HITCHIN MODULI SPACES AND RAMIFIED GEOMETRIC LANGLANDS

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Contents

Let G be a connected reductive group over $\mathbb{C}$. In the series of lectures, for each positive rational number $\nu = \frac{d}{m}$, we are going to construct some symplectic algebraic spaces $\mathcal{M}_\gamma$ with Lagrangian $\mathrm{Fl}_\gamma \subset \mathcal{M}_\gamma$. They are affine analogues of resolutions of Slodowy slices $\widetilde{\mathcal{S}}_e$ which admit Springer fibers $\mathcal{B}_e \subset \widetilde{\mathcal{S}}_e$ as conical Lagrangians. In the end of the lectures, we will formulate a version of ramified geometric Langlands for $\mathbb{P}^1$ using the moduli spaces $\mathcal{M}_\gamma$ and their non-abelian Hodge relatives.

1. Homogeneous elements

Analogues to the situation that Slodowy slices $\widetilde{\mathcal{S}}_e$ are parametrized by nilpotent elements $e \in \mathfrak{g}$, the affine analogues $\mathcal{M}_\gamma$ are parametrized by homogeneous elements $\gamma \in L\mathfrak{g} = \mathfrak{g}(\mathbb{C}((t)))$, which we are going to introduce in this section.

1.1. Slodowy slices.

We first recall the classical story of Slodowy slices.

Let $\mathfrak{g} = \operatorname{Lie} G$ and $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone. For each element $e \in \mathcal{N}$, by Jacobson-Morosov theorem, it extends uniquely to a $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{g}$ up to conjugation. The *Slodowy slice* is defined as $\mathcal{S}_e^{\mathfrak{g}} := e + \mathfrak{g}^f$ and $\mathcal{S}_e = \mathcal{S}_e^{\mathfrak{g}} \cap \mathcal{N}$.

Example 1.1. When $e = 0$, one has $S_e^{\mathfrak{g}} = \mathfrak{g}$ and $S_e = \mathcal{N}$.

Let $\mathcal{B} = \{\text{Borel subgroups of } G\}$ be the flag variety of G. For $B \in \mathcal{B}$, let N_B be the unipotent radical of B and $\mathfrak{n}_B = \operatorname{Lie} N_B$. Let $\widetilde{\mathcal{N}} = \{(x, B) | x \in \mathcal{N}, B \in \mathcal{B}, x \in \mathfrak{n}_B\}$. One has $\widetilde{\mathcal{N}} \cong T^*\mathcal{B}$ and the Springer resolution $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ defined by $\pi(x, B) = x$. Define the resolution of Slodowy slice $\widetilde{\mathcal{S}}_e := \mathcal{S}_e \times_{\mathcal{N}} \widetilde{\mathcal{N}}$ and the Springer fiber $\mathcal{B}_e = \widetilde{\mathcal{N}} \times_{\mathcal{N}} \{e\}$. The construction above can be summarized into the diagram

$$\begin{array}{ccccc} \mathcal{B}_e & \longrightarrow & \widetilde{S}_e & \longrightarrow & \widetilde{\mathcal{N}} \\ \downarrow & & \downarrow{\scriptstyle \pi_e} & & \downarrow{\scriptstyle \pi} \\ \{e\} & \longrightarrow & S_e & \longrightarrow & \mathcal{N} \end{array}$$

in which the two squares are Cartesian.

The following are standard results for Slodowy slices:

Fact 1.2. *The following are true:*

- *The map π_e is a resolution of singularities.*
- *$\widetilde{S}_e$ is a symplectic variety.*
- *$\mathcal{B}_e$ is a Lagrangian in $\widetilde{S}_e$.*

Example 1.3. Consider $\mathfrak{g} = \mathfrak{sl}_n$ and $e = \operatorname{diag}(J_{n-1}, 1)$ where J_{n-1} is the nilpotent Jordan block of size $n-1$. The element e lies in the subregular nilpotent orbit. The Springer fiber $\mathcal{B}_e$ is a chain of $\mathbb{P}^1$ with dual graph A_{n-1}. The symplectic surface $\widetilde{\mathcal{S}}_e$ is resolution of $\mathcal{S}_e$ with A_{n-1}-surface singularity.

There is a $\mathbb{G}_m$-action on objects constructed above. Regard $h : \mathbb{G}_m \to G_{\text{ad}}$. On $\mathcal{S}_e^{\mathfrak{g}} = e + \mathfrak{g}^f$, let $s \in \mathbb{G}_m$ acts by $s^2 \cdot \operatorname{Ad}_{h(s^{-1})}$. Since e has weight 2 under the action of $h(s)$ and $\mathfrak{g}^f$ has non-positive weights under the action of $h(s)$, the $\mathbb{G}_m$-action preserves and contracts $\mathcal{S}_e^{\mathfrak{g}}$ to $\{e\}$.

Note that the $\mathbb{G}_m$-action extends (induces) action on $\mathcal{S}_e$, $\widetilde{\mathcal{S}}_e$. It contracts $\mathcal{S}_e$ to $\{e\}$ and contracts $\widetilde{\mathcal{S}}_e$ to $\mathcal{B}_e$, making $\mathcal{B}_e$ a conical Lagrangian of $\widetilde{\mathcal{S}}_e$.

1.2. **Affine analogue.** Now we move to affine Lie algebras, which means changing from $\mathfrak{g}$ to $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}(\!(t)\!)$ and from $e \in \mathcal{N}$ to a topological nilpotent element $\gamma \in L\mathfrak{g}$. The Springer fiber $\mathcal{B}_e$ will be replaced by the affine Springer fiber Fl_γ.

First, we would like to find an analogue of the $\mathfrak{sl}_2$-triple (e, h, f), under which we can define a symplectic variety $\mathcal{M}_\gamma$ which is an analogue of $\widetilde{\mathcal{S}}_e$.

1.2.1. *Topologically nilpotent element.* Regardless of the $\mathfrak{sl}_2$-triple, the role of nilpotent elements in $\mathfrak{g}$ will be replaced by topological nilpotent elements in $L\mathfrak{g}$.

Example 1.4. When $\mathfrak{g} = \mathfrak{sl}_n$, topologically nilpotent elements are those elements $\gamma \in L\mathfrak{g}$ such that $\gamma^N \to 0$ in the t-adic topology when $N \to \infty$. Equivalently speaking, one require eigenvalues of γ to have positive valuations (i.e. eigenvalues lie in $\cup_{m \geq 1} \mathbb{C}(\!(t^{1/m})\!)$). Examples include $\gamma = \operatorname{diag}(a_1 t^{e_1}, \cdots, a_n t^{e_n})$ with $e_i > 0$ and $\gamma = \begin{pmatrix} & & & t \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix}$.

This motivates the following general definition:

Definition 1.5. Let G be a semisimple algebraic group. An element $\gamma \in L\mathfrak{g}$ is called *topologically nilpotent* if all eigenvalues of $\operatorname{ad}_\gamma : L\mathfrak{g} \to L\mathfrak{g}$ (as a linear map over $\mathbb{C}(\!(t)\!)$) have positive valuations.

1.2.2. *Homogeneous elements.* We would like to find analogues of the triple (e, h, f) for $\gamma \in L\mathfrak{g}$. From now on, we assume γ is regular semisimple as an element of $L\mathfrak{g}$ over $\mathbb{C}(\!(t)\!)$.

In finite-dimensional case, the element h can be regarded as a map $h : \mathbb{G}_m \to G_{\text{ad}}$ such that $\operatorname{Ad}_{h(s)} \cdot e = s^2 \cdot e$. As a first attempt in the affine case, one can try to find $\theta : \mathbb{G}_m \to LG_{\text{ad}}$ under which $\operatorname{Ad}_{\theta(s)} \cdot \gamma = s^d \cdot \gamma$ for some $d \in \mathbb{Z}$. However, such a map θ usually does not exist. For example, when $\mathfrak{g} = \mathfrak{sl}_n$, consider the characteristic polynomial $P_\gamma(x)$ of γ. If such θ exists, one has $P_\gamma(x) = P_{s^d \cdot \gamma}(x) = s^{nd} P_\gamma(s^{-d} x)$. This forces $P_\gamma(x) = x^n$, hence, γ is nilpotent.

One can remedy this as follows: Note that one has an extended action $LG_{\mathrm{ad}} \rtimes \mathbb{G}_m^{\mathrm{rot}}$ on $L\mathfrak{g}$ in which the second factor scales t (we denote this action by rot). We can look for $\theta : \mathbb{G}_m \to LG_{\mathrm{ad}} \rtimes \mathbb{G}_m^{\mathrm{rot}}$ and elements γ satisfying $\theta(s) \cdot \gamma = s^d \cdot \gamma$ instead.

Example 1.6. When $\mathfrak{g} = \mathfrak{sl}_n$, suppose $\theta = (\lambda, m) \in \mathrm{Hom}(\mathbb{G}_m, LG_{\mathrm{ad}}) \times X_*(\mathbb{G}_m)$. The condition above implies that $P_{\mathrm{rot}(s^m)\cdot\gamma}(x) = s^{nd} P_\gamma(s^{-d}x)$ for $s \in \mathbb{G}_m$.

