generality.

Let $G = \mathrm{SL}_2$. Then $G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})K_f$, where we take $K_f = \prod_{v \text{ finite}} G(\mathbb{Z}_v)$. Note $G(\mathbb{Z}_v)$ is a maximal compact for each v. Thus $G(\mathbb{R}) \twoheadrightarrow G(\mathbb{Q})\backslash G(\mathbb{A})/K_f$ is onto.

Claim. $G(\mathbb{Z})\backslash G(\mathbb{R}) \leftrightarrow G(\mathbb{Q})\backslash G(\mathbb{A})/K_f$ is a one-one correspondence.

Proof. Suppose $g_\infty, g'_\infty \in G(\mathbb{R})$ map to the same point in the double coset. Then

$$g'_\infty = \gamma g_\infty k_f$$

where $\gamma \in G(\mathbb{Q})$ and $k_f \in K_f$. We can write $\gamma = \gamma_\infty \cdot \gamma_f$ for the diagonal embedding $G(\mathbb{Q}) \hookrightarrow G(\mathbb{A})$, i.e., $\gamma_\infty = (\gamma, 1, 1, \cdots)$ and $\gamma_f = (1, \gamma, \gamma, \cdots)$. We have

$$g'_\infty = \gamma g_\infty k_f = \gamma_\infty \gamma_f g_\infty k_f = \gamma_\infty g_\infty \gamma_f k_f.$$

Since g'_∞ and $\gamma_\infty g_\infty$ are 1 at all finite places, we must have $\gamma_f = k_f^{-1}$, so

$$\gamma_f \in \prod_{v \text{ finite}} \mathrm{SL}_2(\mathbb{Z}_v) \cap \mathrm{SL}_2(\mathbb{Q}) = \mathrm{SL}_2(\mathbb{R})$$

and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. ☐

We have already gone from $f : \Gamma\backslash\mathbb{H} \to \mathbb{C}$, to $\phi_f : \Gamma\backslash G(\mathbb{R}) \to \mathbb{C}$. Finally, we can go from ϕ_f to a function on $G(\mathbb{Q})\backslash G(\mathbb{A})/K_f$ by defining

$$F(g_{\mathbb{Q}} g_\infty k_f) = \phi_f(g_\infty).$$

The claim above shows that this is well-defined.

Remember the point of this course is to compute constant terms. What happens to them under this identification? We have

$$\int_0^1 f(x+iy)\,dx = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} \phi_f(ng)\,dn$$

where $N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in G(\mathbb{R}) \right\}$ is the standard unipotent radical, and $dn = dx$. The coordinates are

$$z = x + iy \longleftrightarrow g_z = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}$$

and $ng = \begin{pmatrix} \sqrt{y} & \frac{x+n}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}$.

5.6. **Eisenstein series (adelically).** Recall that

$$E(z,s) = \sum_{\substack{c \geq 0 \\ \gcd(c,d)=1}} \frac{y^{s+\frac{1}{2}}}{|cz+d|^{2s+1}} = \sum_{\pm\Gamma_\infty \backslash \operatorname{SL}_2(\mathbb{Z})} \operatorname{Im}(\gamma z)^{s+\frac{1}{2}}.$$

We lift this to

$$E^{\operatorname{SL}_2}(g,s) := E(gi,s).$$

We do not need to deal with the j-factors because $E(z,s)$ is already invariant. E^{SL_2} is defined on $\operatorname{SL}_2(\mathbb{Z}) \backslash \operatorname{SL}_2(\mathbb{R})$.

Remark. Diagonal matrices $\gamma = \begin{pmatrix} a & \\ & a \end{pmatrix}$ act trivially, so we can lift E^{SL_2} to functions on GL_2 that are invariant under the center as well.

To define the adelic Eisenstein series, let us review the Iwasawa decomposition. In general, $G(k) = N(k)A(k)K(k)$, where N is the unipotent radical, A is the diagonal matrices and K is a maximal compact. K looks significantly different for archimedean and non-archimedean places.

For $g \in \operatorname{GL}_2(\mathbb{R})$,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{\det(g)}{\sqrt{c^2+d^2}} & \frac{ac+bd}{\sqrt{c^2+d^2}} \\ 0 & \sqrt{c^2+d^2} \end{pmatrix} \begin{pmatrix} \frac{d}{\sqrt{c^2+d^2}} & \frac{-c}{\sqrt{c^2+d^2}} \\ \frac{c}{\sqrt{c^2+d^2}} & \frac{d}{\sqrt{c^2+d^2}} \end{pmatrix}$$

where the two matrices are in NA and K respectively. We will need this for calculations later.

Exercise. Prove this.

Then we define

$$E^{\operatorname{GL}_2}(g,s) = E(gi,s) = \sum_{\gamma \in P(\mathbb{Z}) \backslash \operatorname{GL}_2(\mathbb{Z})} |\operatorname{Im}(\gamma g i)|^{s+\frac{1}{2}}$$

where $P = NA = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in G \right\}$ is the parabolic. This can be expressed in terms of the function $\varphi_{s,\infty} : \operatorname{GL}_2(\mathbb{R}) \to \mathbb{C}$ defined by

$$g = g_P g_K \mapsto \left| \frac{\alpha_g}{\gamma_g} \right|^{s+\frac{1}{2}}$$

where $g_P = \begin{pmatrix} \alpha_g & \beta_g \\ 0 & \gamma_g \end{pmatrix}$.

Claim.

$$E(i,s) = \sum_{\gamma \in P(\mathbb{Z}) \backslash G(\mathbb{Z})} \varphi_{s,\infty}(\gamma).$$

Exercise. Prove this.

In general,

$$E^{\mathrm{GL}_2}(g,s) = \sum_{\gamma \in P(\mathbb{Z})\backslash G(\mathbb{Z})} \varphi_{s,\infty}(\gamma g).$$

Now we are ready to work adelically. We will need to define a function at each place. Let p be a prime and define $\varphi_{s,p} : G(\mathbb{Q}_p) \to \mathbb{C}$ by sending

$$g \mapsto \left|\frac{\alpha_g}{\beta_g}\right|_p^{s+\frac{1}{2}}$$

where $g = n_g a_g k_g$ with $a_g = \begin{pmatrix} \alpha_g & \\ & \beta_g \end{pmatrix}$. Finally, $\varphi_s : G(\mathbb{A}) \to \mathbb{C}$ is defined as

$$\varphi_s(g) = \varphi_{s,\infty}(g_\infty) \prod_p \varphi_{s,p}(g_p).$$

Definition 5.5.

$$E(g,\varphi_s) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \varphi_s(\gamma g).$$

We need to convince ourselves that $P(\mathbb{Q})\backslash G(\mathbb{Q}) \leftrightarrow P(\mathbb{Z})\backslash G(\mathbb{Z})$, which is believable by clearing denominators. We will do this next time.

Let me end by describing the constant terms. Given any $f : G(\mathbb{A}) \to \mathbb{C}$, its constant term along the parabolic P is

$$c_P(f,g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} f(ng)\, dn.$$

Proposition 5.6.

$$c_P(E(-,\varphi_s),g) = \varphi_s(g) + \frac{\xi(2s)}{\xi(2s+1)}\varphi_{-s}(g).$$

We will do this next time. This is analogous to the classical constant term

$$y^{s+\frac{1}{2}} + \frac{\xi(2s)}{\xi(2s+1)} y^{\frac{1}{2}-s}.$$

More generally, for any adelic characters $\mu_1, \mu_2 : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}$, we can define

$$\varphi_{\mu_1,\mu_2,s}\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} = \mu_1(\alpha)\mu_2(\beta)\left|\frac{\alpha}{\beta}\right|^{s+\frac{1}{2}}$$

and extend to all of $G(\mathbb{A})$. Then the Eisenstein series $E(g,\varphi_{\mu_1,\mu_2,s})$ has constant term

$$c_P(E(-,\varphi_{\mu_1,\mu_2,s}),g) = \varphi_{\mu_1,\mu_2,s}(g) + \frac{L^*(2s,\mu_1\mu_2^{-1})}{L^*(2s+1,\mu_1\mu_2^{-1})}\varphi_{\mu_2,\mu_1,-s}(g)$$

where L^* is the completed L-function.

6. Lecture 6 (February 10, 2015)

6.1. **Last time.** Last time we finally left the realm of classical forms. We started with $f : \mathbb{H} \to \mathbb{C}$, a Maass form or modular form on $\mathrm{SL}_2(\mathbb{Z})$, and associated to this $\phi_f : \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C}$ by multiplying with the cocycle and f evaluated at i. Using strong approximation, we cooked up a function $\phi : \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$.

We will almost never turn back to the upper half plane picture, unless we need some intuition. Instead we will focus on functions on the adelic group.

6.2. **Eisenstein series (adelic).** Let me start by recalling the Iwasawa decomposition

$$G = NAK.$$

(These decompositions are known before they were so named, at least for special groups.) We will talk about general reductive groups later, but for now, we have $G = \mathrm{GL}(2)$, $N = \left\{\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}\right\} \subseteq G$, $A = \left\{\begin{pmatrix} * & \\ & * \end{pmatrix}\right\} \subseteq G$ and K is a maximal compact. Over any ring, N and A stay the same form but K will change.

Over $\mathbb{R}$, we have

$$G(\mathbb{R}) = N(\mathbb{R})A(\mathbb{R})K(\mathbb{R})$$

where we take $K(\mathbb{R}) = \mathrm{SO}(2)$, not $\mathrm{O}(2)$ because we can change the determinant by A. We write

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = p_g k_g$$

where $p_g \in NA = P$ and $k_g \in K$. More explicitly,

$$g = \begin{pmatrix} \frac{\det(g)}{\sqrt{c^2+d^2}} & \frac{ac+bd}{\sqrt{c^2+d^2}} \\ 0 & \sqrt{c^2+d^2} \end{pmatrix} \begin{pmatrix} \frac{d}{\sqrt{c^2+d^2}} & \frac{-c}{\sqrt{c^2+d^2}} \\ \frac{c}{\sqrt{c^2+d^2}} & \frac{d}{\sqrt{c^2+d^2}} \end{pmatrix}.$$

We will use this in a second.

Exercise. Prove this.

There is an analogous decomposition over any p-adic field. Now we are looking at

$$G(\mathbb{Q}_p) = N(\mathbb{Q}_p)A(\mathbb{Q}_p)K(\mathbb{Q}_p)$$

where $K(\mathbb{Q}_p) = \mathrm{GL}_2(\mathbb{Z}_p)$. Multiplying by an integral matrix on the right corresponds to column operations, so the valuations of c and d play a role. Indeed,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = p_g k_g = \begin{cases} \begin{pmatrix} \frac{\det g}{d} & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{c}{d} & 1 \end{pmatrix} & \text{if } \frac{c}{d} \in \mathbb{Z}_p, \\[2ex] \begin{pmatrix} \frac{\det g}{c} & a \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & \frac{d}{c} \end{pmatrix} & \text{if } \frac{d}{c} \in \mathbb{Z}_p. \end{cases}$$

We will not need this, because this decomposition is not unique and we will start talking about functions that are invariant under the maximal compact.

Exercise. Prove this.

We will start with a function on A, extend to $P = NA$ trivially (note that A normalizes N), and induce to all of G.

We will adopt the following convention: p will denote a finite prime of $\mathbb{Q}$, and v will denote any place of $\mathbb{Q}$ (including ∞[3]).

Definition 6.1. Define $\varphi_s : G(\mathbb{A}) \to \mathbb{C}$ by

$$\varphi_s = \prod \varphi_{s,v}$$

where each $\varphi_{s,v} : G(\mathbb{Q}_v) \to \mathbb{C}$ is given by

$$\varphi_{s,v}(g_v) = \varphi_{s,v}(n_{g_v} a_{g_v} k_{g_v}) = \varphi_{s,v}(a_{g_v}) = \left| \frac{\alpha_{g_v}}{\beta_{g_v}} \right|_v^{s+\frac{1}{2}}$$

where $a_{g_v} = \begin{pmatrix} \alpha_{g_v} & 0 \\ 0 & \beta_{g_v} \end{pmatrix}$.

Definition 6.2.

$$E(g, \varphi_s) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} \varphi_s(\gamma g).$$

This is *ad hoc* notation, which will be improved when we start talking about general Eisenstein series.

Exercise. If we take $g = (g_{\infty,z}, 1, \cdots, 1, \cdots)$ where $g_{\infty,z} = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ & \frac{1}{\sqrt{y}} \end{pmatrix}$, then $E(z, s) = E(g_{\infty,z}, \varphi_{s,\infty})$.

6.3. **Constant term.** The general calculation will follow the same line, modulo a few complications. If we can break the group into double cosets modulo P, then we are in good shape. Recall the Bruhat decomposition

$$\mathrm{GL}_2 = B \sqcup B \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} N$$

where $B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subset \mathrm{GL}_2$.

Exercise. Prove this.

These are straightforward for GL_2. In general, this is essentially Gaussian elimination and we will have

$$G = \coprod_{w \in W} BwN$$

where W is the Weyl group, or even more generally

$$G = \coprod_{w \in W_P} PwN_P$$

for any group with a Tits system.

[3] John Conway likes to call this -1.

One consequence of the Bruhat decomposition is that

$$B\backslash \mathrm{GL}_2 \leftrightarrow \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \sqcup wN$$

where $w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. One needs to check that these are not equivalent under B.

Here B is a Borel, but can be replaced by any parabolic subgroup.

Given a function on any group, we can define its constant term along any parabolic. We can think of parabolics as ways of going to infinity. For GL_2, there is only one way. For GL_3, there are three ways. We can think of a manifold with pinches, where the pinches can have various dimensions as the dimension of the manifold goes higher.

