PROBLEM SET

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Throughout the problem set, we use the following notations:

- G: semisimple algebraic group over $\mathbb{C}$
- $T \subset B$: a choice of maximal torus and Borel subgroup
- $W = N_G(T)/T$: Weyl group
- $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$: corresponding Lie algebras
- $\Phi \subset \mathfrak{t}^*$: roots
- $\chi : \mathfrak{t} \to \mathfrak{a} = \mathfrak{t} /\!\!/ W \cong \mathfrak{g} /\!\!/ G$: Chevalley quotient map
- $\mathfrak{g}^{\mathrm{rs}} \subset \mathfrak{g}$, $\mathfrak{t}^{\mathrm{rs}} \subset \mathfrak{t}$, $\mathfrak{a}^{\mathrm{rs}} \subset \mathfrak{a}$: semisimple loci
- $r = r(G)$: rank of G
- $f_1, \cdots, f_r \in \mathcal{O}(\mathfrak{a}) \cong \mathcal{O}(\mathfrak{t})^W \cong \mathcal{O}(\mathfrak{g})^G$: homogeneous generators
- $F = \mathbb{C}(\!(t)\!)$, $\mathcal{O} = \mathbb{C}[\![t]\!]$
- $\mathbf{I} \subset G(F)$: Iwahori subgroup
- $\mathrm{Fl} = G(F)/\mathbf{I}$: affine flag variety
- $\widetilde{W}$: extended affine Weyl group

1. Homogeneous elements

Exercise 1.1. In this exercise, we study regular elements in W.

For element $w \in W$ and $\zeta \in \mathbb{C}^\times$, define $V(w, \zeta) = \{t \in \mathfrak{t} | wt = \zeta t\} \subset \mathfrak{t}$.

- We say that w is *regular* (of order m) if $V(w, \zeta) \cap \mathfrak{t}^{\mathrm{rs}} \neq \varnothing$ for some primitive m-th root of unity $\zeta \in \mu_m$.
- We say that w is *elliptic* if $\mathfrak{t}^w = 0$.
- We say that m is a *regular number* of W if there exists a regular element of order m.
- We say that m is a *regular elliptic number* if there exists a regular elliptic element of order m.

Fix $w \in W$ a regular element of order m and $\zeta \in \mu_m$ a primitive m-th root of unity.

(1) Show that w has order m as an element of W, and it induces a free action of the cyclic group $\mathbb{Z}/m\mathbb{Z} \cong \langle w \rangle$ on Φ.
(2) Show that there exists a choice of simple roots $S \subset \Phi$ under which $l(w) = |\Phi|/m$. Moreover, when w is elliptic, show that $l(w) \geq |\Phi|/m$ for any choice of S.
(3) Consider $\mathfrak{a}_{1/m} = \cap_{i, m \nmid d_i} V(f_i) \subset \mathfrak{a}$ where $V(f_i) \subset \mathfrak{a}$ is the vanishing locus of $f_i \in \mathcal{O}(\mathfrak{a})$. Show that $\chi|_{\mathfrak{t}}^{-1}(\mathfrak{a}_{1/m}) = \cup_{w' \in W} V(w', \zeta)$.
(4) Define $a(m) = |\{1 \leq i \leq r : m \mid d_i\}| = \dim \mathfrak{a}_{1/m}$. Show that $\chi|_{\mathfrak{t}}^{-1}(\mathfrak{a}_{1/m})$ is equi-dimensional of dimension $a(m)$, and W acts transitively on the set of irreducible components of $\chi|_{\mathfrak{t}}^{-1}(\mathfrak{a}_{1/m})$.
(5) Show that $\dim V(w, \zeta) = a(m)$. Conclude that any two regular elements of order m are conjugate in W.
(6) Show that the eigenvalues of w as an automorphism of $\mathfrak{t}$ are $\{\zeta^{1-d_i}\}_{1 \leq i \leq r}$. (Hint: Consider the basis $\{e_i\}_{1 \leq i \leq r}$ of $\mathfrak{t}$ consisting of eigenvectors of w. Assume $e_1 \in \mathfrak{t}^{\mathrm{rs}}$. Consider the Jacobian $J = \det(\partial_{e_i} f_j)$. Show that $J(e_1) \neq 0$, which implies that there exists a permutation $\sigma \in S_r$ such that $(\partial_{e_i} f_{\sigma(i)})(e_1) \neq 0$ for any i.)
(7) For Weyl groups of type A and C, determine the regular numbers and single out the elliptic ones.