For the element

$$\gamma = \begin{pmatrix} & & & t \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix},$$

one has $P_\gamma(x) = x^n \pm t$. Note that $P_{\mathrm{rot}(s^m)\cdot\gamma}(x) = x^n \pm s^m t$ and $s^{nd} P_\gamma(s^{-d}x) = x^n \pm s^{nd} t$. The condition above reads as $\frac{1}{n} = \frac{d}{m}$.

Definition 1.7. A regular semisimple element $\gamma \in L\mathfrak{g}$ is called *homogeneous* if there exists a group homomorphism $\theta = (\lambda, m) : \mathbb{G}_m \to LG_{\mathrm{ad}} \rtimes \mathbb{G}_m^{\mathrm{rot}}$ such that $\theta(s) \cdot \gamma = s^d \cdot \gamma$ for any $s \in \mathbb{G}_m$. We call $\nu = \frac{d}{m}$ the slope of γ.

The following gives an easy criterion for homogeneous element:

Fact 1.8. *Consider the Chevalley quotient $\mathfrak{c} = \mathfrak{g} /\!/ G$ and the map $\chi : \mathfrak{g} \to \mathfrak{c}$. An element $\gamma \in L\mathfrak{g}$ is homogeneous of slope $\gamma = \frac{d}{m}$ if and only if $\chi(\mathrm{rot}(s^m) \cdot \gamma) = s^d \cdot \chi(\gamma)$. Here $\mathbb{G}_m$-acts on $\mathfrak{c}$ via the standard weighted action such that $\chi : \mathfrak{g} \to \mathfrak{c}$ is $\mathbb{G}_m$-equivariant.*

Example 1.9. When $\mathfrak{g} = \mathfrak{sl}_n$, an element $\gamma \in L\mathfrak{g}$ is homogeneous of slope $\frac{d}{n}$ if and only if $P_\gamma(x) = x^n + at^d$ for $a \in \mathbb{C}^\times$.

We would like to address the following questions:

- How to construct homogeneous elements?
- How many homogeneous elements are there?

To do this, we restrict ourselves to consider $\theta : \mathbb{G}_m \to T_{\mathrm{ad}} \times \mathbb{G}_m^{\mathrm{rot}}$ in which $s \mapsto (\lambda(s), s^m)$ for $\lambda \in X_*(T_{\mathrm{ad}})$.

Note that θ induces an action fo $\mathbb{G}_m$ on $L\mathfrak{g}$, hence, a weight decomposition $L\mathfrak{g} = \widehat{\bigoplus}_{i\in\mathbb{Z}} (L\mathfrak{g})_{i/m}$ in which $(L\mathfrak{g})_{i/m} = \{\gamma \in L\mathfrak{g} | \theta(s) \cdot \gamma = s^i \cdot \gamma\}$. We denote $(L\mathfrak{g})^{\mathrm{rs}}_{i/m} \subset (L\mathfrak{g})_{i/m}$ to be the subset of regular semisimple elements. To see these weight subspaces more concretely, consider the root space decomposition $L\mathfrak{g} = \mathfrak{h} \oplus \widehat{\bigoplus}_{\alpha\in\Phi_{\mathrm{aff}}} \mathfrak{g}_\alpha$ where Φ_{aff} is the set of affine roots and $\mathfrak{h} = \mathrm{Lie}\, T$. For non-imaginary roots $\alpha = \overline{\alpha} + n\delta \in \Phi_{\mathrm{aff}}$, we have $(L\mathfrak{g})_\alpha = \mathfrak{g}_{\overline{\alpha}} \cdot t^n$. For imaginary roots $\alpha = n\delta$, we have $(L\mathfrak{g})_{n\delta} = \mathfrak{h} \cdot t^n$. This gives us $(L\mathfrak{g})_{i/m} = \widehat{\bigoplus}_{\langle\lambda/m,\overline{\alpha}\rangle+n=i/m} (L\mathfrak{g})_\alpha$. Viewing α as an affine linear function on $X_*(T)_\mathbb{R}$, one can write $(L\mathfrak{g})_{i/m} = \widehat{\bigoplus}_{\alpha(\lambda/m)=i/m} (L\mathfrak{g})_\alpha$. This gives explicit construction of (possibly not regular semisimple) homogeneous elements of slope $\frac{d}{m}$.

Now we study the existence of (regular semisimple) homogeneous elements of slope ν. The answer will be related to regular elements of the Weyl group W, which we introduce as follows:

Definition 1.10 ([Spr74]). Consider the action of Weyl group W on the Cartan subalgebra $\mathfrak{h}$. An element $w \in W$ is called *regular* if the action of w on $\mathfrak{h}$ has an eigenvector in the regular semisimple locus $\mathfrak{h}^{\mathrm{rs}} \subset \mathfrak{h}$. Moreover, the map

$$\{\text{regular conjugacy classes in } W\} \to \mathbb{Z}_{\geq 1}$$

given by

$$[w] \mapsto \mathrm{ord}(w)$$

is an injection, and we call the image *regular numbers* for W.

Example 1.11. For $\mathfrak{g} = \mathfrak{sl}_n$, the regular elements are those conjugate to $(12\cdots n)^d$ or $(12\cdots(n-1))^d$ for some d.

Example 1.12. When $\mathfrak{g}$ is a simple Lie algebra with simple reflections $\{s_1, \cdots, s_r\}$, the Coxeter element $w_{\mathrm{cox}} = s_1 \cdots s_r$ (which is well-defined up to conjugacy) is regular of order h_G (called the Coxeter number of G). When $G = E_8$, one has $h_G = 30$, and there are 12 regular conjugacy classes in W.

Theorem 1.13 ([RY14][OY16])**.** *The following are true:*

- *$L\mathfrak{g}$ has a homogeneous element of slope $\nu = \frac{d}{m}$ if and only if m is a regular number of W.*
- *When m is a regular number of W, consider $\theta : \mathbb{G}_m \to T_{\mathrm{ad}} \times \mathbb{G}_m^{\mathrm{rot}}$ defined by $s \mapsto (\check{\rho}(s), s^m)$ in which $\check{\rho} = \frac{1}{2}\sum_{\check{\alpha}\in\check{\Phi}_{>0}} \alpha \in X_*(T_{\mathrm{ad}})$. Then any homogeneous element of slope ν can be conjugated to an element in $(L\mathfrak{g})_\nu^{\mathrm{rs}}$.*

In the theorem above, the relation between the homogeneous element $\gamma \in L\mathfrak{g}$ and the regular element $w \in W$ can be seen as follows: For $\gamma \in L\mathfrak{g}$, consider $T_\gamma = C_G(\gamma)$ which is a maximal torus of G defined over $F = \mathbb{C}((t))$. By a standard Galois cohomology calculation, conjugacy classes of maximal tori of G defined over F are in one-to-one correspondence with conjugacy classes in W. One takes associated conjugacy class $[w] \subset W$ to be the conjugacy class corresponding to the maximal torus T_γ.

Example 1.14. When $G = \mathrm{Sp}_{2n} = \mathrm{Sp}(V,\omega)$ and consider slope $\nu = \frac{1}{2}$, the corresponding regular element is the longest element $w_0 \in W$. Assume $V = \mathrm{Span}(e_1, \cdots, e_n, f_n, \cdots, f_1)$ in which $\omega(e_i, f_i) = 1, \omega(e_i, e_j) = \omega(f_i, f_j) = 0$. The theorem above tells us that regular elements of slope $\frac{1}{2}$ can be conjugated to have the form $\gamma = \begin{pmatrix} & P \\ tQ & \end{pmatrix}$ in which P, Q are symmetric matrices in $\mathbb{C}$ in general position.

Homogeneous elements can be understood via finite-dimensional data as follows: For $\theta = (\lambda, m) \in X_*(T_{\mathrm{ad}} \times \mathbb{G}_m^{\mathrm{rot}})$, consider the evaluation map $\bigoplus_{i\in\mathbb{Z}} (L\mathfrak{g})_{i/m} = \mathfrak{g}(\mathbb{C}[t, t^{-1}]) \xrightarrow{\mathrm{ev}_1} \mathfrak{g}$. Consider the $\mathbb{Z}/m\mathbb{Z}$-grading on $\mathfrak{g}$ defined by $\mathfrak{g}_{i/m} = \{X \in \mathfrak{g} : \mathrm{Ad}_{\lambda(\zeta)} \cdot X = \zeta \cdot X, \zeta \in \mu_m\}$. The evaluation map above restricts to a map $(L\mathfrak{g})_{i/m} \xrightarrow{\mathrm{ev}_1} \mathfrak{g}_{i/m}$ which turns out to be an isomorphism. Then $\gamma \in (L\mathfrak{g})_{i/m}$ is regular semisimple if and only if $\overline{\gamma} := \mathrm{ev}_1(\gamma)$ is regular semisimple.

