Beijing-Shanghai Summer School in Mathematics, 2025, exercises for the lecture.

0.1. **Geometry of Springer fibers.** We write $\mathcal{B}$ for the flag space G/B, we also write $\mathcal{B}_n$ for $\mathcal{B}$ if $G = GL(n)$. When working with $GL(n)$ we write V for the n-dimensional space it acts on (the tautological representation).

(1) Let G be a reductive group. Fix a parabolic P and let nilpotent e be chosen generically in the radical of P. (A nilpotent orbit obtained this way is called Richardson. Every nilpotent orbit in $gl(n)$ or $sl(n)$ is Richardson but in general this is not so.)

Show that the set of Borel subalgebras contained in P is a component of the Springer fiber $\mathcal{B}_e$, this component is isomorphic to the flag space of the Levi L.

[You can assume that $G = GL(n)$ if you prefer.]

(2) (a) Let $e \in sl(n)$ be such that $e^2 = 0$, $rank(e) = r$. Check that e is generic in the group of block upper triangular matrices with zero diagonal blocks, where the blocks are of the sizes $(n-r)$ and $r = rank(e)$.

Thus the previous problem gives a component X of $\mathcal{B}_e$ isomorphic to $\mathcal{B}_{n-r} \times \mathcal{B}_r$. Describe another component X' of $\mathcal{B}_e$ isomorphic to a $\mathbb{P}^1$ bundle over $D \subset \mathcal{B}_{n-r} \times \mathcal{B}_r$ where the divisor D parametrizes pairs of flags $(V_0 \subset \cdots \subset V_{n-r}), (V_0' \subset \cdots \subset V_r')$ such that $V_1' \subset V_{n-r-1}$.

(b) For $i = 1, \ldots, n-1$ a line of type i in $\mathcal{B}_n$ is the set of flags $(0 = V_0 \subset \cdots \subset V_n = V)$ with fixed $V_1, \ldots, V_{i-1}; V_{i+1}, \ldots, V_n$.

Let X be a component of Springer fiber, we say that X admits a type i fibration (or i-fibers) if it is a union of lines of type i. Check the following.

If X admits a type i fibration for all i then $X = \mathcal{B}_n$ and $e = 0$. If i is such that X does not i-fiber, show that there exists a divisor $D \subset X$ and another component X' of $\mathcal{B}_e$ which is the union of all type i lines intersecting D.

If X does not admit a type i fibration for a unique i then $e^2 = 0$ and X is as in the previous part of the problem.

[This can also be generalized to any reductive group].

(c) For $r = 1$ show that $\mathcal{B}_e$ has $(n-1)$ components where the i-th component X_i fibers over $Gr(i-1, n-2)$ with fiber $\mathcal{B}_{i-1} \times \mathcal{B}_{n-i-1}$.

(3) Let $e \in sl(n)$ be the nilpotent with Jordan blocks of size $(1, n-1)$ (the subregular nilpotent).

Show that the Springer fiber is a union of $(n-1)$ projective lines, describe their intersection pattern and the explicit matrix by which a simple reflection $s_i = (i, i+1)$ acts on top homology in the basis of components.

Bonus problem: generalize this to $Sp(2n)$ and $SO(n)$ (in the latter case consider the case of odd and even n separately), i.e. find a nilpotent with a one dimensional Springer fiber and describe this fiber explicitly.

(4) The centralizer Z_e of e acts on $\mathcal{B}_e$. This problem gives an example where this action has infinitely many orbits.

Let $e \in sl(8)$ be the nilpotent with two equal Jordan blocks. Consider the set X of flags $(V_0 \subset \cdots \subset V_8)$ such that $V_2 = Ker(e)$, $V_4 = Ker(e^2)$, $V_6 = Ker(e^3)$, this is a Z_e-invariant closed subvariety in $\mathcal{B}_e$. Set $U =$

$Ker(e)$. Construct a Z_e-invariant onto map $\mathcal{B}_e \to \mathbb{P}(U)^4$. Conclude that the action of Z_e on X and hence on $\mathcal{B}_e$ has infinitely many orbits.

One can optimize this example as follows: let $e' \in sl(6)$ of Jordan type $(2,4)$ and let $H \subset V$ be the unique 5-dimensional e'-invariant subspace such that $e'|_H$ has Jordan type $(3,2)$ (namely, $H = Im(e') + Ker(e')^2$). We let Y be the set of e-invariant flags such that $V_5 = H$, $V_3 = e'(H)$, $V_1 = (e')^2(H)$, this is a $Z_{e'}$-invariant closed subvariety in $\mathcal{B}_{e'}$. Let $U' = H/e'H$. Show that $Z_{e'}$ acts on $\mathbb{P}(U')$ fixing two points: the lines $Im(e')/e'H$ and $Ker(e')^2/e'H$. Construct a $Z_{e'}$ invariant onto map $Y \to \mathbb{P}(U')^2$, deduce that $Z_{e'}$ acts on Y, and hence on $\mathcal{B}_{e'}$ with infinitely many orbits.