Exercise 1.2. In this exercise, we study regular semisimple homogeneous elements in $\mathfrak{g}(F)$ of slope $\nu = \frac{d}{m}$. Here $d, m \in \mathbb{Z}_{\geq 1}$ and $\gcd(d, m) = 1$.

Consider the torus $\mathbb{G}_m^{\mathrm{rot}} \times \mathbb{G}_m^{\mathrm{dil}}$ acting on $\mathfrak{g}(F)$ by loop rotation on the first factor and dilation on the section factor, which induces also an action on $\mathfrak{a}(F)$. We use $\xi \in X_*(T_{\mathrm{ad}})$ to denote the unique element such that $x_m := \xi/m \in X_*(T_{\mathrm{ad}})_{\mathbb{Q}}$ lies in the fundamental alcove and is conjugate to $\check{\rho}/m$ under the affine Weyl group. Consider the Moy-Prasad grading on $\mathfrak{g}(F)$ defined by x_m. Explicitly, it is the $\frac{1}{m}\mathbb{Z}$-grading on $\mathfrak{g}(F)$:

$$\mathfrak{g}(F)_{i/m} = \{X \in \mathfrak{g}(F) : \mathrm{Ad}(\xi(s))X = (s^{-m}, s^i) \cdot X \text{ for all } s \in \mathbb{G}_m\}.$$

This induces a $\frac{1}{m}\mathbb{Z}$-grading on $\mathfrak{a}(F)$:

$$\mathfrak{a}(F)_{i/m} = \{x \in \mathfrak{a} : (s^{-m}, s^i) \cdot x = x \text{ for all } s \in \mathbb{G}_m\}.$$

This gives rise to Moy-Prasad subgroups $\mathbf{P}(\frac{i}{m}) \subset G(F)$ such that $\operatorname{Lie}\mathbf{P}(\frac{i}{m}) = \mathfrak{g}(F)_{\geq i/m}$. [1]

Note that classifying semisimple elements in $\mathfrak{g}(F)$ is equivalent to study elements in $\mathfrak{a}(F)$. More precisely, the map between sets $\mathfrak{g}(F)^{\mathrm{ss}}/G(F) \to \mathfrak{a}(F)$ is bijective. The injectivity follows from [Ste75, Theorem 3.14] and [Ste65, Theorem 1.9]. The surjectivity follows from [Ste65, Theorem 1.7]. Therefore, we are reduced to study $\mathfrak{a}(F)^{\mathrm{rs}}_{\nu}$.

Consider the $\frac{1}{m}\mathbb{Z}/\mathbb{Z}$-grading on $\mathfrak{g}$ defined by

$$\mathfrak{g}_{i/m} = \{X \in \mathfrak{g} : \mathrm{Ad}(\xi(s))X = s^i \cdot_{\mathrm{dil}} X \text{ for all } s \in \mu_m\}.$$

Define

$$\mathfrak{a}_{i/m} = \{x \in \mathfrak{a} : s^i \cdot_{\mathrm{dil}} x = x\}.$$

Then the Chevalley quotient map restricts to $\chi : \mathfrak{g}_{i/m} \to \mathfrak{a}_{i/m}$ for any $i \in \mathbb{Z}$. Note that $\mathfrak{a}_{i/m} = \mathfrak{a}_{\gcd(i,m)/m}$. Define $\overline{\nu} = \nu + \mathbb{Z} \in \frac{1}{m}\mathbb{Z}/\mathbb{Z}$.