Example 1.15. Continuing with Example 1.14. In this case, one can take λ with $d\lambda = \mathrm{diag}(1, \cdots, 1, -1, \cdots, -1)$. Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2}$. One has $\mathfrak{g}_0 \cong \mathfrak{gl}_n$ and $\mathfrak{g}_{1/2} \cong \mathrm{Sym}^2(\mathrm{Std}_n) \oplus \mathrm{Sym}^2(\mathrm{Std}_n^*)$ as a representation of $\mathfrak{g}_0$.

To correspondence between homogeneous elements and regular elements in W can also be seen from finite dimensional data as follows: Consider the maximal torus $T_{\overline{\gamma}} = C_G(\overline{\gamma})$, then the $\mathbb{Z}/m\mathbb{Z}$-grading on $\mathfrak{g}$ induces a $\mathbb{Z}/m\mathbb{Z}$-grading $\mathfrak{h} \cong \mathfrak{t}_{\overline{\gamma}} = \mathrm{Lie}\, T_{\overline{\gamma}} = \bigoplus_{i\in\mathbb{Z}/m\mathbb{Z}} \mathfrak{h}_{i/m}$. This grading corresponds to an automorphism of $\mathfrak{h}$ of order m, which is given by the action of $w \in W$.

Definition 1.16. An homogeneous element γ (or a regular element $w \in W$) is called *elliptic* if $\mathfrak{h}^w = 0$.

1.3. Exercises.

Exercise 1.17. In this exercise, we study regular elements in W.

For element $w \in W$ and $\zeta \in \mathbb{C}^\times$, define $V(w, \zeta) = \{t \in \mathfrak{h} | wt = \zeta t\} \subset \mathfrak{h}$. Recall the following definitions:

- We say that w is *regular* (of order m) if $V(w, \zeta) \cap \mathfrak{h}^{\mathrm{rs}} \neq \varnothing$ for some primitive m-th root of unity $\zeta \in \mu_m$.
- We say that w is *elliptic* if $\mathfrak{h}^w = 0$.
- We say that m is a *regular number* of W if there exists a regular element of order m.
- We say that m is a *regular elliptic number* if there exists a regular elliptic element of order m.

Fix $w \in W$ a regular element of order m and $\zeta \in \mu_m$ a primitive m-th root of unity.

(1) Show that w has order m as an element of W, and it induces a free action of the cyclic group $\mathbb{Z}/m\mathbb{Z} \cong \langle w \rangle$ on Φ.
(2) When $m > 1$, show that there exists a choice of simple roots $S \subset \Phi$ under which $l(w) = |\Phi|/m$. Moreover, when w is elliptic, show that $l(w) \geq |\Phi|/m$ for any choice of S.
(3) Let $f_i \in \mathcal{O}(\mathfrak{c})$ be the homogeneous generators such that $\deg f_i = d_i$. Consider $\mathfrak{c}_{1/m} = \cap_{i, m\nmid d_i} V(f_i) \subset \mathfrak{c}$ where $V(f_i) \subset \mathfrak{c}$ is the vanishing locus of $f_i \in \mathcal{O}(\mathfrak{c})$. Show that $\chi|_{\mathfrak{h}}^{-1}(\mathfrak{c}_{1/m}) = \cup_{w'\in W} V(w', \zeta)$.
(4) Define $a(m) = |\{1 \leq i \leq r : m \mid d_i\}| = \dim \mathfrak{c}_{1/m}$. Show that $\chi|_{\mathfrak{h}}^{-1}(\mathfrak{c}_{1/m})$ is equi-dimensional of dimension $a(m)$, and W acts transitively on the set of irreducible components of $\chi|_{\mathfrak{h}}^{-1}(\mathfrak{c}_{1/m})$.
(5) Show that $\dim V(w, \zeta) = a(m)$. Conclude that any two regular elements of order m are conjugate in W.
(6) Show that the eigenvalues of w as an automorphism of $\mathfrak{h}$ are $\{\zeta^{1-d_i}\}_{1\leq i\leq r}$. (Hint: Consider the basis $\{e_i\}_{1\leq i\leq r}$ of $\mathfrak{h}$ consisting of eigenvectors of w. Assume $e_1 \in \mathfrak{h}^{\mathrm{rs}}$. Consider the Jacobian

$J = \det(\partial_{e_i} f_j)$. Show that $J(e_1) \neq 0$, which implies that there exists a permutation $\sigma \in S_r$ such that $(\partial_{e_i} f_{\sigma(i)})(e_1) \neq 0$ for any i.)

(7) For Weyl groups of type A and C, determine the regular numbers and single out the elliptic ones.

Exercise 1.18. Let $F = \mathbb{C}(\!(t)\!)$. In this exercise, we study regular semisimple homogeneous elements in $\mathfrak{g}(F)$ of slope $\nu = \frac{d}{m}$. Here $d, m \in \mathbb{Z}_{\geq 1}$ and $\gcd(d, m) = 1$.

Consider $\theta = (\check{\rho}, m) \in X_*(T_{\mathrm{ad}} \times \mathbb{G}_m^{\mathrm{rot}})$. Recall the Moy-Prasad grading on $\mathfrak{g}(F)$:

$$\mathfrak{g}(F)_{i/m} = \{X \in \mathfrak{g}(F) : \theta(s) \cdot X = s^i \cdot X \text{ for all } s \in \mathbb{G}_m\}.$$

This induces a $\frac{1}{m}\mathbb{Z}$-grading on $\mathfrak{c}(F)$:

$$\mathfrak{c}(F)_{i/m} = \{x \in \mathfrak{c} : s^i \cdot \mathrm{rot}(s^{-m})x = x \text{ for all } s \in \mathbb{G}_m\}.$$

This gives rise to Moy-Prasad subgroups $\mathbf{P}_{i/m} \subset LG$ such that $\mathrm{Lie}\,\mathbf{P}_{i/m} = \mathfrak{g}(F)_{\geq i/m}$.

Note that classifying semisimple elements in $\mathfrak{g}(F)$ is equivalent to study elements in $\mathfrak{c}(F)$. More precisely, the map between sets $\mathfrak{g}(F)^{\mathrm{ss}}/G(F) \to \mathfrak{c}(F)$ is bijective. The injectivity follows from [Ste75, Theorem 3.14] and [Ste65, Theorem 1.9]. The surjectivity follows from [Ste65, Theorem 1.7]. Therefore, we are reduced to study $\mathfrak{c}(F)^{\mathrm{rs}}_{\nu}$.

Recall the $\frac{1}{m}\mathbb{Z}/\mathbb{Z}$-grading on $\mathfrak{g}$ defined by

$$\mathfrak{g}_{i/m} = \{X \in \mathfrak{g} : \theta(s) \cdot X = s^i \cdot X \text{ for all } s \in \mu_m\}.$$

Define

$$\mathfrak{c}_{i/m} = \{x \in \mathfrak{c} : s^i \cdot x = x\}.$$

Then the Chevalley quotient map restricts to $\chi : \mathfrak{g}_{i/m} \to \mathfrak{c}_{i/m}$ for any $i \in \mathbb{Z}$. Note that $\mathfrak{c}_{i/m} = \mathfrak{c}_{\gcd(i,m)/m}$. Define $\overline{\nu} = \nu + \mathbb{Z} \in \frac{1}{m}\mathbb{Z}/\mathbb{Z}$.

(1) Show that evaluation at $t = 1$ induces isomorphisms $\mathrm{ev}_1 : \mathfrak{g}(F)_{i/m} \xrightarrow{\sim} \mathfrak{g}_{i/m}$ and $\mathrm{ev}_1 : \mathfrak{c}(F)_{i/m} \xrightarrow{\sim} \mathfrak{c}_{i/m}$ for any $i \in \mathbb{Z}$. Moreover, show that $\mathrm{ev}_1(\mathfrak{g}(F)^{\mathrm{rs}}_{i/m}) = \mathfrak{g}^{\mathrm{rs}}_{i/m}$ and $\mathrm{ev}_1(\mathfrak{c}(F)^{\mathrm{rs}}_{i/m}) = \mathfrak{c}^{\mathrm{rs}}_{i/m}$.

(2) Show that $\mathfrak{c}^{\mathrm{rs}}_{1/m}$ is non-empty if and only if m is a regular number of W. Therefore, a regular semisimple homogeneous element in $\mathfrak{g}(F)$ of slope ν exists if and only if m is a regular number of W.

(3) Assume m is a regular number, show that the map $\mathfrak{g}_{\overline{\nu}} \to \mathfrak{c}_{1/m}$ is surjective. Therefore, any regular semisimple homogeneous element in $\mathfrak{g}(F)$ of slope ν can be conjugated to an element in $\mathfrak{g}(F)^{\mathrm{rs}}_{\nu}$ by $G(F)$.

(4) For Weyl groups of type A and C, describe all possible homogeneous elements and the corresponding Moy-Prasad subgroups.

Exercise 1.19. This exercise studies invariant theory of $\mathfrak{g}_{\overline{\nu}}$ under the action of $G_0 = L_{\mathbf{P}} = \mathbf{P}_0/\mathbf{P}_{i/m}$.