Recall that

$$c_{P,E(-,\varphi_s)}(g) = \int_{N_B(\mathbb{Q})\backslash N_B(\mathbb{A})} E(ng, \varphi_s)\, dn.$$

Classically, the integral is computed over $N(\mathbb{Z})\backslash N(\mathbb{R})$. From now on the argument is straightforward:

$$\begin{aligned} c_{P,E(-,\varphi_s)}(g) &= \int_{N_B(\mathbb{Q})\backslash N_B(\mathbb{A})} \left(\sum_{B(\mathbb{Q})\backslash G(\mathbb{Q})} \varphi_s(\gamma n g) \right) dn \\ &= \int_{N_B(\mathbb{Q})\backslash N_B(\mathbb{A})} \left(\sum_{\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right) \sqcup w N_B(\mathbb{Q})} \varphi_s(\gamma n g) \right) dn \end{aligned}$$

by the Bruhat decomposition, where $w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. Dropping the dependence of N_B on B, we write

$$c_B(g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi_s(ng)\, dn + \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \sum_{\gamma \in wN(\mathbb{Q})} \varphi_s(\gamma n g)\, dn.$$

We will see that these two terms correspond to $y^{s+\frac{1}{2}}$ and $y^{\frac{1}{2}-s}$ in the classical constant term respectively.

The first integral is easy. Since $\varphi_s(ng) = \varphi_s(g)$ for all $n \in N$, we have

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi_s(ng)\, dn = \varphi_s(g) \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} dn.$$

We take the measure normalization $\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} dn = 1$. This is really the classical normalization as done by Tate: $N(\mathbb{A}) = \mathbb{A}$ and $N(\mathbb{Q}) = \mathbb{Q}$, so $\mathbb{Q}\backslash\mathbb{A} \simeq \mathbb{Z}\backslash\mathbb{R} \cdot \hat{\mathbb{Z}}$ with measures dx on $\mathbb{Z}\backslash\mathbb{R}$ and $d\mu$ on $\mathbb{Q}_p$ respectively, normalized such that $\mu(\mathbb{Z}_p) = 1$.

The second integral can be written as

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \sum_{n_0 \in N(\mathbb{Q})} \varphi_s(w n_0 n g)\, dn = \int_{N(\mathbb{A})} \varphi_s(wng)\, dn = \prod_v \int_{N(\mathbb{Q}_v)} \varphi_{s,v}(w n_v g_v)\, dn_v.$$

Again this corresponds to the classical computation

$$\sum_n \int_0^1 \frac{dx}{|x+n+iy|^c} = \int_{\mathbb{R}} \frac{dx}{|x+iy|^c}$$

by unfolding. Note we chose our function φ_s as a product function; not all adelic functions are product functions. Let me call

$$c_{B,v}(g) = \int_{N(\mathbb{Q}_v)} \varphi_{s,v}(wng_v)\, dn.$$

We are left with local computations, which are not very hard.

(1) At $v = \infty$,

$$\begin{aligned} c_{B,\infty}(g) &= \int_{N(\mathbb{R})} \varphi_{s,\infty}(wng_\infty)\, dn \\ &= \int_{\mathbb{R}} \varphi_{s,\infty}\left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} g_\infty\right) dn. \end{aligned}$$

Here is a simple calculation. Say $g_\infty = n_g a_g k_g$.

Claim.

$$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & \\ & B \end{pmatrix} = \begin{pmatrix} \frac{AB}{\sqrt{A^2+N^2B^2}} & \frac{-B^2N}{\sqrt{A^2+N^2B^2}} \\ 0 & \sqrt{A^2+N^2B^2} \end{pmatrix}$$

modulo SO(2) on the right.

Exercise. Prove this (using the Iwasawa decomposition).

This implies that

$$wng_\infty = \begin{pmatrix} \frac{\alpha_g\beta_g}{\sqrt{\alpha_g^2+(n+n_g)^2\beta_g^2}} & * \\ 0 & \sqrt{\alpha_g^2+(n+n_g)^2\beta_g^2} \end{pmatrix}$$

modulo SO(2), where $a_g = \begin{pmatrix} \alpha_g & \\ & \beta_g \end{pmatrix}$, so

$$\varphi_{s,\infty}(wng_\infty) = \left|\frac{\alpha_g\beta_g}{\alpha_g^2+(n_g+n)^2\beta_g^2}\right|_\infty^{s+\frac{1}{2}}$$

and the constant term is

$$c_{B,\infty}(g) = \int_{\mathbb{R}} \frac{|\alpha_g\beta_g|^{s+\frac{1}{2}}}{(\alpha_g^2+n^2\beta_g^2)^{s+\frac{1}{2}}}\, dn.$$

Under $n \mapsto \frac{\alpha_g}{\beta_g} n$, we get

$$\begin{aligned} c_{B,\infty}(g) &= |\alpha_g\beta_g|^{s+\frac{1}{2}}\left|\frac{\alpha_g}{\beta_g}\right| \int_{\mathbb{R}} \frac{dn}{(\alpha_g^2+n^2\alpha_g^2)^{s+\frac{1}{2}}} \\ &= \frac{|\beta_g^{s-\frac{1}{2}}\alpha_g^{s+\frac{3}{2}}|}{|\alpha_g^{2s+1}|} \int_{\mathbb{R}} \frac{dn}{(1+n^2)^{s+\frac{1}{2}}} \end{aligned}$$

$$= \left|\frac{\alpha_g}{\beta_g}\right|^{\frac{1}{2}-s} \int_{\mathbb{R}} \frac{dn}{(1+n^2)^{s+\frac{1}{2}}}$$
$$= \varphi_{-s,\infty}(g_\infty) \cdot \frac{\Gamma(\frac{1}{2})\Gamma(s)}{\Gamma(s+\frac{1}{2})}$$

which comes from a specialization of the beta function

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = 2\int_0^\infty \frac{v^{2a-1}}{(v^2+1)^{a+b}}\, dv.$$

We have seen that the trivial Bruhat cell gives φ_s, and the non-trivial one will give φ_{-s} with an intertwining operation.

(2) At $v = p$,

$$c_{B,p}(g) = \int_{N(\mathbb{Q}_p)} \varphi_{s,p}(wng_p)\, dn.$$

We want to get the Iwasawa decomposition of wng_p.

Claim.

$$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & N \\ & 1 \end{pmatrix}\begin{pmatrix} A & \\ & B \end{pmatrix} = \begin{cases} \begin{pmatrix} \frac{A}{N} & -B \\ 0 & NB \end{pmatrix} & \text{if } \dfrac{A}{NB} \in \mathbb{Z}_p, \\[2ex] \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} & \text{if } \dfrac{NB}{A} \in \mathbb{Z}_p, \end{cases}$$

modulo K_p on the right.

This implies, modulo K_p, that

$$wng_p = \begin{cases} \begin{pmatrix} \frac{\alpha_g}{n+n_g} & -\beta_g \\ 0 & (n+n_g)\beta_g \end{pmatrix} & \text{if } \dfrac{\alpha_g}{(n+n_g)\beta_g} \in \mathbb{Z}_p, \\[2ex] \begin{pmatrix} \beta_g & 0 \\ 0 & \alpha_g \end{pmatrix} & \text{if } \dfrac{(n+n_g)\beta_g}{\alpha_g} \in \mathbb{Z}_p, \end{cases}$$

so

$$\varphi_{s,p}(wng_p) = \begin{cases} \left|\dfrac{\alpha_g}{\beta_g(n+n_g)^2}\right|_p^{s+\frac{1}{2}} & \text{if } \dfrac{\alpha_g}{(n+n_g)\beta} \in \mathbb{Z}_p, \\ \left|\dfrac{\beta_g}{\alpha_g}\right|_p^{s+\frac{1}{2}} & \text{if } \dfrac{(n+n_g)\beta}{\alpha_g} \in \mathbb{Z}_p. \end{cases}$$

This implies

$$\int_{\mathbb{Q}_p} \varphi_{s,p}\left(w\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} g_p\right) dn$$
$$= \int_{v_p(n)\le v_p\left(\frac{\alpha_p}{\beta_p}\right)} \varphi_{s,p}\left(\begin{pmatrix} \frac{\alpha_g}{n} & \\ & n\beta_g \end{pmatrix}\right) dn + \int_{v_p(n)> v_p\left(\frac{\alpha_p}{\beta_p}\right)} \varphi_{s,p}\left(\begin{pmatrix} \beta_g & \\ & \alpha_g \end{pmatrix}\right) dn.$$

Recall that dn is such that $\int_{\mathbb{Z}_p} dn = 1$. The rest is an easy exercise, and we get

$$\left|\frac{\alpha_g}{\beta_g}\right|_p^{\frac{1}{2}-s} \left\{\sum_{k=0}^{\infty} \frac{p^k}{p^{2sk+k}}\left(1-\frac{1}{p}\right)+\frac{1}{p}\right\} = \left|\frac{\alpha_p}{\beta_p}\right|_p^{\frac{1}{2}-s} \frac{1-\frac{1}{p^{2s+1}}}{1-\frac{1}{p^{2s}}}.$$

This is a nice calculation essentially following Tate's thesis. The only thing to keep in mind is that $\int_{\mathbb{Z}_p^\times} dn = 1-\frac{1}{p}$. So the constant term at p is

$$c_{B,p}(g_p) = \varphi_{-s,p}(g_p)\cdot\frac{\zeta_p(2s)}{\zeta_p(2s+1)}.$$

Therefore, the constant term of $E(g,\varphi_s)$ is

$$c_{B,E(-,\varphi_s)}(g) = \varphi_s(g) + \frac{\xi(2s)}{\xi(2s+1)}\varphi_{-s}(g).$$

For $\mathrm{GL}(n)$, the Weyl group is S_n. If we try to do these integrals, for each Weyl element we will bring the 1's to the appropriate row and column using the Iwasawa decomposition.

This boring calculation above is illustrative. It brings out the intertwining operator between s and $-s$ clearly.

At the beginning of the lecture we defined $\varphi_s : G(\mathbb{A}) \to \mathbb{C}$. Let me now define a slightly more general φ_s. Let $\mu_1, \mu_2 : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ be two characters, with $\mu_1 = \bigotimes \mu_{1,v}$ and $\mu_2 = \bigotimes \mu_{2,v}$. Define

$$\varphi_{(\mu_1,\mu_2,s)}(g) = \prod_v \varphi_{(\mu_{1,v},\mu_{2,v},s)}(g_v)$$

where the functions $\varphi_{(\mu_{1,v},\mu_{2,v},s)}$ are again defined using the Iwasawa decompostion $g_v = n_v a_v k_v$.

For v where μ_1 and μ_2 are unramified, i.e., $\mu_1|_{\mathbb{Z}_v^\times} = \mu_2|_{\mathbb{Z}_v^\times} = 1$ (so they depend on the uniformizer only), we have

$$\varphi_{(\mu_{1,v},\mu_{2,v},s)}(g_v) = \mu_{1,v}(\alpha_{g_v})\mu_{2,v}(\beta_{g_v})\left|\frac{\alpha_{g_v}}{\beta_{g_v}}\right|_v^{s+\frac{1}{2}}$$

where $a_v = \begin{pmatrix}\alpha_{g_v} & \\ & \beta_{g_v}\end{pmatrix}$.

Exercise. Calculate

$$\int_{N(\mathbb{Q}_p)} \varphi_{(\mu_{1,p},\mu_{2,p},s)}(wng_p)\,dn = \varphi_{(\mu_{2,p},\mu_{1,p},-s)}(g_p)\cdot\frac{L_p(2s,\mu_1\mu_2^{-1})}{L_p(2s+1,\mu_1\mu_2^{-1})}.$$

In general, for the ramified places, we can consider an analogous decomposition where the maximal compact K is replaced by something smaller.

Next time I will start talking about the general theory of reductive groups.

7. Lecture 7 (February 12, 2015)

7.1. **Algebraic group theory.** Today I will go over things like roots, weights and the structure theory of groups. We will give examples as we go on. This is necessary because the Eisenstein series is defined by some data on the group. If you have not seen this before, the basic objects are characters and cocharacters. Let us fix a field F, which we assume is a finite extension of $\mathbb{Q}$.

There is a theory over F and $\overline{F}$, which can be connected using cohomological descent. We will study the theory over $\overline{F}$ only, i.e., focus on split groups, because that is the case considered in Langlands' book *Euler Products*. Eventually we may do something with quasi-split groups.

Definition 7.1. An algebraic group defined over F is an F-variety with multiplication and inversion F-morphisms.

Algebraic groups naturally split into two categories, namely abelian varieties and linear algebraic groups, but we will not talk about abelian varieties.

Definition 7.2. A linear algebraic group over F is a Zariski closed subgroup of $\mathrm{GL}(N)$.

From now on, whenever we say "algebraic group" it will mean "linear algebraic group".

Example 7.3.

(1) $G = \mathrm{GL}(N)$.
(2) $G = \mathrm{SL}(N)$.

Once we leave the realm of these two, we encounter the problem of representing these groups. We will need to fix a symplectic form, which we might change over time.

(3) $G = \mathrm{Sp}(2n) = \{g \in \mathrm{GL}(2n) \mid {}^t gJg = J\}$, where

$$J = \begin{pmatrix} &&&&&-1 \\ &&&&\iddots& \\ &&&-1&& \\ &&1&&& \\ &\iddots&&&& \\ 1&&&&& \end{pmatrix}.$$

(4) Let K/F be a separable extension, and H be an algebraic group over K. Then there exists an algebraic group $G = \mathrm{Res}_{K/F}(H)$ over F such that $G(F) = H(K)$. There is an actual construction of this using the coordinate ring, but I will tell you a more concrete example.

For example, let K/F be a quadratic extension, so $K = F(\sqrt{d})$ where $d \in F^\times \backslash (F^\times)^2$. Take $H = \mathbb{G}_m$. Then

$$\mathrm{Res}_{K/F}(H)(F) = \left\{ \begin{pmatrix} a & bd \\ b & a \end{pmatrix} \,\middle|\, a^2 - b^2 d \neq 0, a, b \in F \right\}.$$

All tori over quadratic étale algebras arise this way. Note $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) = \mathbb{S}$ is the Deligne torus, which is related to Hodge theory.

(5) Unitary group. For this example, let F/F^+ be a quadratic extension with conjugation c, and V be an n-dimensional vector space over F with a Hermitian pairing $h : V \times V \to \mathbb{C}$, i.e., $h(\alpha, \varphi\beta) = c(\varphi)h(\alpha, \beta)$. Then

$$U(V) = \{g \in \mathrm{GL}(V) \mid h(g\alpha, g\beta) = h(\alpha, \beta) \text{ for all } \alpha, \beta \in V\}.$$

Remark. Given any one of these defined over $\mathbb{Z}$, one can consider its points over any ring.