(1) Show that evaluation at $t = 1$ induces isomorphisms $\mathrm{ev}_1 : \mathfrak{g}(F)_{i/m} \xrightarrow{\sim} \mathfrak{g}_{i/m}$ and $\mathrm{ev}_1 : \mathfrak{a}(F)_{i/m} \xrightarrow{\sim} \mathfrak{a}_{i/m}$ for any $i \in \mathbb{Z}$. Moreover, show that $\mathrm{ev}_1(\mathfrak{g}(F)^{\mathrm{rs}}_{i/m}) = \mathfrak{g}^{\mathrm{rs}}_{i/m}$ and $\mathrm{ev}_1(\mathfrak{a}(F)^{\mathrm{rs}}_{i/m}) = \mathfrak{a}^{\mathrm{rs}}_{i/m}$.

(2) Show that $\mathfrak{a}^{\mathrm{rs}}_{1/m}$ is non-empty if and only if m is a regular number of W. Therefore, a regular semisimple homogeneous element in $\mathfrak{g}(F)$ of slope ν exists if and only if m is a regular number of W.

(3) Assume m is a regular number, show that the map $\mathfrak{g}_{\overline{\nu}} \to \mathfrak{a}_{1/m}$ is surjective. Therefore, any regular semisimple homogeneous element in $\mathfrak{g}(F)$ of slope ν can be conjugated to an element in $\mathfrak{g}(F)^{\mathrm{rs}}_{\nu}$ by $G(F)$.

(4) For Weyl groups of type A and C, describe all possible homogeneous elements and the corresponding Moy-Prasad subgroups.

Exercise 1.3. This exercise studies invariant theory of $\mathfrak{g}_{\overline{\nu}}$ under the action of $G_0 = L_{\mathbf{P}} = \mathbf{P}(0)/\mathbf{P}(\frac{1}{m})$.

(1) For an element $\overline{\psi} \in \mathfrak{g}_{\overline{\nu}}$, show that $\overline{\psi}$ is *polystable* (i.e. the orbit $G_0 \cdot \overline{\psi}$ is closed) if and only if $\overline{\psi}$ is semisimple as an element in $\mathfrak{g}$.

(2) For an element $\overline{\psi} \in \mathfrak{g}_{\overline{\nu}}$, show that $\overline{\psi}$ is *stable* (i.e. $\overline{\psi}$ is polystable and $\mathrm{Stab}_{G_0}(\overline{\psi})$ is finite) if and only if $\overline{\psi} \in \mathfrak{g}^{\mathrm{rs}}_{\overline{\nu}}$ and m is elliptic.

(3) From now on, fix an element $\overline{\psi} \in \mathfrak{g}^{\mathrm{rs}}_{\overline{\nu}}$, consider the centralizer

$$\mathfrak{c}_{\overline{\psi}} = \mathfrak{z}_{\mathfrak{g}}(\overline{\psi}) \subset \mathfrak{g}.$$

Define the *Cartan* subspace $\mathfrak{c}_{\overline{\psi},\overline{\nu}} = \mathfrak{c}_{\overline{\psi}} \cap \mathfrak{g}_{\overline{\nu}}$. Show that $\mathfrak{g}^{\mathrm{rs}}_{\overline{\nu}} \subset G_0 \cdot \mathfrak{c}_{\overline{\psi},\overline{\nu}}$.

(4) Define the *little Weyl group* $W_m = N_{G_0}(\mathfrak{c}_{\overline{\psi},\overline{\nu}})/\mathrm{Stab}_{G_0}(\overline{\psi})$. Show that W_m naturally embeds into W. Moreover, the Chevalley quotient map induces a finite surjective map $\mathfrak{c}_{\overline{\psi},\overline{\nu}} /\!\!/ W_m \to \mathfrak{a}_{\overline{\nu}}$.

(5) Show that the natural map $\mathfrak{c}_{\overline{\psi},\overline{\nu}} /\!\!/ W_m \to \mathfrak{g}_{\overline{\nu}} /\!\!/ G_0$ is an isomorphism.

(6) For each regular number m associated to a Weyl group of type A and C, describe the Cartan subspace and little Weyl group W_m.

[1] Note that the distinction between ξ and $\check{\rho}$ results in conjugate Moy-Prasad subgroups. We choose ξ instead of $\check{\rho}$ to make sure that $\mathbf{I} \subset \mathbf{P}(0)$ and keep our notation consistent with [BAMY22]. Although it does not really help, this distinction can be mostly ignored in the problem set.