(1) For an element $\overline{\gamma} \in \mathfrak{g}_{\overline{\nu}}$, show that $\overline{\gamma}$ is *polystable* (i.e. the orbit $G_0 \cdot \overline{\gamma}$ is closed) if and only if $\overline{\gamma}$ is semisimple as an element in $\mathfrak{g}$.

(2) For an element $\overline{\gamma} \in \mathfrak{g}_{\overline{\nu}}$, show that $\overline{\gamma}$ is *stable* (i.e. $\overline{\gamma}$ is polystable and $\mathrm{Stab}_{G_0}(\overline{\gamma})$ is finite) if and only if $\overline{\gamma} \in \mathfrak{g}^{\mathrm{rs}}_{\overline{\nu}}$ and m is elliptic.

(3) From now on, fix an element $\overline{\gamma} \in \mathfrak{g}^{\mathrm{rs}}_{\overline{\nu}}$, consider the centralizer

$$\mathfrak{t}_{\overline{\gamma}} = \mathfrak{z}_{\mathfrak{g}}(\overline{\gamma}) \subset \mathfrak{g}.$$

Define the *Cartan* subspace $\mathfrak{t}_{\overline{\gamma},\overline{\nu}} = \mathfrak{t}_{\overline{\gamma}} \cap \mathfrak{g}_{\overline{\nu}}$. Show that $\mathfrak{g}^{\mathrm{rs}}_{\overline{\nu}} \subset G_0 \cdot \mathfrak{t}_{\overline{\gamma},\overline{\nu}}$.

(4) Define the *little Weyl group* $W_m = N_{G_0}(\mathfrak{t}_{\overline{\gamma},\overline{\nu}})/\,\mathrm{Stab}_{G_0}(\overline{\gamma})$. Show that W_m naturally embeds into W. Moreover, the Chevalley quotient map induces a finite surjective map $\mathfrak{t}_{\overline{\gamma},\overline{\nu}} /\!\!/ W_m \to \mathfrak{c}_{\overline{\nu}}$.

(5) Show that the natural map $\mathfrak{t}_{\overline{\gamma},\overline{\nu}} /\!\!/ W_m \to \mathfrak{g}_{\overline{\nu}} /\!\!/ G_0$ is an isomorphism.

(6) For each regular number m associated to a Weyl group of type A and C, describe the Cartan subspace and little Weyl group W_m.

2. Affine Springer fibers

The theory of affine Springer fibers was introduced by Kazhdan–Lusztig in [KL88]. We now recall their definition.

2.1. **Generalities on affine Springer fibers.** Consider $\mathbf{I} \subset L^+G \subset LG$ in which $\mathbf{I}$ is the Iwahori subgroup and L^+G is the jet group. One can consider the affine flag variety $\mathrm{Fl} = LG/\mathbf{I}$ which can be equipped with structure of an ind-scheme. This is the affine analogue of the flag variety $\mathcal{B} = G/B$. When G is simply connected, one has $\mathrm{Fl} = \{\text{Iwahori subgroups of } LG\}$. Recall the Springer fibers $\mathcal{B}_e = \{gB \in G/B : \mathrm{Ad}_{g^{-1}}(e) \in \mathfrak{n}_B\}$, as an affine analogue, one makes the following definition:

Definition 2.1. The affine Springer fiber over $\gamma \in L\mathfrak{g}$ is

$$\mathrm{Fl}_\gamma = \{g\mathbf{I} : \mathrm{Ad}_{g^{-1}}(\gamma) \in \mathrm{Lie}\,\mathbf{I}^+\}.$$

Here $\mathbf{I}^+ = \mathrm{Ker}(\mathbf{I} \to T)$ is the pro-unipotent radical of $\mathbf{I}$.

Note that Fl_γ is non-empty if and only if γ is topologically nilpotent.

Fact 2.2. *When $\gamma \in L\mathfrak{g}$ is regular semisimple and topologically nilpotent, the affine Springer fiber Fl_γ is finite-dimensional.*

Example 2.3. Consider the case $\mathfrak{g} = \mathfrak{sl}_2$. The element $\gamma = \begin{pmatrix} t & \\ & -t \end{pmatrix}$ has slope 1. In this case, the affine Springer fiber Fl_γ is an infinite-chain of $\mathbb{P}^1$'s with dual graph equal to the universal covering of the affine Dynkin graph $\widetilde{A}_{n-1}$.

Example 2.4. For $\gamma = \begin{pmatrix} & t^2 \\ t & \end{pmatrix}$ which has slope $\frac{3}{2}$, one can show that Fl_γ is a union of two $\mathbb{P}^1$'s intersecting as a point (with dual graph A_2).

Example 2.5. When $\mathfrak{g} = \mathfrak{sl}_3$, the element $\gamma = \begin{pmatrix} & & 1 \\ t & & \\ & t & \end{pmatrix}$ has slope $\frac{2}{3}$. In this case, Fl_γ is a union of three $\mathbb{P}^1$'s intersecting at a common single point.

Example 2.6. For a simple Lie algebra $\mathfrak{g}$, consider slope $\nu = \frac{1}{h_G}$. One can take homogeneous element $\gamma = \sum_{i=0}^r x_i$ where $0 \neq x_i \in \mathfrak{g}_{\alpha_i}$ for $i \neq 0$ and $0 \neq x_0 \in t\mathfrak{g}_{-\theta}$ where $\theta \in \Phi_{>0}$ is the highest root. Then $\mathrm{Fl}_\gamma \cong \pi_0(\mathrm{Fl})$.

Theorem 2.7 (Special case of [Bez96], conjectured by [KL88])**.** *When γ is homogeneous of slope ν, one has*

$$\dim \mathrm{Fl}_\gamma = \frac{\nu|\Phi| - c_w}{2}$$

where $\nu = \frac{d}{m}$ for $\gcd(d, m) = 1$, $w \in W$ is a regular element of order m, $c_w = \dim \mathfrak{h}/\mathfrak{h}^w$.

An open locus of Fl_γ is controlled by the action of LT_γ. Here one considers the maximal torus $T_\gamma = C_G(\gamma) \subset G$ defined over F.

Example 2.8. Assuming $\gcd(d, n) = 1$, consider $\gamma = \begin{pmatrix} & & & t \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix}^d \in \mathfrak{sl}_n$. Let $E = F(\gamma)$ which is a degree n extension of F. One has $T_\gamma(F) = E^\times \cap \mathrm{SL}_n(F)$.

There is a natural action of LT_γ on Fl_γ by conjugation. It has an open orbit dense in all irreducible components. Moreover, this action induces an action of $\pi_0(LT_\gamma) = X_*(T)_w$ on the set of irreducible components of Fl_γ, which is a free action with finitely many orbits.

2.2. **Torus action.** When γ is homogeneous of slope $\nu = \frac{d}{m}$, the affine Springer fiber Fl_γ admits a $\mathbb{G}_m$-action: Consider the $\mathbb{G}_m$-action on LG by $g \mapsto \mathrm{Ad}_{\lambda(s)}(\mathrm{rot}(s^m) \cdot g)$. This action preserves $\mathbf{I}$, hence, induces an action on Fl, which further preserves $\mathrm{Fl}_\gamma \subset \mathrm{Fl}$ and gives the desired action.

We study the fixed point and contracting locus of this $\mathbb{G}_m$-action. Recall the Bruhat decomposition $\mathrm{Fl} = \cup_{w \in \widetilde{W} = X_*(T) \rtimes W} \mathbf{I}w\mathbf{I}/\mathbf{I}$. Let $\mathbf{P} \subset LG$ be the connected subgroup with $\mathrm{Lie}\,\mathbf{P} = (L\mathfrak{g})_{\geq 0}$ (called a parahoric subgroup). Moreover, one has subgroups $\mathbf{P}_{i/m} \subset \mathbf{P}$ where $\mathrm{Lie}\,\mathbf{P}_{i/m} = (L\mathfrak{g})_{\geq i/m}$ for $i \in \mathbb{Z}_{\geq 0}$. Note that the $\mathbb{G}_m$-action on LG contracts $\mathbf{P}$ to a Levi subgroup $L_\mathbf{P}$ which satisfies $\mathrm{Lie}\,L_\mathbf{P} = (L\mathfrak{g})_0$. In the parahoric version of Bruhat decomposition $\mathrm{Fl} = \cup_{\overline{w}\in W_{\mathbf{P}}\backslash\widetilde{W}} \mathbf{P}\overline{w}\mathbf{I}/\mathbf{I}$, one sees that the $\mathbb{G}_m$-action contracts each strata $\mathbf{P}\overline{w}\mathbf{I}/\mathbf{I}$ to $L_{\mathbf{P}}\overline{w}\mathbf{I}/\mathbf{I} = L_{\mathbf{P}}/L_{\mathbf{P}} \cap^{\overline{w}} \mathbf{I}$, which is a partial flag variety of $L_{\mathbf{P}}$. Therefore, the decomposition of Fl_γ into attracting loci is $\mathrm{Fl}_\gamma = \cup_{w\in W_{\mathbf{P}}\backslash\widetilde{W}} \mathrm{Fl}_\gamma \cap (\mathbf{P}\overline{w}\mathbf{I})/\mathbf{I}$, in which the stratum $\mathrm{Fl}_\gamma \cap (\mathbf{P}\overline{w}\mathbf{I})/\mathbf{I}$ contracts to $\mathrm{Fl}_\gamma \cap (L_{\mathbf{P}}\overline{w}\mathbf{I})/\mathbf{I}$. We define $\mathrm{Hess}_\gamma(\overline{w}) := \mathrm{Fl}_\gamma \cap (L_{\mathbf{P}}\overline{w}\mathbf{I})/\mathbf{I}$. These are called *Hessenberg varieties.*