Example 7.4. $\mathrm{GL}_N(R) = \{g \in \mathrm{Mat}_n(R) \mid \det(g) \in R^\times\}$.

Now we will talk about the radical of a group, because we want to avoid the solvable part.

Definition 7.5 (Radical)**.** Let G be a connected algebraic group. The radical $R(G)$ of G is a maximal connected solvable normal subgroup of G.

Definition 7.6 (Unipotent radical)**.** The unipotent radical $R_u(G)$ is a maximal connected unipotent normal subgroup of G.

Example 7.7.

- For $G = \mathrm{GL}(N)$, $R(G) = Z^0(G)$ and $R_u(G) = 1$.
- For $G = \mathrm{SL}(N)$, $R(G) = 1$ and $R_u(G) = 1$.
- For B the Borel subgroup of $\mathrm{GL}(N)$ consisting of the upper triangular matrices, $R(B) = B$ and $R_u(B)$ is the set of upper triangular matrices with 1's on the diagonal.

Note $R_u(G)$ is always a subgroup of $R(G)$. We defined these in order to give the

Definition 7.8. An algebraic group G is semisimple if $R(G) = 1$, and reductive if $R_u(G) = 1$.

Thus GL_n is reductive but not semisimple, SL_n and PGL_n are semisimple, and B is neither reductive nor semisimple.

For any such groups, we have the Levi decomposition, which will be used all the time in the parabolic case. Let G be connected over F. Then there exists a reductive M such that

$$G = M \cdot R_u(G).$$

This is a semi-direct product, where M normalizes $R_u(G)$.

If G is reductive, then $G = R(G) \cdot G'$ where $G' = [G, G]$ is the derived group and we have that $R(G) \cap [G, G]$ is finite. For example, for $\mathrm{GL}(N)$, this is just the roots of unity.

Example 7.9 (Levi decomposition)**.** For the Borel, $B = M \cdot R_u(B)$ where M is the set of diagonal matrices and $R_u(B)$ is the set of upper triangular matrices with 1's on the diagonal.

Example 7.10. For $G = P_{2,3,1} \subseteq \mathrm{GL}(6)$ consisting of block upper triangular matrices with block sizes $(2, 3, 1)$, M is the block diagonal matrices and $R_u(G)$ is the subgroup of matrices with identity matrices on the block diagonal, i.e.,

$$\left(\begin{array}{cc|ccc|c} * & * & & * & & * \\ * & * & & & & \\ \hline & & * & * & * & \\ & & * & * & * & * \\ & & * & * & * & \\ \hline & & & & & * \end{array}\right) = \left(\begin{array}{cc|ccc|c} * & * & & & & \\ * & * & & & & \\ \hline & & * & * & * & \\ & & * & * & * & \\ & & * & * & * & \\ \hline & & & & & * \end{array}\right) \cdot \left(\begin{array}{cc|ccc|c} 1 & & & * & & * \\ & 1 & & & & \\ \hline & & 1 & & & \\ & & & 1 & & * \\ & & & & 1 & \\ \hline & & & & & 1 \end{array}\right).$$

Now comes the important piece: the torus. The structure theory goes through this.

Definition 7.11 (Algebraic torus)**.** An algebraic group T defined over F is called a torus if $T \simeq \mathbb{G}_m^k$ over $\overline{F}$.

Example 7.12.

- Diagonal matrices in $\mathrm{GL}(N)$.
- $\mathrm{Res}_{K/F}(\mathbb{G}_m)$.

Definition 7.13 (Split torus)**.** A torus is called split over F if $T \simeq \mathbb{G}_m^k$ over F.

I will say a couple of their properties in a second.

Definition 7.14 (Borel)**.** A Borel subgroup is a maximal connected solvable subgroup of G.

This may not exist over the base field, but always exists over $\mathbb{C}$.

Definition 7.15 (Quasi-split group)**.** A group is called quasi-split over F if the Borel is defined over F.

Everything I wrote down is quasi-split so far. The theory of automorphic forms goes into two parts: quasi-split groups, and the inner forms of the group. We will not talk about the inner forms. In order to define the Eisenstein series, we need the Borel.

Note that tori and Borel subgroups are not canonically defined, but they are all conjugate to each other. More precisely, we have the following properties:

(1) All Borels are F-conjugate. (More generally, all minimal parabolics are F-conjugate.)
(2) All maximal split tori are F-conjugate.

Definition 7.16 (Rank)**.** The F-rank of a group is the dimension of a (hence all) maximal split torus over F.

Once we have all these definitions, we will start talking about weights and roots attached to them. We linearize the theory by looking at the Lie algebra, which can be decomposed into subspaces depending on the characters of the maximal torus. Since they are nice vector spaces, we can calculate things with them instead of looking at the group itself.

7.2. **Roots, weights, etc.**

Definition 7.17 (F-character lattice)**.** $X_F^*(G) = \mathrm{Hom}_F(G, \mathbb{G}_m)$.

Example 7.18. Take $G = \mathbb{G}_m$. Then $X^*(G) \simeq \mathbb{Z}$ is given by $t \in \mathbb{G}_m(F) \mapsto t^k$.

Example 7.19. Take $K = \mathbb{C}$ and $F = \mathbb{R}$, and consider $G = \mathrm{Res}_{K/F}(\mathbb{G}_m)$. Then

$$G(\mathbb{R}) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \;\middle|\; a, b \in \mathbb{R}, a^2 + b^2 \neq 0 \right\} \simeq \mathbb{R}_+ \cdot \mathrm{SO}(2) \simeq \mathbb{C}^\times$$

and $G(\mathbb{C}) \simeq \mathbb{G}_m^2$. We have $X_\mathbb{R}^*(G) \simeq \mathbb{Z}$ and $X_\mathbb{C}^*(G) \simeq \mathbb{Z}^2$.

Definition 7.20 (F-anisotropic)**.** A torus is called F-anisotropic if $X_F^*(T) = \{1\}$.

Example 7.21. $\mathrm{SO}(2, \mathbb{R})$ is anisotropic, because all (non-trivial) characters are defined over $\mathbb{C}$.

Every torus has an anisotropic (compact) part and a split part.

Definition 7.22 (Cocharacter lattice)**.** $X_*(T) = \mathrm{Hom}_F(\mathbb{G}_m, T)$.

Example 7.23. $X_*(\mathbb{G}_m) \simeq \mathbb{Z}$, and more generally $X_*(\mathrm{GL}_n) \simeq \mathbb{Z}^n$ given by

$$\alpha \mapsto \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \alpha & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}.$$

Weyl group will lead to the classification of roots and weights.

Definition 7.24 (Weyl group)**.** Let T be a maximal F-split torus of G. Then

$$W_F(G) := N_G(T)/Z_G(T).$$

Every $s \in N_G(T)$ acts on T via $s \mapsto (w_s : t \mapsto sts^{-1})$.

Example 7.25.

- For $G = \mathrm{GL}(N)$, $W_F(G) = S_n$.
- For $G = \mathrm{Sp}(2n)$, $W_F(G) = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$.
- For $G = \mathrm{SO}(2n+1)$, $W_F(G) = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$.

For the rest of the class, I will give a classification of Chevalley groups. Langlands' book is on Chevalley groups, which are defined over $\mathbb{Z}$. They are split, and essentially one of the simplest groups one can consider, but also fairly general.

From now on, we fix the following notations:

- G will be a connected reductive group over F.
- $\mathrm{Lie}(G)$ is the Lie algebra of G. One can define it using derivations, or more canonically as
$$\mathrm{Lie}(G) = \ker\left(G(F[t]/t^2) \to G(F)\right).$$
- T is a maximal split torus of G. Then T acts on $\mathrm{Lie}(G)$ by $X \mapsto tXt^{-1}$.

We get

$$\mathrm{Lie}(G) = \mathfrak{g}_0^T \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha^T$$

where

$$\mathfrak{g}_\alpha^T = \{X \mid tXt^{-1} = \alpha(t)X\}$$

for $\alpha \in X^*(T)$.

Definition 7.26 (Roots)**.** The elements of Φ are called F-roots.

Example 7.27. Take $G = \mathrm{GL}(n)$ and T the torus consisting of all diagonal matrices. Let $\alpha_j : T \to \mathbb{C}^\times$ be $\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \mapsto a_j$. Then the roots are given by $e_{ij} = \alpha_i - \alpha_j$ for all $1 \leq i \neq j \leq n$, and $\mathfrak{g}_{e_{ij}}$ is the matrix with 1 at the (i,j)-entry and 0's elsewhere.

Example 7.28. Take $G = \mathrm{Sp}(4)$ and $T = \left\{ \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2^{-1} & \\ & & & t_1^{-1} \end{pmatrix} \right\}$. Define

$$\alpha_1 \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2^{-1} & \\ & & & t_1^{-1} \end{pmatrix} = t_1 \text{ and } \alpha_2 \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2^{-1} & \\ & & & t_1^{-1} \end{pmatrix} = t_2.$$

Then the roots are

$$\Phi = \{\pm(\alpha_1 \pm \alpha_2), \pm 2\alpha_1, \pm 2\alpha_2\}.$$

The Lie algebra of G is

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \,\middle|\, A = \begin{pmatrix} u & v \\ w & x \end{pmatrix}, D = \begin{pmatrix} -x & -v \\ -w & -u \end{pmatrix}, B = \begin{pmatrix} \alpha & \beta \\ \beta' & \alpha \end{pmatrix}, C = \begin{pmatrix} \gamma & \theta \\ \theta' & \gamma \end{pmatrix} \right\},$$

and we have

$$\mathfrak{g}_{\alpha_1 - \alpha_2} = \begin{pmatrix} 0 & v & & \\ 0 & 0 & & \\ & & 0 & -v \\ & & 0 & 0 \end{pmatrix}.$$

Note that $X_F^*(T)$ has a Weyl action. Take $\chi \in X_F^*(T)$. Then $\chi \circ w_s = \chi(s \bullet s^{-1})$. Weyl group moves the Weyl chambers around, so if we study one chamber we will know something about the whole space. Choosing a Weyl chamber means fixing a Borel subgroup.

There is a natural pairing between X^* and X_*. Recall that $\mathrm{Hom}_F(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$ by $\alpha \mapsto \alpha^k$. Define $\langle \cdot, \cdot \rangle : X^* \times X_* \to \mathbb{Z}$ by $\langle \alpha, \mu \rangle = k$, where $\alpha \circ \mu : \mathbb{G}_m \to \mathbb{G}_m$ so there exists k such that $\alpha \circ \mu(t) = t^k$, i.e.,

$$\alpha \circ \mu(t) = t^{\langle \alpha, \mu \rangle}.$$

Definition 7.29 (Coroots)**.** For $a \in \Phi$, let $a^\vee \in X_*(T)$ be such that $\langle \alpha, \alpha^\vee \rangle = 2$ (essentially).

Example 7.30. For $G = \mathrm{GL}(n)$, $e_{ij} = \alpha_i - \alpha_j$ sends $\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto \dfrac{t_i}{t_j}$. Then the dual $e_{ij}^\vee : \mathbb{G}_m \to T$ sends $t \mapsto \begin{pmatrix} \ddots & & & & \\ & t & & & \\ & & \ddots & & \\ & & & t^{-1} & \\ & & & & \ddots \end{pmatrix}$ (where t and t^{-1} appear on rows i and j respectively).

Example 7.31. For $\mathrm{Sp}(4)$, we have

$$(\alpha_1 - \alpha_2)^\vee : t \mapsto \begin{pmatrix} t & & & \\ & t^{-1} & & \\ & & t & \\ & & & t^{-1} \end{pmatrix}$$

and

$$(2\alpha_1)^\vee : t \mapsto \begin{pmatrix} 1 & & & \\ & t & & \\ & & t^{-1} & \\ & & & 1 \end{pmatrix}.$$

Definition 7.32 (Root system)**.** Let G be semisimple with maximal torus T. Then $(X^*(T), \Phi_T, W_T)$ is called a root system.

Definition 7.33 (Simple roots)**.** A root $\alpha \in \Phi$ is called simple if $\alpha \neq \beta + \gamma$ for $\beta, \gamma \in \Phi_+$. Denote the set of simple roots by Δ.

We are not going to define the positive roots Φ_+, but a down-to-earth way of thinking about them is that they give the Borel. For example, for $\mathrm{GL}(n)$ they are e_{ij} for $i \geq j$.

Theorem 7.34 (Chevalley)**.**

(1) *Given any abstract root system (X, Φ, W), there exists a connected semisimple G such that $X = X^*(G)$ and $\Phi = \Phi_G$.*

(2) *If (X_1, Φ) and (X_2, Φ) are such that $X_1 \hookrightarrow X_2$ fixing $\Phi \xrightarrow{\mathrm{id}} \Phi$, then there exists an isogeny $G_1 \xrightarrow{\phi} G_2$.*

Theorem 7.35 (Classification)**.** *There are four infinite families and five exceptional irreducible root systems (X, Φ, W):*

- $A_n = \mathrm{SL}(n+1), B_n = \mathrm{SO}(2n+1), C_n = \mathrm{Sp}(2n), D_n = \mathrm{SO}(2n)$*;*
- E_6, E_7, E_8, F_4, G_2*.*

We know all these groups. We will look at their parabolic subgroups and define the Eisenstein series.

8. Lecture 8 (February 17, 2015)

8.1. **Root systems.** Last time we had an overview of the theory of algebraic groups. To summarize, we classified the Chevalley groups over any fields.

We did not have time to define an abstract root system Φ:

- Φ is a finite set, and $V = \mathrm{Span}_{\mathbb{R}}(\Phi)$.
- For every $\alpha \in \Phi$, there exists a reflection s_α such that:
  - $s_\alpha(\Phi) \subseteq \Phi$.
  - s_α fixes a codimension 1 subspace.
- If $\alpha, \beta \in \Phi$, then $s_\alpha(\beta) - \beta \in \alpha\mathbb{Z}$. (So we can define the coroot $\alpha^\vee$ by $s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha$.)

We say that

- Φ is reduced if $\alpha \in \Phi$ and $n \cdot \alpha \in \Phi$ $(n \in \mathbb{Z})$ imply $n = \pm 1$.
- Φ is irreducible if $\Phi \neq \Phi_1 \perp \Phi_2$ with $\Phi_i \neq \emptyset$.