2. Affine Springer fibers

For each regular semisimple topologically nilpotent element $\psi \in \mathfrak{g}(F)^{\mathrm{rs}}$, define the affine Springer fiber

$$\mathrm{Fl}_\psi = \{g\mathbf{I}/\mathbf{I} \in \mathrm{Fl} \mid \mathrm{Ad}_{g^{-1}}(\psi) \in \mathrm{Lie}\,\mathbf{I}\}.$$

When ψ is homogeneous of slope ν, we assume $\psi \in \mathfrak{g}(F)^{\mathrm{rs}}_\nu$ by conjugation. Define the $\mathbb{G}_m$-action on Fl_ψ by

$$s \cdot g\mathbf{I}/\mathbf{I} = s^m \cdot_{\mathrm{rot}} \xi(s) g\mathbf{I}/\mathbf{I}$$

for $s \in \mathbb{G}_m$.

Exercise 2.1. Classify regular semisimple homogeneous elements ψ for semisimple algebraic groups such that $\dim \mathrm{Fl}_\psi = 1$. You may want to use the dimension formula $\dim \mathrm{Fl}_\psi = (\nu|\Phi| - \dim(\mathfrak{t}/\mathfrak{t}^w))/2$ where $w \in W$ is any regular element of order m. You can find the list of regular numbers for Weyl groups of exceptional type in [Spr74, §5.4].

Exercise 2.2. This exercise proves that Hessenberg varieties arsing as connected components of $\mathrm{Fl}_\psi^{\mathbb{G}_m}$ are smooth projective.

Consider a reductive group L and a finite dimension representation $V \in \mathrm{Rep}(L)$. Fix a vector $v \in V_0$, a subspace $V_0 \subset V$, and a parabolic subgroup $Q \subset L$ which stabilizes $V_0 \subset V$. Define the associated *Hessenberg variety* to be

$$\mathcal{H}_v(Q \subset L, V_0 \subset V) = \{lQ/Q \in L/Q : l^{-1}v \in V_0\}.$$

For $\psi \in \mathfrak{g}(F)^{\mathrm{rs}}_\nu$, recall that $\mathrm{Fl}_\psi^{\mathbb{G}_m} = \coprod_{w \in W_{\mathbf{P}} \backslash \widetilde{W}} \mathcal{H}_\psi(w)$ where $\mathcal{H}_\psi(w) = (L_{\mathbf{P}} w\mathbf{I}/\mathbf{I}) \cap \mathrm{Fl}_\psi$. Note that $\mathcal{H}_\psi(w) = \mathcal{H}_\psi(L_{\mathbf{P}} \cap \mathrm{Ad}(w)\mathbf{I} \subset L_{\mathbf{P}}, \mathfrak{g}(F)_\nu \cap \mathrm{Ad}(w)\,\mathrm{Lie}\,\mathbf{I} \subset \mathfrak{g}(F)_\nu)$.

1. Show that $\mathcal{H}_v(Q \subset L, V_0 \subset V)$ is smooth if the following condition is satisfied: For any $v' \in L \cdot v \cap V_0$, one has $\mathfrak{l} \cdot v' + V_0 = V$.
2. For any $\psi' \in \mathfrak{g}(F)^{\mathrm{rs}}_\nu \cap \mathrm{Ad}(w)\,\mathrm{Lie}\,\mathbf{I}$, show that $[\mathfrak{g}(F)_0, \psi'] + \mathfrak{g}(F)_\nu \cap \mathrm{Ad}(w)\,\mathrm{Lie}\,\mathbf{I} = \mathfrak{g}(F)_\nu$. Conclude that $\mathcal{H}_\psi(w)$ is smooth projective for any $w \in W_{\mathbf{P}_\nu} \backslash W$. (Hint: You may first show the following: For any $\mathbb{G}_m \subset G_0$ and $\overline{\psi} \in \mathfrak{g}^{\mathrm{rs}}_{\overline{\nu}}$, one has $[\mathfrak{g}_0, \overline{\psi}] + \mathfrak{g}^{\geq 0}_{\overline{\nu}} = \mathfrak{g}_{\overline{\nu}}$. Here $\mathfrak{g}^{\geq 0}_{\overline{\nu}}$ is the part with non-negative weight with respect to the action of $\mathbb{G}_m$.)