(5) $(*)$ Let $e \in sl(n)$ be a nilpotent element, its Jordan type defines a Young diagram λ. Given a flag $(V_0 \subset \cdots \subset V_n = V)$ invariant under e one can consider the Jordan types of $e|_{V_i}$ for $i = 1, \ldots, n$, this gives a nested collection of Young diagrams $\lambda_1 \subset \lambda_2 \subset \cdots \subset \lambda$, this collection corresponds to a standard tableau of shape λ: we place i at the unique box in λ_i which is not in λ_{i-1}. This defines a map from $\mathcal{B}_e$ to the set of standard tableaux of shape λ. Show that the preimage of every tableau is isomorphic to the affine space $\mathbb{A}^d$, $d = \dim(\mathcal{B}_e)$, the closure of this preimage is a component of $\mathcal{B}_e$, and this defines a bijection between the set of components and the set of standard tableaux of shape λ.

0.2. **Character sheaves.** The next few problems give basic examples of calculations with character sheaves.

(6) Let A be a connected abelian algebraic group over $\mathbb{F}_q$ (for example, the additive group $\mathbb{G}_a$ or the multiplicative group $\mathbb{G}_m$). The Artin-Schreier homomorphism $\alpha : A \to A$ is given by $\alpha = Fr - Id$ where Fr is the Frobenius morphism. Thus α is an etale covering with Galois group $A(\mathbb{F}_q)$, so $A(\mathbb{F}_q)$ acts on the direct image of the constant sheaf under α, inducing a decomposition as a direct sum indexed by characters of $A(\mathbb{F}_q)$. For a character χ of $A(\mathbb{F}_q)$ let $\mathcal{F}_\chi$ be the corresponding summand. Show that the trace of Frobenius function $f_{\mathcal{F}_\chi}$ equals χ.

(7) Consider the set U_{reg} of regular unipotent elements in $G = SL(n)$. Describe order n rank one local systems on U_{reg} and compute their trace of Frobenius functions.

[One can show that extending such a local system by zero to G one gets an irreducible perverse sheaf, in fact, a character sheaf].

(8) Let $G = GL(n)$. The principal series irreducible representations of $G(\mathbb{F}_q)$ (i.e. representations appearing in $\mathbb{C}[G/B]$) are in bijection with irreducible representations of the symmetric group. For an irr. rep ρ of S_n the character of the corresponding irr rep R_ρ of $G(\mathbb{F}_q)$ is the trace of Frobenius on the ρ-isotypic component of the Grothendieck-Springer sheaf (the direct image of the constant sheaf under the Grothendieck-Springer map $\tilde{G} \to G$ where $\tilde{G} \subset G \times \mathcal{B}$ is given by $\tilde{G} = \{(g,x) \mid g(x) = x\}$.)

Use this to compute explicitly dimensions of all principal series irr. reps of $GL(n)$ for $n \leq 4$. For any n describe the character of R_{sgn} and $R_{\tau\otimes sgn}$ restricted to the set of unipotent elements; here sgn is the sign character and τ is the standard reflection representation of S_n.

[You can use that for $G = GL(n)$ the restriction map $H^*(\mathcal{B}) \to H^*(\mathcal{B}_e)$ is onto for all e].

(9) Let G be a reductive group and $T \subset G$ a maximal torus, recall that conjugacy class of T corresponds to a conjugacy class in the Weyl group $w_T \in W$. For a character θ of $T(\mathbb{F}_q)$ Deligne and Lusztig defined a virtual representation $R_{T,\theta}$ of G (it is an irreducible representation if θ is generic). The character value of $R_{T,\theta}$ at a unipotent element $u \in G$ does not depend on θ, it equals $Tr([q] \circ w, H^*(\mathcal{B}_u))$, where $[q]$ is the operator acting by q^i on $H^{2i}(\mathcal{B}_u)$ and w is acting via the Springer representation.

Use this to compute dimensions of all DL representations of $GL(n)$ for $n \leq 4$, as well as their character values at a regular and subregular unipotent element u.