Theorem 8.1. *There is a one-to-one correspondence*

$$\{\textit{reduced irreducible root systems}\} \leftrightarrow \{A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2\},$$

where

- $A_n = \mathrm{SL}(n+1)$*,* $n \geq 1$*,*

- $B_n = \mathrm{SO}(2n+1)$, $n \geq 1$,
- $C_n = \mathrm{Sp}(2n)$, $n \geq 3$,
- $D_n = \mathrm{SO}(2n)$, $n \geq 4$.

Note that all of these can be realized by semisimple split groups (in fact by Chevalley groups). A very similar classification exists for reductive groups. Let G be a reductive group and T be a maximal torus.

Definition 8.2 (Root datum)**.** A root datum is $\Psi = (X, Y, \Phi, \Phi^\vee)$ where X and Y are free abelian groups of the same rank with a pairing $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{Z}$ (so that $X \cong \mathrm{Hom}(Y, \mathbb{Z})$ and $Y \cong \mathrm{Hom}(X, \mathbb{Z})$), Φ and $\Phi^\vee$ are finite subsets with a bijection $\Phi \leftrightarrow \Phi^\vee$, and there is an action of Weyl group.

This is an abstract root datum. For reductive groups, we will have

$$(X, Y, \Phi, \Phi^\vee) = \Psi(G, T) = (X^*, X_*, \Phi(G, T), \Phi(G, T)^\vee).$$

A root datum is reduced if $2\alpha \notin \Phi$ whenever $\alpha \in \Phi$.

Theorem 8.3 (Classification)**.** *Given $(X, Y, \Phi, \Phi^\vee)$, there exists a unique (up to isomorphism) connected reductive group G over $\overline{\mathbb{Q}}$ and maximal torus T such that $\Psi = \Psi(G, T)$.*

Langlands only considered split Chevalley groups in his book. In 1967–68, the theory of reductive groups was not very well understood. To talk about Eisenstein series, one needs to do parabolic induction and talk about points over $\mathbb{Z}_p$. These groups are all defined over $\mathbb{Z}$, so we can immediately go on to the calculation of constant terms without any linear algebraic group theory.

8.2. **Parabolic (sub)groups.** Informally, a parabolic subgroup is a subgroup that measures infinity. It is a measure of the obstruction to G being anisotropic (informally, compact). If P is a parabolic subgroup, then G/P will be compact (for example, look at the Iwasawa decomposition).

G will always be a connected reductive group.

Definition 8.4 (Borel subgroup)**.** A maximal connected solvable closed subgroup B of G is called a Borel subgroup.

Definition 8.5 (Parabolic subgroup)**.** A parabolic subgroup P is a closed subgroup containing a Borel. (More intrinsically, a parabolic subgroup is one for which G/P is projective.)

The standard example of a Borel is the upper triangular matrices, and a parabolic is any subgroup containing it.

The main properties of parabolic subgroups P are:

- P is connected.
- $N_G(P) = P$.
- If P is conjugate to P' with $P, P' \supset B$, then $P = P'$.
- Bruhat decomposition: $G = \coprod_{w \in W} BwB$.

All Borels are conjugate in $G(F)$ whenever they are defined over F. In a sense this says that Borels are intrinsic to infinity. The same is true for all minimal parabolics (which might not be the Borel, if the Borel is not defined).

It would take a whole course to prove these results, so instead I will put up some references. Some standard ones are:

- Humphreys, *Linear Algebraic Groups.*
- Springer, *Linear Algebraic Groups.*
- Borel, *Linear Algebraic Groups.*

Example 8.6.

- Let $G = \mathrm{GL}(n)$. We can take B to be the set of all upper triangular matrices
$$B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \in \mathrm{GL}(n) \right\}.$$
An example of a parabolic is
$$P = \left\{ \begin{pmatrix} \mathrm{GL}(n_1) & * \\ 0 & \mathrm{GL}(n_2) \end{pmatrix} \in \mathrm{GL}(n) \;\middle|\; n_1 + n_2 = n \right\}.$$
- Let $G = \mathrm{SL}(n)$, and B be the set of upper triangular matrices. Then
$$P = \left\{ \begin{pmatrix} \mathrm{GL}(n_1) & * \\ 0 & \mathrm{GL}(n_2) \end{pmatrix} \in \mathrm{SL}(n) \;\middle|\; n_1 + n_2 = n \right\}$$
is a parabolic. Even if we are studying semisimple groups, reductive groups show up naturally.

Parabolic subgroups are subsets of simple roots. One should think of each conjugacy class of parabolics as a boundary component of the group. We are interested in functions on symmetric spaces, such as G/K. In $\mathrm{GL}(2)$, there is only one way to go to infinity:
$$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$
In $\mathrm{GL}(3)$, there is more than one way. In order to decompose $\mathcal{L}^2(\Gamma\backslash G)$, we need to understand the behavior at infinity, so we want to know the constant terms, which are integrals over parabolics.

Simple roots are a subset $\Delta \subseteq \Phi$ such that:

- Δ spans Φ.
- For any $\beta \in \Phi$, we have $\beta = \sum_{\alpha\in\Delta} c_\alpha \alpha$ where c_α are either all ≥ 0 or all ≤ 0.

Given simple roots Δ, the positive roots are $\Phi^+ = \{\sum_{\alpha\in\Delta} c_\alpha\alpha \mid c_\alpha \geq 0\}$.

Let us look at the prime example of $\mathrm{GL}(n)$.

Example 8.7. Let $G = \mathrm{GL}(n)$. Let $\alpha_i : T \to \mathbb{C}^\times$ be the root $\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_i$ and $e_{ij} : T \to \mathbb{C}^\times$ be $\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto \dfrac{t_i}{t_j}$. (Eventually we will linearize everything and look at these in additive notations. It is sufficient to look at the tangent space (Lie algebra) because the fundamental group does not really matter to the behavior at infinity.) Then we can take
$$\Phi = \{e_{ij} \mid 1 \leq i \neq j \leq n\},$$

$$\Delta = \{e_{i,i+1} \mid 1 \le i \le n-1\},$$
$$\Phi^+ = \{e_{ij} \mid 1 \le i < j \le n\}.$$

We have the following facts:

(1) The choice of a Borel is the same thing as fixing Φ^+. Note that Φ^+ involves ordering the roots and looking at a positive cone. This is because the unipotent radical can be written as
$$R_u(B) = \prod_{\alpha \in \Phi^+} U_\alpha.$$
For $\mathrm{GL}(n)$, $U_{e_{ij}} = U_{ij}$ is the set of unipotent matrices with 1 at the (i,j)-th entry. Then
$$\prod_{e_{ij} \in \Phi^+} U_{ij} = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \right\}.$$

(1') Levi decomposition: $B = T \ltimes R_u(B)$, where the Levi component is a split torus T.

(1") Given a base Δ, the group generated by T and U_α ($\alpha \in \Phi^+$) is a Borel.

(2) The Borels containing a fixed T are in one-one correspondence with the set of bases of Φ.

There are a lot of choices in the theory. We started with a reductive group, with the choice of a torus. Then we choose a Borel. But once we have fixed these, we can start talking about things which are standard to those choices.

Definition 8.8 (Standard parabolics)**.** Fix a Borel (or a minimal parabolic) B. Any P containing B is called standard.

The structure of $\mathcal{L}^2(\Gamma\backslash G)$ is very inductive: it will be composed of pieces, each of which corresponds to a parabolic subgroup. This was understood in the 1960's by Gelfand and his troop. They did this for GL(2) and probably for SL(n). The Eisenstein series tells us that for each parabolic, there is a map into $\mathcal{L}^2$. These form the continuous spectrum, and the intertwining maps come from the Eisenstein series.

8.3. **Decompositions of parabolics.** Fix G, T, B, Δ as before.

8.3.1. *Levi decomposition.* Fix a parabolic $P \supseteq B$. Then
$$P = M_P N_P$$
where M_P is the Levi component and $N_P = R_u(P)$ is the unipotent radical. Moreover,

- M_P normalizes N_P.
- N_P is uniquely determined by P.
- M_P is unique up to conjugation by P.

The same thing works if B is replaced by any minimal parabolic.

8.3.2. *Langlands decomposition.*

Definition 8.9 (Split component)**.** A split component of G is a maximal split torus of the connected component of the center of G.

This is an annoying definition. For $\mathrm{GL}(n)$, the split component is $Z^0(G)$. For $\mathrm{SL}(n)$, the split component is trivial. The reason for looking at this is the following.

By the Levi decomposition $P = M_P N_P$. This can be further broken down into

$$P = M_P^1 A_P N_P$$

where $M_P^1 A_P = M_P$ and A_P is the split component of M_P.

Example 8.10. For $\mathrm{GL}_n(\mathbb{R})$, if P is the parabolic of block upper triangular matrices with block sizes $(n_1, \cdots, n_r)$, then

$$M_P = \begin{pmatrix} \mathrm{GL}_{n_1} & & & \\ & \mathrm{GL}_{n_2} & & \\ & & \ddots & \\ & & & \mathrm{GL}_{n_r} \end{pmatrix} \text{ and } N_P = \begin{pmatrix} I_{n_1} & & & * \\ & I_{n_2} & & \\ & & \ddots & \\ & & & I_{n_r} \end{pmatrix}$$

are the block diagonal matrices and block unipotent matrices respectively. In this case, we have

$$A_P = \begin{pmatrix} z_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & z_1 & & & & & & \\ & & & z_2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & z_2 & & & \\ & & & & & & \ddots & & \\ & & & & & & & z_r & \\ & & & & & & & & \ddots \\ & & & & & & & & & z_r \end{pmatrix} \text{ and } M_P^1 = \begin{pmatrix} g_1 & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_r \end{pmatrix}$$

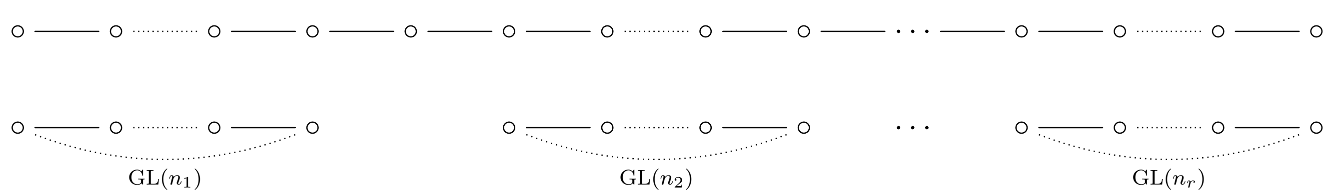
where $z_i > 0$ and $g_i \in \mathrm{GL}(n_i)$ with $\det(g_i) = \pm 1$.

8.3.3. *Explicit construction of the Langlands decomposition.* Fix G, T, B. For each $P \supseteq B$, there exists $\Delta_P \subseteq \Delta$ such that

$$P = G(\Sigma_{\Delta_0^P}) T_{\Delta_0^P} U_{\Delta_0^P} = M_P^1 A_P N_P$$

where $\Delta_0^P = \Delta \backslash \Delta_P$. This notation is in accordance with Arthur's article on the trace formula.

Example 8.11. Consider the Dynkin diagram of A_{n-1} corresponding to $\mathrm{GL}(n)$. Each parabolic is a block of $\mathrm{GL}(n_i)$'s, and we are removing the roots between $\mathrm{GL}(n_i)$ and $\mathrm{GL}(n_{i+1})$.



In general, Δ_P is the set of roots that we are removing from the Dynkin diagram. We set

$$\Sigma_{\Delta_0^P} = \{ n\alpha \mid \alpha \in \Delta_0^P, n \in \mathbb{Z} \} \cap \Phi$$

and

$$\Sigma^+_{\Delta_0^P} = \{n\alpha \mid \alpha \in \Delta_0^P, n \in \mathbb{Z}\} \cap \Phi^+.$$

Then the Langlands decomposition can be constructed as:

- $A_P = \left(\bigcap_{\alpha \in \Delta_0^P} \ker(\alpha) \right)^0.$
- $G(\Sigma_{\Delta_0^P})$ is the analytic subgroup over $\mathbb{R}$ corresponding to $\mathfrak{m}_1$ in the Lie algebra decomposition
$$\mathfrak{m} = \mathfrak{a} \oplus \mathfrak{m}_1$$
where $\mathfrak{a}$ is the Lie algebra of A_P.
- $U_{\Delta_0^P} = \prod_{\alpha \in \Phi_P} U_\alpha$, where $\Phi_P = \Phi^+ \backslash \Sigma^+_{\Delta_0^P}$.

They satisfy the following properties:

(1) M_P is the centralizer of A_P.
(2) $G(\Sigma_{\Delta_0^P})^0 = [M_P, M_P]^0$.
(3) $A_P \cap G(\Sigma_{\Delta_0^P})$ is finite.

Example 8.12. Let $G = \mathrm{GL}(n)$, T be the diagonal matrices, B be the upper triangular matrices, $\Phi = \{e_{ij} \mid 1 \leq i \neq j \leq n\}$, and $\Delta = \{e_{i,i+1} \mid 1 \leq i \leq n-1\}$. If we take $\Delta_P = \Delta$, then $\Delta_0^P = \emptyset$, i.e., we are taking out all the roots. Then

$$A_P = T^0,$$

$$M_P^1 = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$$

with each $* \in \{\pm 1\}$, and

$$N_P = \prod_{\alpha \in \Phi^+} U_\alpha = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \right\}.$$

9. Lecture 9 (February 19, 2015)

9.1. **Last time.** Last time we talked about parabolics, which correspond to subsets of the simple roots.

Let G be a connected reductive group, T be a maximal torus and B a Borel (minimal parabolic), Φ be the roots and Δ be the simple roots. There is a one-one correspondence

$$\{\text{standard parabolics}\} \leftrightarrow \{\text{subsets } \Delta_P \subseteq \Delta\}.$$

The correspondence goes like this. For Δ_P, we set $\Delta_0^P = \Delta \backslash \Delta_P$ and define

$$\Sigma^+_{\Delta_0^P} = (\mathbb{Z}\text{-span of } \Delta_0^P) \cap \Phi^+.$$

Then

$$A_P = \left(\bigcap_{\alpha \in \Delta_0^P} \ker(\alpha) \right)^0, N_P = \prod_{\Phi^+ \setminus \Sigma^+_{\Delta_0^P}} U_\alpha.$$

We have also defined M_P^1.