Exercise 2.3. Consider $G = \mathrm{SL}_2$ with slope $\nu = \frac{1}{2} + k$ where $k \in \mathbb{Z}_{\geq 0}$. Take the homogeneous element $\psi = \begin{pmatrix} & t^k \\ t^{k+1} & \end{pmatrix} \in \mathfrak{sl}_2(F)$.

1. Describe the fixed point locus $\mathrm{Fl}_\psi^{\mathbb{G}_m}$.
2. For each connected component of $\mathrm{Fl}_\psi^{\mathbb{G}_m}$, describe the corresponding attracting locus.
3. Describe the affine Springer fiber Fl_ψ.

Exercise 2.4. Consider $G = \mathrm{Sp}_4 = \mathrm{Sp}(V)$ and $\nu = \frac{1}{2}$. Fix a polarization $V = V_0 \oplus V_0^*$. Take the homogeneous element $\psi = \begin{pmatrix} & P \\ tQ & \end{pmatrix} \in \mathfrak{sp}_4(F)$ in which $P : V_0^* \xrightarrow{\sim} V_0$ and $Q : V_0 \xrightarrow{\sim} V_0^*$ are two non-degenerate quadratic forms on V_0 in general position.

1. Show that Fl_ψ can be identified with $\mathbb{P}^1$'s with dual graph D_4.
2. Determine the cross-ratio of the four points on the central $\mathbb{P}^1$. (The answer should depend on P, Q)

Exercise 2.5. Consider $G = \mathrm{Sp}_6$ and $\nu = \frac{1}{2}$. Take the homogeneous element as in the previous exercise.

1. Show that there exists a unique non-rational connected component $E_\psi \subset \mathrm{Fl}_\psi^{\mathbb{G}_m}$, which is an elliptic curve.
2. Compute the j-invariant of E_ψ. (The answer should depend on P, Q)
3. Show that there exists a unique non-rational irreducible component of Fl_ψ, which is a $\mathbb{P}^1 \times \mathbb{P}^1$-fibration over E_ψ.

3. Hitchin moduli spaces for homogeneous elements

Define $F_0 = F = \mathbb{C}(\!(t)\!)$ and $F_\infty = \mathbb{C}(\!(t^{-1})\!)$. Let D_0, D_∞ be the formal disc around $0, \infty \in \mathbb{P}^1$. Fix $\psi \in \mathfrak{g}(F)^{\mathrm{rs}}_\nu$, which automatically lies in $\mathfrak{g}(\mathbb{C}[t, t^{-1}])_\nu$. Take $\mathbf{I}_0 \subset G(F_0)$ to be the standard Iwahori subgroup. Take $\mathbf{P}_\infty \subset G(F_\infty)$ to be standard parahoric subgroup such that $\mathrm{Lie}\,\mathbf{P}_\infty = \mathfrak{g}(F_\infty)_{\leq 0}$. It is equipped with a filtration by Moy-Prasad subgroup $\mathbf{P}_\infty(\frac{i}{m}) \subset \mathbf{P}_\infty$ such that $\operatorname{Lie}\mathbf{P}_\infty(\frac{i}{m}) = \mathfrak{g}(F_\infty)_{\leq -i/m}$. Define $\mathbf{C}_\infty^+ \subset \mathbf{P}_\infty(\frac{1}{m})$ to be the connected subgroup such that $\operatorname{Lie}\mathbf{C}_\infty^+ = \mathfrak{z}_{\mathfrak{g}(F_\infty)}(\psi)_{\leq -1/m}$. Take $\mathbf{K}_\infty = \mathbf{P}_\infty(\nu)\cdot\mathbf{C}_\infty^+$. Identify $\mathfrak{g} \cong \mathfrak{g}^*$.