(10) If W is a Weyl group and ρ its irreducible representation that the formal degree d_ρ is a polynomial in q given by $d_\rho(q) = \sum_i [A_i : \rho] q^i$. Here $A = \mathbb{C}[\mathfrak{t}]/(\mathbb{C}[\mathfrak{t}]^W_+) \cong H^*(G/B)$ is the coinvariants algebra, A_i is its component of degree ρ and $[V : \rho] = \dim Hom(\rho, V)$ is the multiplicity.

Let W be the Weyl group of type C_2 (=B_2) a.k.a. the order 8 dihedral group (symmetries of the square). Let ρ_1, ρ_2, ρ_3 be its representations other than the sign and the trivial one, with ρ_1 being the 2-dimensional reflection representation, let d_1, d_2, d_3 be their formal degrees.

Compute d_i and check that the quantities $\frac{1}{2}(d_1+d_2+d_3)$, $\frac{1}{2}(d_1+d_2-d_3)$, $\frac{1}{2}(d_1+d_3-d_2)$ are the dimensions of the principal series irr. reps, while $\frac{1}{2}(d_1-d_2-d_3)$ is the dimension of the unipotent cuspidal representation as listed in the lecture.

(11) Let $G = \mathbf{G}(\mathbb{F}_q)$ be a finite Chevalley group with a Borel subgroup B and a character θ of $B = T \cdot U$ factoring through T. Let $H = \mathbb{C}[U \backslash G/U]^T$ be the space of functions invariant under the diagonal torus action and $H_\theta \subset \mathbb{C}[U \backslash G/U]$ be the subspace of functions transforming under θ with respect to both left and right action, let $Z = \mathbb{C}[G]^G$. We have maps $A : Z \to H$ and $C : H \to Z$ given, respectively, by left averaging over U and conjugation averaging over G/B.

(a) $G = SL(2)$. Let $\sigma \in \mathbb{C}[G]^G$ be the trace of Frobenius function for the local system $\mathcal{S}$ on the regular unipotent locus introduced in the lecture; thus $\sigma(1, a; 0, 1) = \pm 1$; where the sign is plus iff a is a square. Show that $A(\sigma) \in H_\theta$ for the quadratic character θ and compute it explicitly.

(b) Describe analogous categories of sheaves $\mathcal{H}$, $\mathcal{H}_\theta$, $\mathcal{Z}$ and functors $\mathcal{A}$, $\mathcal{C}$ acting on them.

(c) Describe the analogue of a) for sheaves, show that $\mathcal{A}(\mathcal{S})$ is an irreducible perverse sheaf.

(d) (*) Generalize this to $SL(n)$. In particular, let χ be an order n character of $\mathbb{F}_q^\times$ and consider the character $\chi_n = (1, \chi, \chi^2, \cdots, \chi^{n-1})$ of $(\mathbb{F}_q^\times)^n$. Let θ be the restriction of χ_n to the maximal torus of $SL(n)$. Show that $\mathcal{H}_\theta$ has n irreducible objects, for every such object $\mathcal{F}$ the sheaf $\mathcal{C}(\mathcal{F})$ is a sum of n distinct irreducible perverse sheaves. For each irreducible summand $\mathcal{G}$ in $\mathcal{C}(\mathcal{F})$ we have $\mathcal{A}(\mathcal{G}) \cong \mathcal{F}$.

(12) Let W be a Weyl group and $\tilde{W} = W \ltimes \Lambda$ be the (extended) affine Weyl group.

Let $[w]$ be a conjugacy class in W. Let $[\tilde{W}]_w$ be the set of lifting of w to a conjugacy class in $\tilde{W}$.

(a) Let $\delta = \det(1-w, \mathfrak{t})$. Show that $\#[\tilde{W}]_w = \infty$ if $\delta = 0$ and $\#[\tilde{W}]_w = |\delta|$ otherwise.

(b) If $\delta \neq 0$, construct a bijection between $\#[\tilde{W}]_w$ and the finite abelian group $T_{\mathbb{C}}^w$ (it may help to use the exponential isomorphism $T_{\mathbb{C}} \cong \mathfrak{t}/\Lambda$).

(c) Recall that two algebraic tori (say, over $\mathbb{C}$) T and T' are dual if their cocharacter lattices are dual. Let w be an automorphism of T with finitely many fixed points and w' the dual automorphism of the dual torus T'. Show that T^w and $(T')^{w'}$ are Pontryagin dual finite abelian groups.