Hessenberg varieties can be understood from the $\mathbb{Z}/m\mathbb{Z}$-grading on $\mathfrak{g}$. Indeed, $\mathrm{Hess}_\gamma(\overline{w}) = \{lL_{\mathbf{P}} \in L_{\mathbf{P}}/L_{\mathbf{P}} \cap^w \mathbf{I} : \mathrm{Ad}_{l^{-1}}(\overline{\gamma}) \in \mathfrak{g}_{\overline{\nu}} \cap^w \mathrm{Lie}\,\mathbf{I}^+\}$. These are like generalizations of Springer fibers, with the adjoint representation replaced by a general representation of G.

Remark 2.9. It is expected that the Springer fibers $\mathcal{B}_e$ can be paved by affine spaces. In contrast, this fails for affine Springer fibers. There is a famous example given by Bernstein in [KL88, Appendix] (see Exercise 2.14).

2.3. **Exercises.**

Exercise 2.10. Classify regular semisimple homogeneous elements γ for semisimple algebraic groups such that $\dim \mathrm{Fl}_\gamma = 1$. You may want to use the dimension formula 2.7. You can find the list of regular numbers for Weyl groups of exceptional type in [Spr74, §5.4].

Exercise 2.11. This exercise proves that Hessenberg varieties arsing as connected components of $\mathrm{Fl}_\gamma^{\mathbb{G}_m}$ are smooth projective.

Consider a reductive group L and a finite dimension representation $V \in \mathrm{Rep}(L)$. Fix a vector $v \in V_0$, a subspace $V_0 \subset V$, and a parabolic subgroup $Q \subset L$ which stabilizes $V_0 \subset V$. Define the associated *Hessenberg variety* to be

$$\mathrm{Hess}_v(Q \subset L, V_0 \subset V) = \{lQ/Q \in L/Q : l^{-1}v \in V_0\}.$$

For $\gamma \in \mathfrak{g}(F)_\nu^{\mathrm{rs}}$, recall that $\mathrm{Fl}_\gamma^{\mathbb{G}_m} = \coprod_{w\in W_{\mathbf{P}}\backslash\widetilde{W}} \mathrm{Hess}_\gamma(w)$ where $\mathrm{Hess}_\gamma(w) = (L_{\mathbf{P}}w\mathbf{I}/\mathbf{I}) \cap \mathrm{Fl}_\gamma$. Note that $\mathrm{Hess}_\gamma(w) = \mathrm{Hess}_\gamma(L_{\mathbf{P}} \cap \mathrm{Ad}_w \mathbf{I} \subset L_{\mathbf{P}}, \mathfrak{g}(F)_\nu \cap \mathrm{Ad}_w \mathrm{Lie}\,\mathbf{I} \subset \mathfrak{g}(F)_\nu)$.

1. Show that $\mathrm{Hess}_v(Q \subset L, V_0 \subset V)$ is smooth if the following condition is satisfied: For any $v' \in L \cdot v \cap V_0$, one has $\mathfrak{l} \cdot v' + V_0 = V$.
2. For any $\gamma' \in \mathfrak{g}(F)_\nu^{\mathrm{rs}} \cap \mathrm{Ad}_w \mathrm{Lie}\,\mathbf{I}$, show that $[\mathfrak{g}(F)_0, \gamma'] + \mathfrak{g}(F)_\nu \cap \mathrm{Ad}_w \mathrm{Lie}\,\mathbf{I} = \mathfrak{g}(F)_\nu$. Conclude that $\mathrm{Hess}_\gamma(w)$ is smooth projective for any $w \in W_{\mathbf{P}_\nu}\backslash W$. (Hint: You may first show the following: For any $\mathbb{G}_m \subset G_0$ and $\overline{\gamma} \in \mathfrak{g}_{\overline{\nu}}^{\mathrm{rs}}$, one has $[\mathfrak{g}_0, \overline{\gamma}] + \mathfrak{g}_{\overline{\nu}}^{\geq 0} = \mathfrak{g}_{\overline{\nu}}$. Here $\mathfrak{g}_{\overline{\nu}}^{\geq 0}$ is the part with non-negative weight with respect to the action of $\mathbb{G}_m$.)

Exercise 2.12. Consider $G = \mathrm{SL}_2$ with slope $\nu = \frac{1}{2} + k$ where $k \in \mathbb{Z}_{\geq 0}$. Take the homogeneous element $\gamma = \begin{pmatrix} & t^k \\ t^{k+1} & \end{pmatrix} \in \mathfrak{sl}_2(F)$.

1. Describe the fixed point locus $\mathrm{Fl}_\gamma^{\mathbb{G}_m}$.
2. For each connected component of $\mathrm{Fl}_\gamma^{\mathbb{G}_m}$, describe the corresponding attracting locus.
3. Describe the affine Springer fiber Fl_γ.

Exercise 2.13. Consider $G = \mathrm{Sp}_4$ and $\nu = \frac{1}{2}$. Take the homogeneous element in Example 1.14.

1. Show that Fl_γ can be identified with $\mathbb{P}^1$'s with dual graph $\widetilde{D}_4$.
2. Determine the cross-ratio of the four points on the central $\mathbb{P}^1$. (The answer should depend on P, Q)

Exercise 2.14. Consider $G = \mathrm{Sp}_6$ and $\nu = \frac{1}{2}$. Take the homogeneous element as in the previous exercise.

1. Show that there exists a unique non-rational connected component $E_\gamma \subset \mathrm{Fl}_\gamma^{\mathbb{G}_m}$, which is an elliptic curve.
2. Compute the j-invariant of E_γ. (The answer should depend on P, Q)
3. Show that there exists a unique non-rational irreducible component of Fl_γ, which is a $\mathbb{P}^1 \times \mathbb{P}^1$-fibration over E_γ.

3. Hitchin moduli spaces

In this section, we work over a complete smooth algebraic curve X over $\mathbb{C}$. We introduce the Hitchin moduli spaces $\mathcal{M}_\gamma$ attached to a homogeneous element $\gamma \in L\mathfrak{g}$ which are the main players in this lecture series.

3.1. **Classical story.** We first recall the classical story of Hitchin moduli spaces.

The standard Hitchin moduli stack is $\mathcal{M} = T^* \operatorname{Bun}_G$. Here, Bun_G is the moduli stack of (principal) G-bundles over X, which is a smooth Artin stack with $\dim \operatorname{Bun}_G = (g-1)\dim G$. Let g be the genus of X. For a point $[\mathcal{E}] \in \operatorname{Bun}_G$, deformation theory tells us

$$T_{[\mathcal{E}]} \operatorname{Bun}_G = H^1(X, \operatorname{Ad}(\mathcal{E}))$$

in which $\operatorname{Ad}(\mathcal{E}) = \mathcal{E} \times^G \mathfrak{g}$ is the adjoint vector bundle associated to $\mathcal{E}$. By Serre duality, one has

$$T^*_{[\mathcal{E}]} \operatorname{Bun}_G = H^0(X, \operatorname{Ad}^*(\mathcal{E}) \otimes \omega_X)$$

in which $\operatorname{Ad}^*(\mathcal{E}) = \mathcal{E} \times^G \mathfrak{g}^*$ is the coadjoint vector bundle.

When G is semisimple, one can identify $\mathfrak{g} \cong \mathfrak{g}^*$ via the Killing form. In this case, a point in $\mathcal{M}$ is given by a pair $(\mathcal{E}, \varphi)$ where

- $\mathcal{E} \in \operatorname{Bun}_G$,
- $\varphi \in H^0(X, \operatorname{Ad}(\mathcal{E}) \otimes \omega_X)$.

Such a pair is called a *G-Higgs bundle*, and φ is called a *G-Higgs field*.