Consider the Hitchin moduli space $\mathcal{M}_\psi$ defined as moduli of $(\mathcal{E}, \varphi)$ where

- $\mathcal{E}$ is a G-bundle on $\mathbb{P}^1$ with $\mathbf{K}_\infty$-level structure at ∞ and $\mathbf{I}_0$-level structure at 0. Define $\operatorname{Ad}(\mathcal{E})$ to be the adjoint vector bundle.
- $\varphi \in H^0(\mathbb{P}^1\backslash\{0,\infty\}, \operatorname{Ad}(\mathcal{E}) \otimes \omega_{\mathbb{P}^1})$ such that
  (1) Under any trivialization of $\mathcal{E}|_{D_\infty}$ with $\mathbf{K}_\infty$-level structure, we require
  $$\varphi|_{D_\infty^\times} \in (-\psi + \mathfrak{g}(F_\infty)_{\leq 0})dt/t.$$
  (2) Under any trivialization of $\mathcal{E}|_{D_0}$ with $\mathbf{I}_0$-level structure, we require
  $$\varphi|_{D_0^\times} \in \operatorname{Lie}\mathbf{I}_0^+ dt/t.$$

The usual Hitchin map construction gives a map $\mathcal{M}_\psi \to \oplus_{i=1}^r H^0(\mathbb{P}^1\backslash\{0,\infty\}, \omega_{\mathbb{P}^1}^{d_i})$. We use $\mathcal{A}_\psi$ to denote the image of this map. This gives the Hitchin map $f_\psi : \mathcal{M}_\psi \to \mathcal{A}_\psi$. Define $a_\psi = f_\psi(\mathcal{E}_{\mathrm{triv}}, -\psi dt/t) \in \mathcal{A}_\psi$.

Exercise 3.1. Consider $G = \mathrm{SL}_2$ with homogeneous element $\psi = \begin{pmatrix} t & \\ & -t \end{pmatrix}$ of slope 1.

(1) Describe $\mathcal{A}_\psi$.
(2) Show that $f_\psi^{-1}(a_\psi)$ is isomorphic to an infinite chain of $\mathbb{P}^1$.
(3) Show that the generic fiber of f_ψ is isomorphic to $\mathbb{G}_m$.

Exercise 3.2. Consider $G = \mathrm{SL}_2$ with homogeneous element $\psi = \begin{pmatrix} & t \\ t^2 & \end{pmatrix}$ of slope $\frac{3}{2}$.

(1) Describe $\mathcal{A}_\psi$.
(2) Show that $f_\psi^{-1}(a_\psi)$ is isomorphic to two $\mathbb{P}^1$'s tangent at a point (i.e. has equation $(y - x^2)y = 0$ locally around the intersection point).
(3) Show that the generic fiber of f_ψ is isomorphic to the elliptic curve with complex multiplication by $\mathbb{Z}[i]$.

Exercise 3.3. Consider $G = \mathrm{SL}_3$ with homogeneous element $\psi = \begin{pmatrix} & & 1 \\ t & & \\ & t & \end{pmatrix}$ of slope $\frac{2}{3}$.

(1) Describe $\mathcal{A}_\psi$.
(2) Show that $f_\psi^{-1}(a_\psi)$ is isomorphic to three $\mathbb{P}^1$'s intersect pairwise-transversally at a single point (i.e. has equation $(y - x)xy = 0$ locally at the intersection point).
(3) Show that the generic fiber of f_ψ is isomorphic to the elliptic curve with complex multiplication by $\mathbb{Z}[\omega]$.

Exercise 3.4. Consider $G = \mathrm{Sp}_4$ with homogeneous element $\psi = \begin{pmatrix} & P \\ tQ & \end{pmatrix}$ of slope $\frac{1}{2}$ as in Exercise 2.4. Show that the generic fiber of f_ψ is isomorphic to the elliptic curve which is the double cover of the central $\mathbb{P}^1$ ramified at the four points in Exercise 2.4(2).

Exercise 3.5. Repeat Exercise 3.4 for $G = \mathrm{SO}_8$ with homogeneous element of slope $\frac{1}{4}$.

4. Stokes data and Betti moduli spaces

We use Br_W^+ to denote the semigroup of positive braids. For each $\psi \in \mathfrak{g}(F)_\nu^{\mathrm{rs}}$, we use $\beta_\nu = \beta_\psi \in \mathrm{Br}_W^+$ to denote the associated braid, which is well-defined up to a cyclic shift and only depends on the slope ν. We refer to [BAMY22, §4.3] for the definition of β_ν. See also [BAMY22, §4] for the definition of braid variety $\mathcal{M}(\beta)$ and Betti moduli space $\mathcal{M}_{\mathrm{Bet},\psi}$.

