PROBLEM SET 1

$G = \mathbb{G}_m$.

Problem 1. *Show that χ_σ defined in the first contruction in the lecture*[1] *satisfies the Hecke property.*

Problem 2. *Compare the two constructions of χ_σ in the lecture*[2].

$G = \mathsf{PGL}_2$.

Problem 3. *Show that for $d \geq 1$ the map $\mathsf{Bun}_B^d \to \mathsf{Bun}_G$ is injective on k-points.*

Problem 4. *Show that $\mathsf{Bun}_G^{\mathsf{even}}$ is connected.*

Problem 5. *Find the dimension of Bun_G. Find the dimension of Bun_B^d for any $d \in \mathbb{Z}$. Deduce that there exists*[3] *odd semistable G-bundles on X.*

Problem 6. *For $d << 0$, show that the map $\mathsf{Bun}_B \to \mathsf{Bun}_G$ is smooth.*

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[1] Recall in the lecture, we work in the Betti setting and identify the given rank 1 local system σ with a homomorphism

$$\pi_1(X)^{\mathsf{ab}} \simeq \mathsf{H}^1(X, \mathbb{Z}) \to \mathsf{e}^\times,$$

where the first isomorphism is due to Poincaré duality. Via the isomorphism

$$\pi_1(\mathsf{Jac}(X)) \simeq \mathsf{H}^1(X, \mathbb{Z}),$$

we obtain a local system on $\mathsf{Jac}(X)$, which gives the desired local system χ_σ on $\mathsf{Bun}_{\mathbb{G}_m} \simeq \mathsf{Jac}(X) \times \mathbb{Z} \times \mathbb{B}\mathbb{G}_m$ by taking external product with the constant sheaves.

[2] We recall the second construction as follows. For $d >> 0$, consider the map $\mathsf{add}_d : X^d \to \mathsf{Sym}^d(X)$ and define

$$\sigma^{(d)} := \mathsf{add}_{d,*}(\sigma \boxtimes \cdots \boxtimes \sigma)^{S_d},$$

which is a local system on $\mathsf{Sym}^d(X)$. Since the fibers of the Abel–Jacobi map $\mathsf{Sym}^d(X) \to \mathsf{Bun}_{\mathbb{G}_m}^d$ is simply connected, this local system descends to a local system on $\mathsf{Bun}_{\mathbb{G}_m}^d$. Then we use Hecke property to extend this local system to the entire $\mathsf{Bun}_{\mathbb{G}_m}$.

[3] Challenge: can you write down such a bundle?