The Hitchin moduli spaces $\mathcal{M}$ are some global avatars of affine Springer fibers Fl_γ. To see this relation, consider the affine Grassmanian $\operatorname{Gr} = LG/L^+G$. It admits a moduli interpretation

$$\operatorname{Gr} = \{(\mathcal{E}, \tau) : \mathcal{E} \text{ is a } G\text{-bundle on } D = \operatorname{Spec} \mathbb{C}[\![t]\!],\ \tau \text{ is a trivialization of } \mathcal{E} \text{ on } D^\times = \operatorname{Spec} \mathbb{C}(\!(t)\!)\}.$$

In the affine Grassmanian, one has the variant of affine Springer fiber $\operatorname{Gr}_\gamma = \{gL^+G \in LG/L^+G : \operatorname{Ad}_{g^{-1}}(\gamma) \in L^+\mathfrak{g}\}$ which has moduli interpretation

$$\operatorname{Gr}_\gamma = \{(\mathcal{E}, \tau) \in \operatorname{Gr} : \tau \text{ transforms } \gamma \text{ to a section of } \operatorname{Ad}(\mathcal{E}) \text{ on } D\}.$$

Note that LT_γ acts on Gr_γ by conjugation on LG. We have a map

$$\operatorname{Gr}_\gamma \to \{(\mathcal{E}, \varphi) : \mathcal{E} \text{ is a } G\text{-bundle on } D,\ \varphi \in H^0(D, \operatorname{Ad}(\mathcal{E}))\}$$

which realizes the former as a LT_γ-torsor over a substack of the later. This identifies the stack $[\operatorname{Gr}_\gamma / LT_\gamma]$ with the moduli of local Higgs bundles with the same characteristic polynomial as γ.

3.1.1. *Hitchin fibration.* An important feature of the Hitchin moduli space is that it is equipped with a map $f : \mathcal{M} \to \mathcal{A}$, where $\mathcal{A}$ is called the Hitchin base and $f : \mathcal{M} \to \mathcal{A}$ is called the Hitchin fibration.

Example 3.1. When $G = \operatorname{GL}_n$, the Hitchin base is $\mathcal{A} = \bigoplus_{i=1}^n H^0(X, \omega_X^{\otimes i})$ and the Hitchin map $f : \mathcal{M} \to \mathcal{A}$ is given by $(\mathcal{E}, \varphi) \mapsto$ characteristic polynomial of φ. More precisely, under the correspondence between GL_n-torsors and vector bundles, the pair $(\mathcal{E}, \varphi)$ corresponds to $(\mathcal{V}, \varphi : \mathcal{V} \to \mathcal{V} \otimes \omega_X)$ where $\mathcal{V}$ is a rank n vector bundle on X. Suppose the characteristic polynomial of φ is $y^n + a_1 y^{n-1} + \cdots + a_n$ where $a_i = \pm \operatorname{tr}(\wedge^i \varphi) \in H^0(X, \omega_X^{\otimes i})$, one defines $f(\mathcal{V}, \varphi) = (a_1, \cdots, a_n) \in \mathcal{A}$.

For a general semisimple group G, the Chevalley quotient has the form $\mathcal{O}(\mathfrak{g} /\!\!/ G) \cong \mathcal{O}(\mathfrak{g})^G = \mathbb{C}[f_1, \cdots, f_r]$ where f_i is of degree d_i and are homogeneous generators of $\mathcal{O}(\mathfrak{g} /\!\!/ G)$. Here r is the rank of G. We define the Hitchin base $\mathcal{A} = \prod_{i=1}^r H^0(X, \omega_X^{\otimes d_i})$. The Hitchin map $f : \mathcal{M} \to \mathcal{A}$ is given by $(\mathcal{E}, \varphi) \mapsto (f_1(\varphi), \cdots, f_r(\varphi))$.

Fact 3.2. *The map $f : \mathcal{M} \to \mathcal{A}$ is a Lagrangian fibration.*

In particular, this implies that $\dim \mathcal{A} = \dim \mathcal{M}/2 = \dim \operatorname{Bun}_G = (g-1)\dim G$.

Exercise 3.3. Check the above identity directly.

3.2. **Hitchin moduli space for homogeneous elements.** Now we consider Hitchin moduli spaces attached to homogeneous elements $\gamma \in L\mathfrak{g}$, which are moduli spaces of Higgs bundles over $\mathbb{P}^1$ with Iwahori level structure at 0 and deeper level structure ∞.

3.2.1. *Level at zero.* Given a curve X together with a point $0 \in X$, consider

$$\mathrm{Bun}_G(\mathbf{I}_0) = \{(\mathcal{E}, \mathcal{E}_0^B) : \mathcal{E} \in \mathrm{Bun}_G, \mathcal{E}_0^B \text{ is a } B\text{-reduction of } \mathcal{E}_0\}.$$

Then $T^* \mathrm{Bun}_G(\mathbf{I}_0)$ is the moduli space of triples $(\mathcal{E}, \mathcal{E}_0^B, \varphi)$ in which:

- $(\mathcal{E}, \mathcal{E}_0^B) \in \mathrm{Bun}_G(\mathbf{I}_0)$
- $\varphi \in H^0(X, \mathrm{Ad}(\mathcal{E}) \otimes \omega_X(\underline{0}))$ such that $\mathrm{res}_0(\varphi) \in \mathcal{E}_0^B \times^B \mathfrak{n}_B$.

Example 3.4. When $G = \mathrm{GL}_n$, giving a B-reduction of $\mathcal{V}_0$ is equivalent to choosing a full flag in $\mathcal{V}_0$. The condition $\mathrm{res}_0(\varphi) \in \mathcal{E}_0^B \times^B \mathfrak{n}_B$ amounts to requiring the residue of φ to be strictly upper triangular with respect to the flag.

3.2.2. *Level at* ∞. We motivate our choice of level structure at ∞ by looking back to the construction of Slodowy slices. Recall the Slodowy slice can be constructed as $\mathcal{S}_e^{\mathfrak{g}} = e + \mathfrak{g}^f \cong (e + \mathfrak{g}_{\leq 0})/G_{\leq -2}$. Here the element h in the $\mathfrak{sl}_2$-triple induces a $\mathbb{Z}$-grading on $\mathfrak{g}$ and $\mathrm{Lie}\, G_{\leq -2} = \mathfrak{g}_{\leq -2}$. As an affine analogue, for a homogeneous element $\gamma \in L\mathfrak{g}$ of slope ν, one considers $(\gamma + (L_\infty \mathfrak{g})_{\leq 0})/\mathbf{K}_\gamma$ in which $\mathbf{K}_\gamma = (L_\infty G)_{\leq -\nu} \cdot (L_\infty T_\gamma)_{<0}$. Here, L_∞ is the loop construction with $\mathbb{C}(\!(t)\!)$ replaced by $\mathbb{C}(\!(t^{-1})\!)$, the subgroups $(L_\infty G)_{\leq -\nu}$ and $(L_\infty T_\gamma)_{<0}$ are defined by the $\mathbb{G}_m$-grading on $L\mathfrak{g}$. This suggests us to choose $\mathbf{K}_\gamma$-level structure at ∞.

In classical story, one has a Cartesian square

$$\begin{array}{ccc} \widetilde{\mathcal{S}}_e & \longrightarrow & \widetilde{\mathcal{N}} \\ \downarrow & & \downarrow \\ \mathcal{S}_e^{\mathfrak{g}} & \longrightarrow & \mathfrak{g} \end{array} .$$

In the affine setting, we have analogue of part of the square

$$\begin{array}{ccc} \mathcal{M}_\gamma & \xrightarrow{\mathrm{res}_0} & \mathfrak{n}_B/B \\ \downarrow{\scriptstyle \mathrm{ev}_\infty} & & \\ (\gamma + (L_\infty \mathfrak{g})_{\leq 0})/\mathbf{K}_\gamma & & \end{array} .$$

Definition 3.5. We define Hitchin moduli space $\mathcal{M}_\gamma$ attached to a homogeneous element $\gamma \in L\mathfrak{g}$ of slope ν to be the moduli stack of quadruples $(\mathcal{E}, \mathcal{E}_0^B, \mathcal{E}_\infty^{\mathbf{K}_\gamma}, \varphi)$ in which:

- $\mathcal{E}$ is a G-bundle on $\mathbb{P}^1$,
- $\mathcal{E}_0^B$ is a B-reduction of $\mathcal{E}_0$,
- $\mathcal{E}_\infty^{\mathbf{K}_\gamma}$ is a $\mathbf{K}_\gamma$-level structure of $\mathcal{E}$ at $\infty \in \mathbb{P}^1$,
- $\varphi \in H^0(\mathbb{P}^1 \backslash \{0, \infty\}, \mathrm{Ad}(\mathcal{E}) \otimes \omega_X)$ satisfies:
  - φ has simple pole at $0 \in \mathbb{P}^1$ with $\mathrm{res}_0(\varphi) \in \mathcal{E}_0^B \times^B \mathfrak{n}_B$,
  - Under a (or equivalently, any) trivialization of $\mathcal{E}$ together with the level structure $\mathcal{E}_\infty^{\mathbf{K}_\gamma}$, we have
$$\varphi|_{D_\infty^\times} \in (\gamma + (L_\infty \mathfrak{g})_{\leq 0}) dt/t.$$

Example 3.6. When $\nu = 1$, one can take $\gamma = \gamma_0 \cdot t$ for $\gamma \in \mathfrak{h}^{\mathrm{rs}}$. In this case, one has $\mathbf{K}_\gamma = \mathrm{Ker}(G(\mathbb{C}[\![t^{-1}]\!]) \xrightarrow{\mathrm{ev}_\infty} G)$. The Hitchin moduli $\mathcal{M}_\gamma$ classifies quadruples $(\mathcal{E}, \mathcal{E}_0^B, \tau_\infty, \varphi)$ in which

- $(\mathcal{E}, \mathcal{E}_0^B)$ is the same as before,
- τ_∞ is a trivialization of $\mathcal{E}_\infty$,
- $\varphi \in H^0(\mathbb{P}^1, \mathrm{Ad}(\mathcal{E}) \otimes \omega_{\mathbb{P}^1}(\underline{0} + 2\underline{\infty}))$ such that:
  - $\mathrm{res}_0(\varphi)$ satisfies the same condition as before,
  - Under the trivialization τ_∞, one has
$$\varphi = (\gamma_0 t + \text{higher order terms}) dt/t.$$

3.3. **Properties of the Hitchin moduli.** We would like to address the following questions concerning $\mathcal{M}_\gamma$:

(1) The Hitchin fibration of $\mathcal{M}_\gamma$,
(2) The $\mathbb{G}_m$-action on $\mathcal{M}_\gamma$,
(3) The relation between Fl_γ and $\mathcal{M}_\gamma$,
(4) The symplectic structure on $\mathcal{M}_\gamma$.