Exercise 4.1. Consider $G = \mathrm{SL}_n$ with homogeneous element

$$\psi = \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ t & & & \end{pmatrix}^d$$

of slope $\nu = \frac{d}{n}$ for $d \in \mathbb{Z}_{\geq 1}$ such that $\gcd(d, n) = 1$.

(1) Describe the braid $\beta_\nu \in \mathrm{Br}^+_{S^n}$. Identify the braid closure $\widehat{\beta}_\nu$ which is the link obtained from β_ν by connecting startpoints with endpoints in order [2].
(2) For an algebraic curve $C \subset \mathbb{C}^2$ passing through the origin $0 \in \mathbb{C}^2$, the *algebraic link* associated to C is defined as $C \cap S^3_\epsilon \subset S^3_\epsilon = \{(x, y) \in \mathbb{C}^2 | |x|^2 + |y|^2 = \epsilon\}$ for sufficiently small $\epsilon > 0$. Show that the algebraic link associated to $V(x^n - y^d)$ is equivalent to the link $\widehat{\beta}_\nu$.

Exercise 4.2. Consider $G = \mathrm{SL}_n$ with homogeneous element

$$\psi = \begin{pmatrix} & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ t & & & & \\ & & & & 0 \end{pmatrix}^d$$

of slope $\nu = \frac{d}{n-1}$ for $d \in \mathbb{Z}_{\geq 1}$ such that $\gcd(d, n-1) = 1$.

(1) Describe the braid $\beta_\nu \in \mathrm{Br}^+_{S^n}$. Identify the braid closure $\widehat{\beta}_\nu$.
(2) Show that the algebraic link associated to $V(x^n - xy^d)$ is equivalent to the link $\widehat{\beta}_\nu$.

Exercise 4.3. This exercise studies the Betti moduli space $\mathcal{M}_{\mathrm{Bet},\psi}$ with slope $\nu = \frac{1}{m}$ where m is a regular elliptic number of W.

For each elliptic element $w \in W$ with minimal length in its conjugacy class, consider $Z_w = \langle T^w, U_\alpha : w\alpha = \alpha, \alpha \in \Phi \rangle$ and $U_w = U \cap wU^- w^{-1}$. Here U_α is the root subgroup of α, U is the unipotent radical of B, and U^- is the opposite of U. Define the *multiplicative transversal slice* $\Sigma_w = U_w Z_w w$, which satisfies the following properties:

- The map
$$U \times \Sigma_w \to U Z_w w U$$
$$(u, s) \mapsto usu^{-1}$$
is an isomorphism.
- Σ_w is transversal to conjugacy classes in G.

The result above is proved in [HL12]. See [Dua24] for a generalization to non-elliptic case.

(1) Show that $\beta_{1/m} \in \mathrm{Br}^+_W$ is a minimal length representative of a regular elliptic element of order m in W.
(2) Choose $w \in W$ a minimal length representative of a regular elliptic element of order m. Consider $\Sigma^\circ_w = U_w w$. Show that $\mathcal{M}_{\mathrm{Bet},\psi} \cong \Sigma^\circ_w \times_G \widetilde{\mathcal{U}}$ where $\widetilde{\mathcal{U}} \to \mathcal{U}$ is the Springer resolution. Conclude that $\mathcal{M}_{\mathrm{Bet},\psi}$ is a classical smooth algebraic variety.
(3) Show that $\mathcal{M}_{\mathrm{Bet},\psi}$ is a point when $m = h$ is the Coxeter number.
(4) For $G = \mathrm{Sp}_4$ and $\nu = \frac{1}{2}$, show that $\mathcal{M}_{\mathrm{Bet},\psi}$ can be identified with a resolution of $V(x^2 + (y+z)^2 + xyz) \subset \mathbb{A}^3$. Note that the later has A_3-singularity at the origin as its only singular point.

[2] You may find more background on links and link invariants in [GKS21].

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