For $\mathrm{GL}(N)$, we have the following picture:

Dynkin diagram	Δ_P	P
$\emptyset$	Δ	B
○ ⋯⋯ ○ $\alpha_{n_1}-\alpha_{n_1+1}$ ○ ⋯⋯ ○ $\cdots \alpha_{n_r}-\alpha_{n_r+1}$ ○ ⋯⋯ ○	$\{\alpha_{n_1} - \alpha_{n_1+1}, \cdots, \alpha_{n_r} - \alpha_{n_r+1}\}$	$P_{n_1,\cdots,n_r}$
○ ⋯⋯ ○ $\alpha_i - \alpha_{i+1}$ ○ ⋯⋯ ○	$\{\alpha_i - \alpha_{i+1}\}$	$P_{i,n-i}$
○ —— ○ —— ○ ⋯⋯ ○ —— ○ —— ○	$\emptyset$	G

Example 9.1. $\mathrm{Sp}(4) = \{g \in \mathrm{GL}(4) \mid {}^t g J_0 g = J_0\}$ where $J_0 = \begin{pmatrix} 0 & -J \\ J & 0 \end{pmatrix}$ and $J = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. More explicitly, we can write its Lie algebra as

$$\mathfrak{sp}(4) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| A = \begin{pmatrix} u & v \\ w & x \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{11} \end{pmatrix}, C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{11} \end{pmatrix}, D = \begin{pmatrix} -x & -v \\ -w & -u \end{pmatrix} \right\}$$

and we choose

$$\mathfrak{t} = \begin{pmatrix} u & & & \\ & x & & \\ & & -x & \\ & & & -u \end{pmatrix}.$$

The roots are

$$\Phi = \{\pm(\alpha_1 \pm \alpha_2), \pm 2\alpha_1, \pm 2\alpha_2\},$$

and some of the root spaces (corresponding to the unipotent subgroups U_α) are:

α	$\mathfrak{g}_\alpha$
$\alpha_1 - \alpha_2$	$\begin{pmatrix} 0 & v & & \\ 0 & 0 & & \\ & & 0 & -v \\ & & 0 & 0 \end{pmatrix}$
$\alpha_2 - \alpha_1$	$\begin{pmatrix} 0 & 0 & & \\ w & 0 & & \\ & & 0 & 0 \\ & & -w & 0 \end{pmatrix}$
$\alpha_1 + \alpha_2$	$\begin{pmatrix} & & b & 0 \\ & & 0 & b \\ & & & \\ & & & \end{pmatrix}$
$2\alpha_1$	$\begin{pmatrix} & & 0 & b \\ & & 0 & 0 \\ & & & \\ & & & \end{pmatrix}$
$2\alpha_2$	$\begin{pmatrix} & & 0 & 0 \\ & & b & 0 \\ & & & \\ & & & \end{pmatrix}$

Let B be the upper triangular matrices in $G = \mathrm{Sp}(4)$, corresponding to the simple roots $\Delta = \{\alpha_1 - \alpha_2, 2\alpha_2\}$. Then the standard parabolics are:

Δ_0^P	Δ_P	P
$\emptyset$	Δ	B
$\{2\alpha_2\}$	$\{\alpha_1 - \alpha_2\}$	$\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{pmatrix}$
$\{\alpha_1 - \alpha_2\}$	$\{2\alpha_2\}$	$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}$ (Siegel parabolic)
Δ	$\emptyset$	$G = \mathrm{Sp}(4)$

When one learns Lie theory, the root spaces are usually one-dimensional, but this is not always the case.

Example 9.2 (Non-quasisplit)**.** Fix a base field F and consider the quadratic form

$$Q_F = x_1 x_n + x_2 x_{n-1} + \cdots + x_q x_{n-q+1} + Q_0(x_{q+1}, \cdots, x_{n-q}),$$

where Q_0 is anisotropic. The matrix is

$$\begin{pmatrix} &&&&&&1\\ &&&&&\cdot^{\cdot^{\cdot}}&\\ &&&&1&&\\ &&&Q_0&&&\\ &&1&&&&\\ &\cdot^{\cdot^{\cdot}}&&&&&\\ 1&&&&&& \end{pmatrix}.$$

A maximal F-split torus of $\mathrm{SO}(Q)$ is

$$T = \begin{pmatrix} t_1 &&&&&&&& \\ & \ddots &&&&&&& \\ && t_q &&&&&& \\ &&& 1 &&&&& \\ &&&& \ddots &&&& \\ &&&&& 1 &&& \\ &&&&&& t_q^{-1} && \\ &&&&&&& \ddots & \\ &&&&&&&& t_1^{-1} \end{pmatrix}.$$

The Levi component is the centralizer of the maximal torus:

$$C_G(T) = \begin{pmatrix} * &&&&&& \\ & \ddots &&&&& \\ && * &&&& \\ &&& \mathrm{SO} &&& \\ &&&& * && \\ &&&&& \ddots & \\ &&&&&& * \end{pmatrix}.$$

The minimal parabolic is

$$P_{\min} = \begin{pmatrix} * &&&& * \\ & \ddots &&& \\ && \square && \\ &&& \ddots & \\ &&&& * \end{pmatrix}.$$

Note that the Borel is not defined over F.

Remark. Root spaces are not always 1-dimensional. In this example, they are either 1-dimensional or $(n-2q)$-dimensional.

9.2. **Character spaces.** Now I will start defining spaces which will be some linearization of groups, like the Lie algebra. The notations will be heavy.

From now on (G, B) (and all related data) will be fixed. Let P be a standard parabolic, with Levi decomposition

$$P = M_P N_P = A_P M_P^1 N_P.$$

Recall the characters $X^*(G) = \mathrm{Hom}(G, \mathbb{G}_m)$.

Definition 9.3.

$$\begin{aligned} \mathfrak{a}_P^* &= X^*(M_P) \otimes \mathbb{R}, & \mathfrak{a}_{P,\mathbb{C}}^* &= \mathfrak{a}_P^* \otimes_{\mathbb{R}} \mathbb{C}, \\ \mathfrak{a}_P &= \mathrm{Hom}(X^*(M_P)_{\mathbb{Q}}, \mathbb{R}), & \mathfrak{a}_{P,\mathbb{C}} &= \mathfrak{a}_P \otimes_{\mathbb{R}} \mathbb{C}. \end{aligned}$$

Example 9.4. Take $P = G = \mathrm{GL}(N)$. Then $X^*(\mathrm{GL}(N)) = \langle \det \rangle$, so $\mathfrak{a}_P^* \simeq \mathbb{R}$.

Example 9.5. Take $P = P_{n_1,n_2,n_3}$. Then $X^*(M_P) = \langle \det(n_1), \det(n_2), \det(n_3) \rangle$, so $\mathfrak{a}_P^* \simeq \mathbb{R}^3$.

Remark. $X^*(A_P) \supseteq X^*(P)$ as $\mathbb{Z}$-modules, but are not necessarily equal. These are equal for the Borel, but that is essentially the only case.

Example 9.6. Take $P = G = \mathrm{GL}(N)$. Then

$$A_P = \begin{pmatrix} t & & \\ & \ddots & \\ & & t \end{pmatrix} \text{ and } X^*(A_P) = \left\langle \begin{pmatrix} t & & \\ & \ddots & \\ & & t \end{pmatrix} \mapsto t \right\rangle,$$

whereas $X^*(\mathrm{GL}(N)) = \langle \det \rangle$. So the index is $[X^*(A_P) : X^*(\mathrm{GL}(N))] = n$, but $X^*(G)_{\mathbb{Q}} = X^*(A_G)_{\mathbb{Q}}$.

As a final observation, if $P_1 \subseteq P_2$, then $\mathfrak{a}_{P_2}^* \hookrightarrow \mathfrak{a}_{P_1}^*$ and $\mathfrak{a}_{P_2} \hookrightarrow \mathfrak{a}_{P_1}$. Here is an example of the first inclusion.

Example 9.7. Take $P_1 = P_{3,2,1} \subseteq \mathrm{GL}(6)$ and $P_2 = P_{3,3}$. Then

$$\mathfrak{a}_{P_1}^* = \langle \det(n_1), \det(n_2), \det(n_3) \rangle \simeq \mathbb{R}^3$$

and

$$\mathfrak{a}_{P_2}^* = \langle \det(n_1'), \det(n_2') \rangle \simeq \mathbb{R}^2.$$

The inclusion map $\mathfrak{a}_{P_2}^* \hookrightarrow \mathfrak{a}_{P_1}^*$ is induced by

$$(x, y) \mapsto (x, y, y),$$

by looking at the restriction map.

9.3. **The H_p function.** Let F_v be a local field, and P be a standard parabolic, with Levi decomposition $P = M_P N_P = M_P^1 A_P N_P$. Recall that $\mathfrak{a}_P = \mathrm{Hom}(X^*(M_P)_{\mathbb{Q}}, \mathbb{R})$.

We will first bring out the intuitive idea, and then define by pairing.

Definition 9.8. $H_{M_P,v} : M_P(F_v) \to \mathfrak{a}_P$ is given by $m \mapsto (\phi_m : X^*(M_P) \to \mathbb{R})$, where $\phi_m : \chi \mapsto \log |\chi(m)|$, i.e.,

$$\exp\langle \chi, H_{M_P,v}(m) \rangle = |\chi(m)|_v.$$

This measures the "complexity" of an element. For SL(2), this measures how high an element is on the upper half plane, i.e., how close it is to the cusp.

Definition 9.9. $H_{P,v} : P(F_v) \to \mathfrak{a}_P$ is defined by extending $H_{M_P,v}$ trivially on N_P, i.e., for $p = mn$, we have $H_{P,v}(p) = H_{M_P,v}(m)$.

Definition 9.10. $H_P : P(\mathbb{A}) \to \mathfrak{a}_P$ is given by

$$H_P(m) = \prod H_{P,v}(m_v).$$

This is well-defined and trivial on the $\mathbb{Q}$-points $P(\mathbb{Q})$.

Example 9.11. Take $G = \mathrm{GL}(n)$ and $P = P_{n_1,\cdots,n_p} \supseteq B$, corresponding to $\mathfrak{a}_P$ and $\mathfrak{a}_0 := \mathfrak{a}_B$. As before, $\mathfrak{a}_P \hookrightarrow \mathfrak{a}_0$ is given by

$$(t_1, \cdots, t_p) \mapsto (t_1, \cdots, t_1, t_2, \cdots, t_2, \cdots, t_p, \cdots, t_p)$$

where t_i appears n_i times. Then

$$X^*(M_{P,v})_{\mathbb{Q}} = \langle |\det(m_1)|_v, \cdots, |\det(m_p)|_v \rangle$$

and

$$X^*(A_{P,v})_{\mathbb{Q}} = \langle |\det(m_1)|_v^{1/n_1}, \cdots, |\det(m_p)|_v^{1/n_p} \rangle.$$

We are using multiplicative notations just because we are thinking in terms of groups. To view these as linear spaces, one should take the logarithm of everything. Take as basis for $\mathfrak{a}_P$

$$\left\{ \frac{\log|\det(m_i)|}{n_i} \right\}.$$

Call $|\det(m_i)| = \chi_i$. Then

$$\exp\langle \chi_i, H_p(m) \rangle = |\det(m_i)|_v^{1/n_i}$$

and hence

$$H_{P,v}(m) = \left(\frac{\log|\det(m_1)|_v}{n_1}, \frac{\log|\det(m_2)|_v}{n_2}, \cdots, \frac{\log|\det(m_p)|_v}{n_p} \right).$$

In particular,

$$H_{P,v}(a) = (\log|a_1|_v, \log|a_2|_v, \cdots, \log|a_p|_v).$$

The above normalizations are chosen so that this is nice.

This is the example one should keep in mind about H_P. Let me give another non-trivial example: the Siegel case.

Example 9.12. Consider $\mathrm{Sp}(4)$. Let $P = \begin{pmatrix} * & * & * & * \\ & \boxed{\begin{matrix} * & * \\ * & * \end{matrix}} & & * \\ & & & * \\ & & & * \end{pmatrix}$ (i.e., the intersection of $P_{1,2,1} \subseteq \mathrm{GL}(4)$ and $\mathrm{Sp}(4)$), with $\Delta_P = \{\alpha_1 - \alpha_2\}$. Then

$$H_P(m) = (\log|a_1|)$$

where $m = \begin{pmatrix} a_1 & & & \\ & * & * & \\ & * & * & \\ & & & a_1^{-1} \end{pmatrix}$.

Very classically, if we take $G = \mathrm{SL}_2(\mathbb{R})$ and $P = B$, then $M_P = \left\{ \begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix} \right\}$ and $H_P(m) = \log \alpha$. For $z = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ & 1/\sqrt{y} \end{pmatrix}$, $H_P(z) = \log y$. This tells us how high z is on the upper half plane.

Definition 9.13. Let $P = MAN$. Define Φ_P to be the roots of P with respect to A, i.e., A acts on the unipotent radical with

$$\mathfrak{n}_P = \bigoplus_{\alpha \in \Phi_P} \mathfrak{n}_\alpha,$$

where

$$\mathfrak{n}_\alpha = \{X \in \mathfrak{n}_P \mid \mathrm{Ad}(a)X = \alpha(a)X \text{ for all } a \in A\}.$$

Example 9.14. Take $G = \mathrm{GL}(9)$, $P = P_{3,2,3,1}$. The adjoint action of A looks like

$$\begin{pmatrix} 1 & t_1/t_2 & t_1/t_3 & t_1/t_4 \\ & * & t_2/t_3 & t_2/t_4 \\ & & * & t_3/t_4 \\ & & & * \end{pmatrix}.$$

The dimensions of the eigenspaces are:

	6	9	3
		6	2
			3

Fix (G, B) and the standard parabolic P.