(13) In this problem we consider the group $G = SL(2, F)$ for a local nonArchimedean field F with residue field $\mathbb{F}_q$. Let $\mathcal{B}$ be the Bruhat-Tits building of G, so $\mathcal{B}$ is an infinite tree whose nodes are in bijection with lattice in F^2 up to dilation where two nodes are connected if they are represented by lattices L, L' with $L' \subset L$, $L/L' \cong \mathbb{F}_q$. If v is a vertex of $\mathcal{B}$ we let G_v be its stabilizer in G and G_v^+ be its pro-p radical; for an edge e the groups G_e, G_e^+ are defined similarly. Thus $G_v/G_v^+ \cong SL(2, \mathbb{F}_q)$, $G_e/G_e^+ \cong \mathbb{F}_q^\times$.

(a) For $g \in G^{rs}$ the fixed point set $\mathcal{B}^g$ is either a ball or the set of points at distance $\leq r$ to a fixed infinite path in $\mathcal{B}$ depending on whether g is elliptic or split (here G^{rs} is the set of regular semisimple elements).

(b) Let (ρ, V) be an admissible smooth representation of G. For a node $v \in \mathcal{B}^g$ we write $\tau_\rho(g, v) = Tr(g, V^{G_v^+})$ and similarly for an edge.
For a compact $S \subset \mathcal{B}^g$ set

$$\tau_\rho(g, S) := \sum_{v \in S} Tr(g, V^{G_v^+}) - \sum_{e \subset S} Tr(g, V^{G_e^+}).$$

By a Theorem of [2] (see also [1]), if g is elliptic, $S = \mathcal{B}^g$ and V is generated by G_v^+ invariant vectors for a vertex v (i.e. ρ has depth zero) then $\tau_\rho(g, S)$ equals the character value $\chi_\rho(g)$. Use this to compute the character of the Steinberg module $V = C_c^\infty[\mathbb{P}_F^1]/\mathbb{C} \cdot 1_{\mathbb{P}_F^1}$; here C_c^∞ is the space of locally constant functions and $1_{\mathbb{P}_F^1}$ stands for the constant function.

(c) $(*)$ find a similar formula for a split element $g \in G^{rs}$.
[Notice that $\mathcal{B}^g$ has an action of a cyclic group $\mathbb{Z}$ in the centralizer of g with a compact quotient, so a natural guess is to let S be a fundamental domain for that action. Explain why this is not the right answer by evaluating this expression for the trivial representation.]

(14) According to standard heuristics (Harish-Chandra principle) characters share many properties with Fourier transform of delta distribution on a coadjoint orbit. In this problem we explore this for $SL(2)$

We write FT for Fourier transform.

(a) Let $G = SL(2, \mathbb{F}_q)$ $(p > 2)$, identify the set U of unipotent elements in G with nilpotent elements in $\mathfrak{g} = sl(2)$. Consider the principal series representation I_θ and the Deligne-Lusztig representation R_θ for a character θ of $T_s = (\mathbb{F}_{q^2}^\times)_1$. Compare restrictions of these representation to U with restriction of Fourier transform of characteristic functions of a split and non-split regular semisimple orbits respectively.

For the Steinberg representation $S = \mathbb{C}[\mathbb{P}^1_{\mathbb{F}_q}]/\mathbb{C} \cdot 1_{\mathbb{P}^1_{\mathbb{F}_q}}]$ find a function on the nilpotent cone whose Fourier transform restricted to nilpotent elements equals $\chi_S|_U$.

(b) We now switch to the local field $F = \mathbb{R}$. Consider $sl(2,\mathbb{R}) = \mathbb{R}^3$ and let X be one sheet of a two sheet hyperboloid a.k.a. as an ellitptic (co)adjoint orbit. Let X' be the second sheet of that hyperboloid, so $X \cup X'$ is the intersection of $sl(2,\mathbb{R})$ with an $SL(2,\mathbb{C})$ orbit.
Show that $FT(\delta_X)$ has full support while $FT(\delta_X - \delta_{X'})$ vanishes on an open set, describe this set.

(c) $(*)$ Compute $FT(\delta_X - \delta_{X'})$ explicitly by using an appropriate complex integral expression for $FT(\delta_X)$ and expressing $FT(\delta_X - \delta_{X'})$ as a residue. Convince yourself that the answer has a simpler form than that for $FT(\delta_X)$.

References

[1] R. Bezrukavnikov, *Homological properties of representations of p-adic groups related to geometry of the group at infinity,* https://arxiv.org/math/0406223

[2] P. Schneider, U. Stuhler, *Representation theory and sheaves on the Bruhat-Tits building*, Inst. Hautes Études Sci. Publ. Math. No. **85** (1997), 97–191.