3.3.1. *Slope one case.* Now we consider the case $\nu = 1$.

We start from studying the Hitchin fibration. In this case,

$$f_i(\varphi) \in H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}(2\underline{\infty} + \underline{0})^{\otimes d_i}) \cong H^0(\mathbb{P}^1, \mathcal{O}(\underline{\infty})^{\otimes d_i}) = H^0(\mathbb{P}^1, \mathcal{O}(d_i)).$$

Here we are using the trivialization of $\omega_{\mathbb{P}^1}(\underline{0} + \underline{\infty})$ given by the section dt/t. In other word, one can regard $f_i(\varphi) \in \mathbb{C}[t]_{\deg \leq d}$. Moreover, the condition at 0 implies that the constant term of $f_i(\varphi)$ is zero, while the condition at ∞ implies that the leading coefficient of $f_i(\varphi)$ is $f_i(\gamma_0)$. Therefore, we can define the Hitchin base to be

$$\mathcal{A}_\gamma = \prod_{i=1}^{r} \mathbb{C}[t]_{\deg \,\leq\, d_i, \text{ leading coefficient } = f_i(\gamma_0), \text{ constant coefficient } = 0}.$$

The space $\mathcal{A}_\gamma$ is an affine space of dimension $\sum_{i=1}^r (d_i - 1) = \dim \mathcal{B}$. We get the desired Hitchin fibration $f_\gamma : \mathcal{M}_\gamma \to \mathcal{A}_\gamma$.

Example 3.7. When $G = \mathrm{GL}_n$, Hitchin fibrations can be understood via spectral curves. In this case, the moduli stack $\mathcal{M}_\gamma$ classifies quadruples $(\mathcal{V}, F_\bullet, \tau_\infty, \varphi)$ in which

- $\mathcal{V}$ is a rank n vector bundle on $\mathbb{P}^1$,
- $F_\bullet$ is a full flag of $\mathcal{V}_0$,
- τ_∞ is a trivialization of $\mathcal{V}_\infty$,
- $\varphi : \mathcal{V} \to \mathcal{V} \otimes \omega_{\mathbb{P}^1}(2 \cdot \underline{\infty} + \underline{0}) \cong \mathcal{V}(\underline{\infty})$.

Fix a point $a = (a_1, \cdots, a_n) \in \mathcal{A}_\gamma$ where $a_i \in H^0(\mathbb{P}^1, \mathcal{O}(d_i))$. The equation $y^n + a_1 y^{n-1} + \cdots a_n = 0$ defines a curve $Y_a \subset \mathrm{Tot}(\mathcal{O}(1))$ equipped with the natural projection $p_a : Y_a \to \mathbb{P}^1$. The reduced structure of the fiber $p_a^{-1}(0)$ is a single point, and around ∞ the curve Y_a is cut out by the charcteristic polynomial of γ_0. In this case, $f_\gamma^{-1}(a)$ is the moduli space of triples $(\mathcal{L}, F_\bullet, \tau_\infty)$ in which:

- $\mathcal{L} \in \overline{\mathrm{Pic}}(Y_a)$. Here $\overline{\mathrm{Pic}}(Y_a)$ is the compactified Jacobian of Y_a which classifies torsion-free sheaves on Y_a generically of rank 1.
- $F_\bullet$ is a complete flag of $(p_{a,*}\mathcal{L})_0$.
- τ_∞ is a system of basis of $\mathcal{L}|_{\infty'}$ where ∞' runs over points above ∞.

Here, for each triple $(\mathcal{L}, F_\bullet, \tau_\infty)$, the corresponding Higgs bundle is $(\mathcal{V} = p_{a,*}\mathcal{L}, \varphi = y \cdot -, F_\bullet, \tau_\infty)$.

For $G = \mathrm{SL}_n$, one further adds the data of a trivialization $\det \mathcal{V} \cong \mathcal{O}$ compatible with τ_∞ under which $\mathrm{tr}(\varphi) = 0$.

Now we study the $\mathbb{G}_m$-action on the Hitchin moduli space $\mathcal{M}_\gamma$. Note that there is special point $a_\gamma = (f_1(\gamma_0)t^{d_1}, \cdots, f_r(\gamma_0)t^{d_r}) \in \mathcal{A}_\gamma$. There is a $\mathbb{G}_m$-action on $\mathcal{A}_\gamma$ contracting the entire space to $a_\gamma \in \mathcal{A}_\gamma$ defined such that $s \in \mathbb{G}_m$ acts by $s \cdot \mathrm{rot}(s^{-1})$. There is a compatible $\mathbb{G}_m$-action on $\mathcal{M}_\gamma$ by rotation via s^{-1} and scaling the Higgs field by $(\varphi \mapsto s \cdot \varphi)$.

In this case, one can construct a map $\mathrm{Fl}_\gamma \to \mathcal{M}_\gamma$, which we spell out explicitly in the following example:

Example 3.8. Continuing the Example 3.7, the spectral curve Y_{a_γ} is a union of n-copies of $\mathbb{P}^1$ intersecting at a point (this point lies over $0 \in \mathbb{P}^1$). There is a natural map $\mathrm{Fl}_\gamma \to f_\gamma^{-1}(a_\gamma)$ which is a bijection on $\mathbb{C}$-points defined as follows: Recall that Fl_γ classifies a periodic chain of $\mathcal{O}_F$-lattices $\{\Lambda_\bullet\}$ in F^n with $\gamma \cdot \Lambda_i \subset \Lambda_{i-1}$. Given a point $\{\Lambda_\bullet\} \in \mathrm{Fl}_\gamma$, one can glue $\Lambda_\bullet$ with $\mathcal{O}_{Y_{a_\gamma}}|_{\mathbb{P}^1 \setminus \{0\}}$ and equip it with the canonical trivialization at ∞. This defines a point in $f_\gamma^{-1}(a_\gamma)$. This procedure defines a map $\mathrm{Fl}_\gamma \to f_\gamma^{-1}(a_\gamma)$.

Now we come to the symplectic structure on $\mathcal{M}_\gamma$. Recall the construction of Hamiltonian reduction: For an algebraic group H acting on a symplectic variety X equipped with a H-equivariant moment map $\mu : T^*X \to \mathfrak{h}^*$. For any $\zeta \in \mathfrak{h}^{*,H}$, one can consider $T^*X /\!\!/_\zeta H := [\mu^{-1}(\zeta)/H]$. When $\zeta = 0$, one has $T^*X /\!\!/_0 H = T^*(X/H)$. Varieties obtained via Hamiltonian reductions are equipped with induced symplectic structures.

In our case, consider

$$\mathbf{K}_1 = \mathrm{Ker}(G(\mathbb{C}[\![t^{-1}]\!]) \xrightarrow{\mathrm{ev}_\infty} G))$$

and

$$\mathbf{K}_2 = \mathrm{Ker}(G(\mathbb{C}[\![t^{-1}]\!]) \xrightarrow{\bmod t^{-2}} G(\mathbb{C}[t^{-1}]/(t^{-2}))).$$

Consider the map $\mathrm{Bun}_G(\mathbf{I}_0, \mathbf{K}_2) \to \mathrm{Bun}_G(\mathbf{I}_0, \mathbf{K}_1)$ which is a $\mathfrak{g} \cong \mathbf{K}_2/\mathbf{K}_1$-torsor. Consider the natural moment map $\mu : T^* \mathrm{Bun}_G(\mathbf{I}_0, \mathbf{K}_2) \to \mathfrak{g}^* \cong \mathfrak{g}$, one has the following:

Fact 3.9. *We have a natural isomorphism $T^*\operatorname{Bun}_G(\mathbf{I}_0,\mathbf{K}_2)/\!\!/_{\gamma_0}\mathfrak{g}\cong\mathcal{M}_\gamma$. In particular, this equips $\mathcal{M}_\gamma$ with a symplectic structure.*

3.3.2. *General case.* Now we come to the general slope ν. Most of arguments in the slope one case generalizes directly. Only the last part (the symplectic structure on $\mathcal{M}_\gamma$) acquires a significant generalization.