Definition 9.15 (Half-sum of positive roots)**.**

$$\rho_P = \frac{1}{2} \sum_{\alpha \in \Phi_P} (\dim(\mathfrak{n}_\alpha))\alpha.$$

Example 9.16. Take $P_{3,2,3,1}$ as in Example 9.14. Then

$$\rho_{P_{3,2,3,1}} = \frac{1}{2}(18\alpha_1 + 2\alpha_2 - 12\alpha_3 - 8\alpha_4).$$

Example 9.17. Take $G = \mathrm{GL}(n)$ and $P = B$. Then

$$\rho_B = \sum_{i=1}^{n} \left(\frac{n-1}{2} - i \right) \alpha_i.$$

This is related to representation theory over local fields: these numbers are the exponents in

$$|\cdot|^{\frac{n-1}{2}} \otimes \cdots \otimes |\cdot|^{\frac{1-n}{2}}$$

in normalized induction.

In general, the left and right Haar measures on P are related by

$$d_\ell p = e^{2\rho_P(H_P(p))}\, d_r p.$$

10. Lecture 10 (February 24, 2015)

10.1. **Last time.** Let G be a quasi-split connected reductive group G over F and B be a Borel (if G is not quasi-split, take B as a minimal parabolic). Take a standard parabolic $B \subset P \subset G$. We defined

$$\mathfrak{a}_P^* = X^*(M_P) \otimes_{\mathbb{Z}} \mathbb{R}$$

and

$$\mathfrak{a}_P = \mathrm{Hom}(X^*(M_P)_{\mathbb{Q}}, \mathbb{R}).$$

Suppose we have a function on a non-compact domain, and we want to show some analytic property or behavior at infinity of this function. We can define the Eisenstein series as functions of intertwining operators. As operators, they are only defined on some domain and we want them extended to some bigger domain. These $\mathfrak{a}_P$ and $\mathfrak{a}_P^*$ are gadgets for studying how the Eisenstein series behave at infinity.

We defined the function $H_P : G \to \mathfrak{a}_P$ by

$$g = pk \mapsto \log(\chi \mapsto \chi(p)),$$

and so

$$\exp\langle \chi, H_P(g)\rangle = |\chi(p_g)|.$$

This will tell us how high g is, i.e. how close g is to infinity. Of course one can take the Hilbert–Schmidt norm $|g|^2 = \mathrm{Tr}(g\,{}^t g^*)$. Recall the Cartan decomposition $G = KAK$. This norm is left- and right-invariant under K, so the size is essentially measured by A. This is essentially what $\mathfrak{a}_p$ and $\mathfrak{a}_p^*$ do, and we will explain more on this next time.

Here is a correction from last time:

$$d_\ell p = e^{2\rho_p}\, d_r p \tag{3}$$

where

$$\rho_p = \frac{1}{2} \sum_{\alpha \in \Phi_P} m_\alpha \alpha$$

and $m_\alpha = \dim(\mathfrak{n}_\alpha)$ is the dimension of the eigenspace of $\mathfrak{n}_P$ corresponding to α. As the parabolic grows, the eigenspace also grows: for the parabolic $P_{3,3,2}$, the eigenspaces for t_1/t_2, t_1/t_3 and t_2/t_3 have dimensions 9, 6, 6 respectively:

	9	6
		6

For our purpose, 3 will be the crucial relation.

10.2. **More on parabolics.** Today we will define the Eisenstein series, but before that let me say a few words about the general Bruhat decomposition.

Recall that classically, we have

$$G = \coprod_{w \in W} BwB.$$

This is a disjoint union of Bruhat cells. We can imagine that this is useful if we are working with functions defined over B and induced up. But we will define functions on P and then induce, so we need a more general decomposition.

Let G be connected reductive over F, P_0 be a minimal parabolic, n_W be a set of representatives of $W_F = N_G(T_F)/C_G(T_F)$ in $N(T_F)$, where T_F is the maximal F-split torus.

(1) The Bruhat decomposition is

$$G_F = \coprod_{w\in W_F} P_0 w P_0 = \coprod_{w\in W_F} N_{P_0} w P_0.$$

(2) More generally, let
- P_i be a standard parabolic,
- $\Delta_{P_i} \subseteq \Delta$ be the roots that we take out of the Dynkin diagram to get P_i,
- $\Delta_0^{P_i} = \Delta \backslash \Delta_{P_i}$, and
- $W_{\Delta_0^{P_i}} = \langle S_\alpha \mid \alpha \in \Delta_0^{P_i} \rangle$.

Then the general Bruhat decomposition is

$$P_2 \backslash G / P_1 = W_{\Delta_0^{P_2}} \backslash W / W_{\Delta_0^{P_1}}. \tag{4}$$

Let us compare this with $P_i = B$. In this case $\Delta_B = \Delta$, so $\Delta_0^B = \emptyset$ which implies $W_{\Delta_0^B} = \{1\}$. Thus $B\backslash G/B \leftrightarrow W$. (4) will be useful for constant term computations.

In a second we will define functions associated to each parabolic P. There are many parabolics and maps between them, so we want the functions to have the same kind of symmetries the parabolics have. The Weyl sets will be useful in studying symmetries of Eisenstein series.

Definition 10.1 (Weyl sets)**.** Let P_1, P_2 be parabolics. Then

$$W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_2}) = \left\{ \sigma : \mathfrak{a}_{P_1} \to \mathfrak{a}_{P_2} \;\middle|\; \begin{array}{l} \sigma \text{ is a linear isomorphism, obtained by restricting elements} \\ \text{for the Weyl group } W(G,B) \end{array} \right\}.$$

Example 10.2. Take $G = \mathrm{GL}(n)$, $P_1 = (n_{1,1}, \cdots, n_{1,t})$ and $P_2 = (n_{2,1}, \cdots, n_{2,s})$. Then

$$W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_2}) = \begin{cases} \emptyset & \text{if } t \neq s, \\ \{\sigma \in S_{t=s} \mid n_{1,j} = n_{2,\sigma(j)}\} & \text{if } t = s. \end{cases}$$

For example, if $P_1 = (1,2,1)$ and $P_2 = (1,1,2)$, then $W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_2}) = \{(2\,3), (1\,2\,3)\} \subset S_3$.

Definition 10.3 (Associated parabolics)**.** P_1 and P_2 are associated if $W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_2}) \neq \emptyset$.

We are cutting down parabolics into equivalence classes using this definition. The reason behind this is that Eisenstein series for associated parabolics contain the same amount of (spectral) information.

Example 10.4. $P_{3,2,1}$ and $P_{1,2,3}$ are associated, but $P_{3,2,1}$ and $P_{5,1}$ are not.

Definition 10.5 (Opposite parabolics)**.** P_1 and P_2 are called opposite if $P_1 \cap P_2$ is the Levi subgroup of each other.

Example 10.6. $P_1 = \begin{pmatrix} \mathrm{GL}(2) & * \\ & \mathrm{GL}(2) \end{pmatrix}$ and $P_2 = \begin{pmatrix} \mathrm{GL}(2) & \\ * & \mathrm{GL}(2) \end{pmatrix}$ are opposite.

10.3. **Parabolic induction.** Let G, T, B over F be as before, P be a standard parabolic. Recall

$$\mathfrak{a}_P^* = X^*(M_P) \otimes_{\mathbb{Z}} \mathbb{R},$$
$$\mathfrak{a}_P = \mathrm{Hom}(X^*(M_P)_{\mathbb{Q}}, \mathbb{R}),$$

and we complexify them to get

$$\mathfrak{a}^*_{P,\mathbb{C}} = \mathfrak{a}^*_P \otimes_{\mathbb{R}} \mathbb{C},$$
$$\mathfrak{a}_{P,\mathbb{C}} = \mathfrak{a}_P \otimes_{\mathbb{R}} \mathbb{C}.$$

For example, if $P = G = \mathrm{GL}(n)$, then $\mathfrak{a}_P$ and $\mathfrak{a}^*_P$ are both isomorphic to $\mathbb{R}$, and $\mathfrak{a}_{P,\mathbb{C}}$ and $\mathfrak{a}^*_{P,\mathbb{C}}$ are both isomorphic to $\mathbb{C}$. We should think of the isomorphism as given by $s \mapsto |\det(g)|^s$.

Before we define parabolic induction, here is some motivation. $G(\mathbb{A})$ acts on $\mathcal{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ by the right regular representation R_G. We want to see how it decomposes. For each equivalence class of parabolics, we get an intertwining operator.

$$\mathcal{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A})) = \bigoplus_P \int_{\mathfrak{a}^*_P} \mathcal{L}^2_{\mathrm{disc}}(M_P(\mathbb{Q})\backslash M_P(\mathbb{A}))\, d\mu.$$

Note the Eisenstein series are not in $\mathcal{L}^2$! This is the same kind of statement as saying

$$\mathcal{L}^2(\mathbb{R}) = \int_{i\mathbb{R}} e(z).$$

Let F be a global field and $\mathbb{A} = \mathbb{A}_F$ be its ring of adeles.

Definition 10.7 (Parabolic induction)**.** Let $P \subseteq G$ be a parabolic, $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$. Then

$$\mathcal{I}_P(\lambda, \cdot) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(R_{M^1_P,\mathrm{disc},\lambda} \otimes 1_{N_p})$$

where $\cdot \in G(\mathbb{A})$.

Here $R_{M^1_P,\mathrm{disc}}$ is the action of $M^1_P(\mathbb{A})$ on $\mathcal{L}^2_{\mathrm{disc}}(M_P(\mathbb{Q})\backslash M^1_P(\mathbb{A}))$ by right multiplication (note $M_P(\mathbb{Q}) = M^1_P(\mathbb{Q})$ by the product formula), and

$$R_{M^1_P,\mathrm{disc},\lambda}(\cdot) = R_{M^1_P,\mathrm{disc}}(\cdot) \otimes \exp\langle \lambda, H_P(\cdot)\rangle$$

for $\cdot \in M^1_P(\mathbb{A})$. Here is an example to illustrate this in human language.

Example 10.8. Let $P = P_{3,3,2}$. Then

$$M_P = \begin{pmatrix} \mathrm{GL}(3) & & \\ & \mathrm{GL}(3) & \\ & & \mathrm{GL}(2) \end{pmatrix}$$

and

$$M^1_P = \left\{ \begin{pmatrix} \boxed{m_1} & & \\ & \boxed{m_2} & \\ & & \boxed{m_3} \end{pmatrix} \;\middle|\; \det(m_i) = \pm 1 \right\}.$$

For any $m \in M^1_P$ as above and $\lambda = (s_1, s_2, s_3) \in \mathfrak{a}^*_{P,\mathbb{C}}$, we have

$$R_{M^1_P,\mathrm{disc},\lambda}(m) = R_{M_P,\mathrm{disc}}(m)|\det(m_1)|^{s_1}|\det(m_2)|^{s_2}|\det(m_3)|^{s_3}.$$

Remark.

- Note that we are embedding λ as part of the induction, rather than part of the space $\mathcal{L}^2$.
- $R_{M^1_P,\mathrm{disc},\lambda}$ is unitary if and only if λ is completely imaginary.
- In general, $M^1_P = \{m \in M_P \mid H_p(m) = 0\}$.

More explicitly, parabolic induction is as follows.

- The space of $\mathcal{I}_P(\lambda, y)$, denoted V_P (by Shahidi) or $\mathcal{H}_P$ (by Arthur), is the space of functions
$$\phi : N_P(\mathbb{A})M_P(F)A_P^0(\mathbb{R})\backslash G(\mathbb{A}) \to \mathbb{C}$$
such that for any $x \in G(\mathbb{A})$, the function $\phi_x : M_P(F)\backslash M_P^1(\mathbb{A}) \to \mathbb{C}$ given by
$$\phi_x(m) = \phi(mx)$$
is in $\mathcal{L}^2(M_P(F)\backslash M_P^1(\mathbb{A}))$.
- The action of $y \in G(\mathbb{A})$ is given by
$$\left(\mathcal{I}_P(\lambda, y)\phi\right)(g) = \phi(gy)\exp\langle \lambda + \rho, H_P(gy)\rangle \exp\langle -(\lambda+\rho), H_P(g)\rangle.$$

This is the confusing part. The first two terms are fairly okay. The last term is not that trivial. In a sense, when we take $y = 1$, we expect this action to be trivial on ϕ. This will take some convincing.

The whole point is that we have a bunch of parabolics P_i. For each λ_i we get an $\mathcal{I}_{P_i}(\lambda_i, \cdot)$. $\mathfrak{a}^*_{P_i,\mathbb{C}}$. The information is part of the induction, but not the space.

10.4. **Eisenstein series.**

Definition 10.9 (Eisenstein series)**.** Let $g \in G(\mathbb{A})$, $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ and $\phi \in \mathcal{H}_P = V_P$. Then
$$E(g, \phi, \lambda) := \sum_{\delta \in P(F)\backslash G(F)} \phi(\delta g)\exp\langle \lambda + \rho, H_p(\delta g)\rangle.$$

In the next lecture I will talk about how this is an intertwining operator. Basically we will see that
$$E(g, \mathcal{I}_P(\lambda, y)\phi, \lambda) = R_G(y)E(g, \phi, \lambda).$$

Example 10.10. Let $G = \mathrm{SL}_2$ and $P = B = \begin{pmatrix} \alpha_1 & * \\ & \alpha_2 \end{pmatrix}$. Then $\mathfrak{a}^*_P = \mathfrak{a}^*_B \simeq \mathbb{R}$ with coordinate $t \mapsto |\frac{\alpha_1}{\alpha_2}|^t$. Take $\lambda = it \in \mathfrak{a}^*_{P,\mathbb{C}} = \mathbb{C}^*$. Let $\mu_{it} : B \to \mathbb{C}^\times$ be
$$\begin{pmatrix} \alpha_1 & * \\ & \alpha_2 \end{pmatrix} \mapsto \left|\frac{\alpha_1}{\alpha_2}\right|^{it}$$
and $\phi_{it} : G(\mathbb{A}) \to \mathbb{C}$ be extended trivially to $G = P \cdot K$.

For $\alpha \in \mathfrak{a}^*_P$, we have
$$\exp\langle \alpha, H_B(p \cdot k)\rangle = |\alpha(p)|.$$
and $\rho_B = \frac{1}{2}$. Finally,
$$E(g, \phi_{it}, \nu_s) = \sum_{\delta \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \left|\frac{a_1(\delta g)}{a_2(\delta g)}\right|^{it+s+\frac{1}{2}} = E(g, s+it)$$
where $\nu_s \in \mathfrak{a}^*_{P,\mathbb{C}}$ is $\alpha \mapsto |\frac{a_1(\alpha)}{a_2(\alpha)}|^s$.

We defined the Eisenstein series, but we don't know if it makes sense. It only converges when λ is regular. In the example, this means $\mathrm{Re}(s) > \frac{1}{2}$.

Remark (by Jacquet). The space of parabolic induction can be viewed as sections of a fiber bundle over G/P. There are different ways to trivialize the sections. Over K this gives our formula.

11. Lecture 11 (February 26, 2015)

11.1. **Last time.** Let G be a connected reductive group over a number field F, B be a fixed Borel with roots Δ, and $P \supseteq B$ be a standard parabolic. Recall

$$\mathfrak{a}_{P,\mathbb{C}}^* = X^*(M_P) \otimes_{\mathbb{Z}} \mathbb{C}.$$

We defined the parabolic induction

$$\mathrm{Ind}_P^G(R_{M_P,\mathrm{disc},\lambda} \otimes 1_{N_P}) = \mathcal{I}_P(\lambda, \cdot)$$

for $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$ and $\cdot \in G(\mathbb{A})$, whose underlying space is given by $\mathcal{H}_P$, the set of functions

$$\phi : N_P(\mathbb{A})M_P(F)A_P^0(\mathbb{R})\backslash G(\mathbb{A}) \to \mathbb{C}$$

such that for all $x \in G(\mathbb{A})$, the function $\phi_x : M_P^1(\mathbb{A}) \to \mathbb{C}$ defined by $m \mapsto \phi(mx)$ is in $\mathcal{L}^2_{\mathrm{disc}}(M_p(F)\backslash M_P^1(\mathbb{A}))$.

For function fields, we need to modify by some valuation issue to take care of degrees.

This space is simple, so the complication lies in the action:

$$(\mathcal{I}_P(\lambda, g)\phi)(x) = \phi(xg) \exp\langle \lambda + \rho_P, H_p(xg)\rangle \exp\langle -(\lambda + \rho_P), H_P(x)\rangle.$$

For $g \in G(\mathbb{A})$, $\phi \in \mathcal{H}_P$ and $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$, the Eisenstein series is

$$E(g, \phi, \lambda) = \sum_{\delta \in P(F)\backslash G(F)} \phi(\delta g) \exp\langle \lambda + \rho, H_P(\delta g)\rangle.$$

Today I will do the following:

(1) give concrete examples;
(2) interpret the Eisenstein series as intertwining operators;
(3) decompose $\mathcal{L}^2$;
(4) discuss convergence in terms of ϕ and λ.

11.2. **A concrete example.** Let $G = \mathrm{SL}_2$ and $P = B = \left\{\begin{pmatrix} * & * \\ & * \end{pmatrix}\right\}$. Then $A_B = \left\{\begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix}\right\}$ with characters $\alpha \mapsto |\alpha|_{\mathbb{A}}^n$, so $\mathfrak{a}_B^* \simeq \mathbb{R}$ with 1 corresponding to $\alpha \mapsto |\alpha|_{\mathbb{A}}$ and $\mathfrak{a}_{B,\mathbb{C}}^* \simeq \mathbb{C}$ with s corresponding to $\alpha \mapsto |\alpha|_{\mathbb{A}}^s$. The half sum of positive roots is $\rho_B = 1 \in \mathfrak{a}_{B,\mathbb{C}}^*$.

Take:

- ϕ a constant function (which is in $\mathcal{H}_P$ because of the M_P^1!),
- $\lambda_s = s \in \mathfrak{a}_{B,\mathbb{C}}^*$,
- $g \in \mathrm{SL}_2(\mathbb{A})$.

Then

$$E(g, \phi_0, \lambda_s) = \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \exp\langle \lambda_s + \rho_B, H_B(\delta g)\rangle = \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} |a(\delta g)|^{s+1} = E(g, s)$$

(where $a(g)$ is defined by $g = namk$) is the classical Eisenstein series. $E(g, s)$ converges for $\mathrm{Re}(s) > 1$.

11.3. **Intertwining.** What do these Eisenstein series do? Let us forget about convergence issues for now, and consider formally

$$\begin{aligned} E(g, \mathcal{I}_P(\lambda, y)\phi, \lambda) &= \sum_{\delta \in P(F)\backslash G(F)} (\mathcal{I}_P(\lambda, y)\phi)(\delta g) \cdot \exp\langle \lambda + \rho_P, H_P(\delta g)\rangle \\ &= \sum_{\delta \in P(F)\backslash G(F)} \phi(\delta g y) \cdot \exp\langle \lambda + \rho_P, H_P(\delta g y)\rangle \\ &= E(gy, \rho, \lambda) = R_G(y) E(g, \phi, \lambda). \end{aligned}$$

We see that the Eisenstein series are intertwining operators between spaces.

11.4. **Decomposition of $\mathcal{L}^2(G(F)\backslash G(\mathbb{A}))$.** The basic idea is the following. G has parabolics, each of which has a parabolic rank, and there are equivalence classes of associated parabolics. For each class, we get a spectrum of $\mathcal{L}^2$. From all classes, we build up the spectrum of the whole space. The Eisenstein series serve as intertwining operators between chunks of the spectrum.

Definition 11.1 (Intertwining operators)**.** Suppose P and P' are associated, i.e., $W(\mathfrak{a}_P^*, \mathfrak{a}_{P'}^*) \neq \emptyset$, and let $s \in W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ with representative $w_s \in N_{A_P}/C_{A_P}$. For $\lambda \in \mathfrak{a}_{P,\mathbb{C}}$, we define

$$M(s, \lambda) : \mathcal{H}_P \to \mathcal{H}_{P'}$$

by

$$(M(s,\lambda)\phi)(x) = \int_{N_{P'}(\mathbb{A}) \cap w_s N_P(\mathbb{A}) w_s^{-1} \backslash N_{P'}(\mathbb{A})} \phi(w_s^{-1} n x) \exp\langle \lambda + \rho_P, H_P(w_s^{-1} n x)\rangle \exp\langle -s\lambda + \rho_P, H_{P'}(x)\rangle \, dn.$$

This formula is not very important. This intertwines spaces associated to associated parabolics.

Definition 11.2 (Positive cone)**.**

$$(\mathfrak{a}_P^*)^+ = \{\Lambda \in \mathfrak{a}_P^* \mid \Lambda(\alpha^\vee) > 0 \text{ for all } \alpha \in \Delta_P\}.$$

Recall that for any root α, we get a coroot $\alpha^\vee$. For example, if $\alpha : \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \mapsto t^2$, then $\alpha^\vee : t \mapsto \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$.

Theorem 11.3 (Langlands)**.** *Let $\phi \in \mathcal{H}_P^0$ (the space of K-finite vectors), and $\lambda \in \{\lambda \in \mathfrak{a}_{P,\mathbb{C}}^* \mid \mathrm{Re}(\lambda) \in \rho_P + (\mathfrak{a}_P^*)^+\}$. Then both $E(g, \phi, \lambda)$ and $M(s, \lambda)$ converge absolutely and define analytic functions of λ.*

We will try to prove the first part, at least in many cases. Here is the huge theorem.

Theorem 11.4 (Langlands)**.**

1. *$E(g, \phi, \lambda)$ and $M(s, \lambda)$ can be meromorphically continued in λ.*
2. *The Eisenstein series satisfies the functional equation*

$$E(x, M(s,\lambda)\phi, s\lambda) = E(x, \phi, \lambda).$$

3. *$M(ts, \lambda) = M(t, s\lambda) M(s, \lambda)$ for all $t \in W(\mathfrak{a}_{P'}, \mathfrak{a}_{P''})$.*

(4) *(Decomposition) For each class $\mathcal{P}$ of associated parabolics, i.e., a maximal set of the form*

$$\mathcal{P} = \{P_1, \cdots, P_r \mid W(\mathfrak{a}_{P_i}^*, \mathfrak{a}_{P_j}^*) \neq \emptyset\},$$

define $\hat{\mathcal{L}}_{\mathcal{P}}$ to be the space of families of functions

$$F = \{F_P : i\mathfrak{a}_P^* \to \mathcal{H}_P \text{ for each } P \in \mathcal{P}\}$$

such that

$$F_{P'}(s\lambda) = M(s, \lambda) F_P(\lambda)$$

for all $s \in W(\mathfrak{a}_P^*, \mathfrak{a}_{P'}^*)$, and the $\mathcal{L}^2$-norm

$$||F||^2 := \sum_{P \in \mathcal{P}} ||F_P||^2 := \sum_{P \in \mathcal{P}} \frac{1}{n_P} \int_{i\mathfrak{a}_P^*} |F_P(\lambda)|^2 \, d\lambda < \infty,$$

where

$$n_P = \sum_{P_1 \sim P} \frac{1}{|W(\mathfrak{a}_P^*, \mathfrak{a}_{P_1}^*)|}.$$

Then

$$F \mapsto \sum_{P \in \mathcal{P}} \frac{1}{n_P} \int_{i\mathfrak{a}_P^*} E(\cdot, F_P(\lambda), \lambda) \, d\lambda$$

maps $\hat{\mathcal{L}}_{\mathcal{P}}$ unitarily onto a closed subspace of $\mathcal{L}^2(G(F)\backslash G(\mathbb{A}))$, and

$$\mathcal{L}^2(G(F)\backslash G(\mathbb{A})) = \bigoplus_{\mathcal{P}} \hat{\mathcal{L}}_{\mathcal{P}}.$$

This is a huge theorem in the following sense. We would like these functions to be defined when λ is purely imaginary, which is when they are unitary. The first conclusion in (4) gives essentially the Fourier inversion formula. Recall every $f \in \mathcal{L}^2(\mathbb{R})$ can be written as

$$f(x) = \int_{i\mathbb{R}} \hat{f}(\lambda) e(-\lambda x) \, d\lambda,$$

where

$$\hat{f}(\lambda) = \int_{\mathbb{R}} f(x) e(\lambda x) \, dx.$$

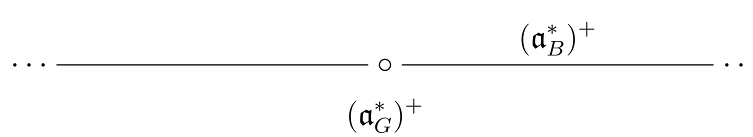
Be careful! G is itself a parabolic, so the decomposition is really

$$\bigoplus_{\mathcal{P} \neq G} \hat{\mathcal{L}}_{\mathcal{P}} \oplus \text{Discrete spectrum of } G.$$

This decomposition doesn't say anything about the discrete spectrum.

11.5. **Drawings.**

11.5.1. $G = \mathrm{SL}_2$. Let us consider $(\mathfrak{a}_P^*)^+$ for the parabolics G and B. Under the identification $\mathfrak{a}_B^* \simeq \mathbb{R}$, $(\mathfrak{a}_B^*)^+$ is the positive real line and $(\mathfrak{a}_G^*)^+$ is the origin.



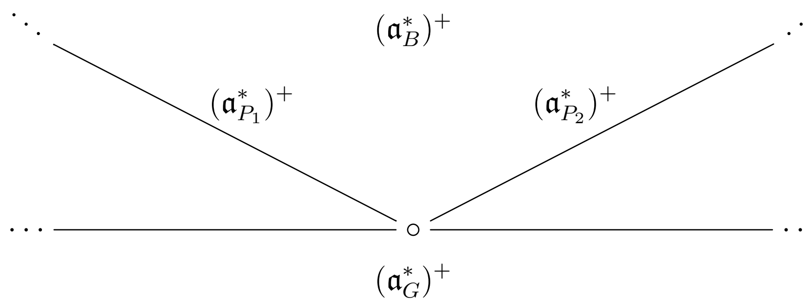
11.5.2. $G = \mathrm{SL}_3$. Look at the parabolics G, P_1, P_2, B. Let α_{12} be the root

$$\alpha_{12}\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} = \frac{t_1}{t_2}$$

and similarly for α_{23}. Then

$$\begin{aligned} \Delta &= \{\alpha_{12}, \alpha_{23}\}, \\ \Delta_{P_1} &= \{\alpha_{12}\}, \\ \Delta_{P_2} &= \{\alpha_{23}\}. \end{aligned}$$

We can identify $\mathfrak{a}_B^* \simeq \mathbb{R}^2 \hookrightarrow \mathbb{R}^3$ as the plane $\{(x, y, z) \mid x + y + z = 0\}$, and $\mathfrak{a}_{P_i}^* \simeq \mathbb{R}$. On this plane, $(\mathfrak{a}_B^*)^+$ is a fan with an obtuse angle. The two edges are $(\mathfrak{a}_{P_i}^*)^+$ and the vertex is $(\mathfrak{a}_G^*)^+$.



The interesting thing is that as the parabolic gets bigger, the picture goes smaller. It looks like $\mathfrak{a}_B$ is giving the main contribution, but it is the opposite. All the way down, the origin gives the most information. This diagram contains a lot of information, except that the picture is "flipped".

The theory of Eisenstein series reduces the study of the spectrum to the study of discrete spectrum for smaller Levis.

12. Lecture 12 (March 3, 2015)

Last time I described the spectral decomposition, and promised to prove the convergence of Eisenstein series today. Before that, we will go over the integration formulas and reduction theory.

12.1. **Integration.** Let G be a connected algebraic group.

Definition 12.1 (Haar measure)**.** A non-zero Borel measure on G that is left (*resp.* right) translational invariant is called a left (*resp.* right) Haar measure, i.e., a left Haar measure satisfies

$$d_\ell \mu(g \cdot) = d_\ell \mu(\cdot)$$

for all $g \in G$, and similarly for right.

Theorem 12.2. *Let G be locally compact and Hausdorff. Then there exists a unique Haar measure up to constant.*

Example 12.3.

- Finite groups: the discrete measure.
- $\mathbb{Q}_p$: $d\mu$ which gives measure 1 to $\mathbb{Z}_p$.
- Borel $B \subset \mathrm{GL}_2(\mathbb{R})$:
$$\begin{pmatrix} a & b \\ & c \end{pmatrix} \rightsquigarrow \frac{1}{|a|} d^\times a \, d^\times c \, db$$
is a left-invariant measure.
- In general, Borel $B \subseteq \mathrm{GL}(n)$:
$$d_\ell g = \frac{\prod_{1 \le i \le j \le n} dx_{ij}}{\prod_{k=1}^n x_{kk}^{n-k+1}} = \frac{1}{\det(g)^{n+1}} \frac{\prod dx_{ij}}{\prod x_{kk}^k}.$$
is a left-invariant measure.
- $G = \mathrm{GL}(n, \mathbb{R})$:
$$dg = \frac{\prod dg_{ij}}{(\det g)^n}$$
is a left and right Haar measure.

Definition 12.4 (Unimodular group)**.** G is called unimodular if $d\mu_\ell = d\mu_r$.

Example 12.5.
- $\mathbb{R}$, $\mathbb{Q}_p$ and every abelian group is unimodular.
- Compact groups.
- Semisimple groups.
- Reductive groups.
- Nilpotent groups.
- Solvable groups are in general not unimodular: B in the previous examples are not unimodular.

Definition 12.6 (Modular function)**.** Define $\Delta_G : G \to \mathbb{R}^+$ by
$$d_\ell(\cdot t) = \Delta_G(t)^{-1} d_\ell(\cdot).$$

We have the following facts:
- $\Delta_G(t) = |\det(\mathrm{Ad}(t))|$.
- Δ_G is a smooth homomorphism.
- $d_\ell(x^{-1}) = \Delta_G(x) d_\ell(x)$.
- $d_r(x^{-1}) = \Delta_G(x^{-1}) d_r(x)$.
- $d_r(t\cdot) = \Delta_G(t) d_r(\cdot)$.
- $\Delta_G = 1$ if and only if G is unimodular.

12.2. **Decomposition of Haar measures.** We want to integrate over a group but we don't want to put coordinates on G in the sense of differential geometry. We want to do group theory, using decompositions like Cartan. We would like to know how these are related to the measures. The order of integration usually does not matter, but the order in which we multiply or decompose elements matters, such as nak or ank in the Iwasawa decomposition.

Theorem 12.7. *Let G be a Lie group with closed subgroups S and T such that:*
- *$S \cap T$ is compact,*
- *$S \times T \to G$ is an open map,*

- *ST exhaust G up to a measure 0 set,*

and let Δ_G and Δ_T be the modular functions of G and T. Then the left Haar measures can be normalized such that

$$\int_G f(g)\, d_\ell g = \int_{S\times T} f(s\cdot t)\cdot \frac{\Delta_T(t)}{\Delta_G(t)}\, d_\ell s\, d_\ell t.$$

Essentially all the integration formulas will follow from this. A detailed proof can be found in Knapp. The proof goes through the action of $S\times T$ on G given by $(s,t)\cdot g = sgt^{-1}$.

Corollary 12.8. *If G is unimodular with S and T satisfying the above, then*

$$d_\ell g = d_\ell s\, d_r t$$

This is intuitive: take the left-invariant measure on the left and right-invariant measure on the right!

12.3. **Applications.**

12.3.1. *Parabolics.* Parabolics are not unimodular, and are essentially the only problems. Let $P = M^1AN$ be the Langlands decomposition. Then we can take

$$\Delta_P(p) = e^{2\rho_P(p)}$$

where

$$\rho_P = \frac{1}{2}\sum_{\alpha\in\Phi_P^+} n_\alpha \alpha$$

is the half sum. Alternatively,

$$\Delta_P(p) = |\det \mathrm{Ad}(p)| = \left|\det_{\mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n}} \mathrm{Ad}(p)\right|.$$

We have

$$\begin{aligned}\Delta_P(m) &= 1,\\ \Delta_P(n) &= 1,\\ \Delta_P(a) &= e^{2\rho_P(a)}.\end{aligned}$$

Then

$$\begin{aligned}d_\ell(man) &= dm\, da\, dn,\\ d_r(man) &= e^{2\rho_P(x)}\, dm\, da\, dn.\end{aligned}$$

Note that M^1, A and N are all unimodular so there is no ambiguity with dm, da and dn here.

For AN, we have

$$\Delta_{AN}(an) = e^{2\rho_p(a)},$$

so

$$\begin{aligned}d_\ell(an) &= da\, dn,\\ d_r(an) &= e^{2\rho_P(a)}\, da\, dn.\end{aligned}$$

12.3.2. *Iwasawa decomposition.* For KAN,

$$dg = e^{2\rho_P(a)}\, dk\, da\, dn.$$

If we use NAK instead,

$$dg = e^{-2\rho_P(a)}\, dk\, da\, dn.$$

12.3.3. *Langlands decomposition.* For $G = KM^1AN$,

$$dg = e^{2\rho_P(a)}\, dk\, dm\, da\, dn.$$

For $G = NAM^1K$,

$$dg = e^{-2\rho_P(a)}\, dk\, dm\, da\, dn.$$

12.3.4. *LU decomposition.* For $G = \overline{N}M^1AN$ (up to measure zero) (the LU decomposition in linear algebra),

$$dg = e^{2\rho_P(a)}\, d\overline{n}\, dm\, da\, dn.$$

The order of integration does not matter; only the order of multiplication does.

Proposition 12.9. *Let $P = M^1AN$ be a parabolic and $G = KM^1AN$. Then for all continuous function $f : K \to \mathbb{C}$ that is right-invariant under $M^1 \cap K$, we can normalize the measures such that*

$$\int_K f(k)\, dk = \int_{\overline{N}} f(k(\overline{n}))d^{-2\rho_P(\overline{n})}\, d\overline{n}.$$

12.4. **Convergence of Eisenstein series.** We will consider SL_2 first.

If f is "suitably regular", $\lambda \in \rho_P + (\mathfrak{a}_P^*)^+ = \{\rho_P + \Lambda \mid \Lambda(\alpha^\vee) > 0 \text{ for all } \alpha \in \Delta_P\}$.

For SL_2, recall that we have $\mathfrak{a}_P^* = \mathbb{R}$ and $\rho_P = 1$, so $\rho_P + (\mathfrak{a}_P^*)^+ = (1, \infty)$. We would like to show the Eisenstein series

$$E(z, s) = \sum_{\delta \in B(\mathbb{Z})\backslash G(\mathbb{Z})} \mathrm{Im}(\delta z)^{s+\frac{1}{2}}.$$

converges for large $\mathrm{Re}(s)$. Let us compare this with an integral.

Here is a fact. Given $w \in \mathbb{H}$ and $\epsilon > 0$, there exists $c(\epsilon, s)$ independent of w such that

$$\int_{\{|z-w|<\epsilon\}} \mathrm{Im}(z)^{s+\frac{1}{2}}\, dz = c(\epsilon, s)\, \mathrm{Im}(w)^{s+\frac{1}{2}}.$$

The idea is that the left hand side is point-pair invariant. As an integral of an eigenfunction for the Laplacian, it is still an eigenfunction.

Given this, for a fixed z we would like to show $E(z, s)$ converges. We will compare $\mathrm{Im}(z)$ with the integral over a disk around z. Given $z_1, z_2 \in \mathcal{F}$ fundamental domain, there does not exist $g \in G(\mathbb{Z})$ such that $g \cdot z_1 = z_2$.

Recall $B(\mathbb{Z}) = \left\{\pm\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}\right\}$, so $\delta \in B(\mathbb{Z})\backslash G(\mathbb{Z})$ can only map z downwards. Note there are infinitely many fundamental domains tessellated near the real axis.

Thus

$$E(z, s) = \frac{1}{c(\epsilon, s)} \sum_{\delta \in B(\mathbb{Z})\backslash G(\mathbb{Z})} \int_{|z-w|<\epsilon} \mathrm{Im}(\delta z)^{s+\frac{1}{2}} \frac{dx\, dy}{y^2} \le y^{s+\frac{1}{2}} + \int_D \mathrm{Im}(z)^{s+\frac{1}{2}} \frac{dx\, dy}{y^2}$$

where we choose ϵ small enough (since the action is properly discontinuous) and D is some region inside $[-\frac{1}{2}, \frac{1}{2}] \times [0,1]$. This integral is

$$\sim \int_0^x y^{s-\frac{3}{2}}\, dy$$

which converges for $\mathrm{Re}(s) > \frac{1}{2}$.

In summary, the three steps are:

(1) compare the function with an integral,
(2) consider translations of disks, and
(3) use integration formulas to calculate the integral.

12.5. **A little bit of reduction theory.** Redution theory came from Gauss, Minkowski, Hermit, Siegel and a lot of other people, before Harish-Chandra and Borel finished it. Suppose we have some congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, we would like to get a fundamental domain. In general this is messy, so we would like to get an approximate fundamental domain in which every element has finitely many translates.

We will do this for general G and Γ.

Definition 12.10 (Siegel domains)**.** Let P be a parabolic with $P = NAM^1$, and K a maximal compact such that $G = PK$. Then a Siegel domain for P is a subset $\mathfrak{S}_{t,W} = W \cdot A_t \cdot K$, where $W \subseteq N(\mathbb{A})M^1(\mathbb{A})$ is compact, $t \in \mathbb{R}^+$ and $A_t = \{p \in A_P(\mathbb{A}) \mid |\alpha_P(p)| \geq t \text{ for all } \alpha_P \in \Delta_P\}$.

Example 12.11. For $G = \mathrm{SL}_2$, $\Delta_B = \{e_1 - e_2\}$. We have $M^1 = \begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix}, A = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, N = \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}$. Then $A_t = \{a \in \mathbb{R} \mid a^2 > t\}$ and

$$\mathfrak{S}_{t,W} = \left\{ z = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ & 1/\sqrt{y} \end{pmatrix} \,\middle|\, x \in W, y > t \right\}.$$

Theorem 12.12 (Borel–Harish-Chandra, 1961)**.** *Let G be connected reductive, and $G^1 = (\bigcap_{\chi \in X^*(G)} \ker \chi)^0$. Let $P = N_P A_P N_P$ be a parabolic and K be a maximal compact. Then*

(1) *There exists $\mathfrak{S}_{t,W}$ such that $G(\mathbb{A}) = G(F) \cdot \mathfrak{S}_{t,W}$. Moreover, $G^1(\mathbb{A}) = G(F)\mathfrak{S}'_{t,W}$ where $\mathfrak{S}'_{t,W} = W \cdot A_t^1 \cdot K$ and $A_t^1 = \{p \in A_P(\mathbb{A}) \cap G^1(\mathbb{A}) \mid |\alpha_P(p)| \geq t \text{ for all } \alpha_P \in \Delta_P\}$.*
(2) *For any t, W and $g \in G(\mathbb{A})$, the set*

$$\{\gamma \in G(F) \mid \gamma \mathfrak{S}_{t,W} \cap g\mathfrak{S}_{t,W} \neq \emptyset\}$$

is finite.
(3) *The measure of $\mathfrak{S}'_{t,W}$ is finite.*

We consider G^1 because G often has a center in a trivial way. It is a good exercise to try to do this for $\mathrm{SL}_3(\mathbb{Z}) \backslash \mathrm{SL}_3(\mathbb{R})$.

12.6. **Back to Eisenstein series.** We have a function $\Phi : N_P(\mathbb{A})M_P(F)A_P^0(\mathbb{R})\backslash G(\mathbb{A}) \to \mathbb{C}$, $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$, and

$$E(g, \Phi, \lambda) = \sum_{\delta \in P(F)\backslash G(F)} \Phi(\delta g) \exp\langle \lambda + \rho_P, H_P(\delta g)\rangle.$$

Theorem 12.13. *$E(g, \Phi, \lambda)$ converges absolutely and uniformly on compact subsets when $\lambda \in \rho_P + (\mathfrak{a}_P^*)^+$ and Φ is K-finite (i.e., $\mathcal{I}(\lambda, k)\phi$ spans a finite-dimensional space and lies in a sum of finite many irreducibles).*

We are going to mimic the SL_2 case.

Proof. We can make the following reductions:

(1) Can assume $\lambda \in \mathfrak{a}_P^*$. (Convergence does not depend on the imaginary part of λ.)
(2) Can assume $\Phi > 0$.

For g in a compact and $g', g'' \in G(\mathbb{A})$, the ratio

$$\frac{\Phi(g'g) \exp\langle \lambda + \rho_P, H_P(g'g)\rangle}{\Phi(g''g) \exp\langle \lambda + \rho_P, H_P(g''g)\rangle} \leq C(g', g'')$$

for some constant independent of g. Taking $g'' = 1$ gives

$$\Phi(g'g) \exp\langle \lambda + \rho_P, H_P(g'g)\rangle \leq \Phi(g) \exp\langle \lambda + \rho_P, H_P(g)\rangle$$

up to some constant, and and similarly taking $g' = 1$ gives the reverse inequality up to some constant.

This implies that

$$\sum_{\delta \in P(F)\backslash G(F)} \Phi(\delta g) \exp\langle \lambda + \rho_P, H_P(\delta g)\rangle$$

converges if and only if

$$\sum_{\delta \in P(F)\backslash G(F)} \int_C \Phi(\delta g') \exp\langle \lambda + \rho_P, H_P(\delta g')\rangle \, dg'$$

converges for any compact C.

Now choose C small enough so that $\delta C \cap C = \emptyset$ for all $\delta \in G(F)\backslash\{1\}$ (i.e., the radius of the ball is small enough so that its translates do not intersect itself). Then the above becomes

$$\int_{P(F)\backslash G(F)C} \Phi(g) \exp\langle \lambda + \rho_P, H_P(g)\rangle \, dg.$$

Recall that for the decomposition $G = NAM^1K$,

$$dg = e^{-2\rho_P(g)} \, dn \, da \, dm \, dk$$

and that $G(\mathbb{A}) = G(F)\mathfrak{S}_{W,t}$ for some W, t. Then this integral is majorized by

$$\int_{P(F)\backslash G(F)(\mathfrak{S}_{W,0}\backslash \mathfrak{S}_{W,t'})} \exp\langle \lambda + \rho_P, H_P(g)\rangle e^{-2\rho_P(g)} \, dn \, da \, dm \, dk$$

for some t'. This is

$$\ll \prod_{j=1}^{\mathrm{rank} P} \int_0^{t'} a^{\lambda + \rho_P - 2\rho_P} \, d^\times a,$$

which converges if and only if $\lambda \in \rho_P + (\mathfrak{a}_P^*)^+$. ∎