To see the symplectic structure on $\mathcal{M}_\gamma$, we first recall how one sees the symplectic structure on $\widetilde{\mathcal{S}}_e$. One achieves this by writing $\widetilde{\mathcal{S}}_e = T^*X/\!\!/_\zeta H$. Indeed, one can take $X=\mathcal{B}$. When e is even (i.e. $\mathfrak{g}_{\mathrm{odd}}=0$), one takes $H=G_{\le -2}$. Regard e as an element in $\mathfrak{g}^*_{\le -2}$ by $\langle -,e\rangle$, one has $\widetilde{\mathcal{S}}_e = T^*\mathcal{B}/\!\!/_e G_{\le -2}$. For general e, consider the symplectic pairing on $\mathfrak{g}_{-1}$ given by $\langle[-,-],e\rangle$. Taking a Lagrangian subspace $\mathfrak{m}\subset\mathfrak{g}_{-1}$, we can take H such that $\operatorname{Lie}H=\mathfrak{g}_{\le -2}+\mathfrak{m}$ and arrives at $\widetilde{S}_e = T^*\mathcal{B}/\!\!/_e H$.

We would like to find the affine analogue of the above argument. Heuristically speaking, this means that we would like to find a subgroup $\mathbf{J}_\gamma\subset L_\infty G$ such that $(\gamma+(L_\infty\mathfrak{g})_{\le 0})/\mathbf{K}_\gamma\cong(\gamma+(\operatorname{Lie}\mathbf{J}_\gamma)^\perp)/\mathbf{J}_\gamma$ in which one regards γ as an homomorphism $\gamma:\mathbf{J}_\gamma\to\mathbb{C}$. After that, one can write $\mathcal{M}_\gamma = T^*\operatorname{Fl}_\gamma/\!\!/_\gamma\mathbf{J}_\gamma$. However, this involves infinite-dimensional geometry, which we would like to avoid. To work with finite-dimensional geometry, we do the following modification: We look for a pair of subgroups $\mathbf{J}'_\gamma\lhd\mathbf{J}_\gamma\subset LG$ such that γ can be regarded as a character $\mathbf{J}'_\gamma/\mathbf{J}_\gamma\to\mathbb{C}$. In this case, the moduli space $\operatorname{Bun}_G(\mathbf{I}_0,\mathbf{J}'_\gamma)$ is equipped with an action by $\mathbf{J}'_\gamma/\mathbf{J}_\gamma$. We expect $\mathcal{M}_\gamma\cong T^*\operatorname{Bun}_G(\mathbf{I}_0,\mathbf{J}'_\gamma)/\!\!/_\gamma(\mathbf{J}'_\gamma/\mathbf{J}_\gamma)$.

Motivated by the finite-dimensional case, we can take $\mathbf{J}_\gamma=(L_\infty G)_{\le-\nu/2}\cdot(L_\infty T)_{<0}$ when $(L_\infty\mathfrak{g})_{-\nu/2}=0$. In general, take a Lagrangian subspace $\mathfrak{m}\subset(L_\infty\mathfrak{g})_{-\nu/2}/(L_\infty\mathfrak{t}_\gamma)_{-\nu/2}$ and take $\mathbf{J}_\gamma$ to be the preimage of $\mathfrak{m}$ under the natural quotient map $(L_\infty G)_{\le-\nu/2}\cdot(L_\infty T_\gamma)_{<0}\to(L_\infty\mathfrak{g})_{-\nu/2}/(L_\infty\mathfrak{t}_\gamma)_{-\nu/2}$. Viewing $\gamma\in\operatorname{Lie}\mathbf{J}^*_\gamma$, by the choice of $\mathbf{J}_\gamma$, this functional integrates to a map $\gamma:\mathbf{J}_\gamma\to\mathbb{G}_{\mathrm{a}}$. Therefore, we can take $\mathbf{J}'_\gamma=\operatorname{Ker}(\gamma:\mathbf{J}_\gamma\to\mathbb{G}_{\mathrm{a}})$. As a generalization of Fact 3.9, one shows that $\mathcal{M}_\gamma\cong T^*\operatorname{Bun}_G(\mathbf{I}_0,\mathbf{J}'_\gamma)/\!\!/_\gamma\mathbb{G}_{\mathrm{a}}$. This defines the symplectic structure on $\mathcal{M}_\gamma$.

Example 3.10. When $\nu=2$ and $\gamma=\gamma_0\cdot t^2$ for $\gamma_0\in\mathfrak{h}^{\mathrm{rs}}$. We can choose $\mathfrak{m}\subset\mathfrak{g}/\mathfrak{h}$ to be the image of the Borel subalgebra $\mathfrak{b}\subset\mathfrak{g}$. In this case, we have $\mathbf{K}_\gamma=\mathbf{K}_2\cdot(L_\infty T)_{\le -1}$ and $\mathbf{J}_\gamma=\{1+\mathfrak{b}t^{-1}+\cdots\}\subset L_\infty G$.

This explains the symplectic structure on $\mathcal{M}_\gamma$. Now we come to the description of Hitchin base $\mathcal{A}_\gamma$ in the general case.

The Hitchin base for γ is a subspace $\mathcal{A}_\gamma\subset\prod_{i=1}^r H^0(\mathbb{P}^1,\mathcal{O}([d_i\nu]))$ described as follows: An element $a=(a_1,\cdots,a_r)\in\prod_{i=1}^r H^0(\mathbb{P}^1,\mathcal{O}([d_i\nu]))$ lies in the subspace $\mathcal{A}_\gamma$ if and only if

- Each a_i (regarded as a polynomial in t) has zero constant term.
- Each a_i has leading term equals to that of $f_i(\gamma)$, and other terms have degree $\le(d_i-1)\nu$.

Theorem 3.11. *Suppose $\gamma\in L\mathfrak{g}$ is a homogeneous element of slope $\nu>0$.*

(1) The Hitchin moduli $\mathcal{M}_\gamma$ is a smooth algebraic space with a canonical symplectic structure.
(2) The map $f_\gamma:\mathcal{M}_\gamma\to\mathcal{A}_\gamma$ is a Lagrangian fibration.
(3) When γ is elliptic, f_γ is proper.
(4) There is a compatible $\mathbb{G}_m$-action on $\mathcal{M}_\gamma$ and $\mathcal{A}_\gamma$ contracting $\mathcal{A}_\gamma$ to a single point $a_\gamma\in\mathcal{A}_\gamma$.
(5) There is a natural map $\operatorname{Fl}_\gamma\to f_\gamma^{-1}(a_\gamma)$ which is a homeomorphism.
(6) The natural restriction map induces an isomorphism $H^*(\mathcal{M}_\gamma)\cong H^*(\operatorname{Fl}_\gamma)$.

Example 3.12. For $G=\mathrm{SL}_2$ and $\nu=\frac{3}{2}$. The affine springer fiber Fl_γ is isomorphic to two $\mathbb{P}^1$'s intersect at a point, while the special Hitchin fiber $f_\gamma^{-1}(a_\gamma)$ is isomorphic to two $\mathbb{P}^1$'s tangent at a point. This is an example that Fl_γ and $f_\gamma^{-1}(a_\gamma)$ are homeomorphic but not isomorphic.

Example 3.13. For $G=\mathrm{SL}_2$ and $\nu=1$, the special fiber $f_\gamma^{-1}(a_\gamma)$ is isomorphic to the affine Springer fiber Fl_γ which is an infinite chain of $\mathbb{P}^1$'s, while the generic fiber of f_γ is isomorphic to $\mathbb{G}_m$.

Example 3.14. As an evidence for the theorem, note that $\dim\mathcal{A}_\gamma=\sum_{i=1}^r[(d_1-1)\nu]$ and $\dim\operatorname{Fl}_\gamma=\frac{\nu|\Phi|-c_w}{2}$. The theorem implies that $\sum_{i=1}^r[(d_1-1)\nu]=\dim\mathcal{A}_\gamma=\dim\operatorname{Fl}_\gamma=\frac{\nu|\Phi|-c_w}{2}$. As a first approximation, this would require $\sum_{i=1}^r(d_i-1)=\frac{|\Phi|}{2}$, which is easy to check.

3.4. **Exercises.**

Exercise 3.15. Consider $G=\mathrm{SL}_2$ with homogeneous element $\gamma=\begin{pmatrix}t & \\ & -t\end{pmatrix}$ of slope 1.

(1) Describe $\mathcal{A}_\gamma$.
(2) Show that $f_\gamma^{-1}(a_\gamma)$ is isomorphic to an infinite chain of $\mathbb{P}^1$.
(3) Show that the generic fiber of f_γ is isomorphic to $\mathbb{G}_m$.

Exercise 3.16. Consider $G = \mathrm{SL}_2$ with homogeneous element $\gamma = \begin{pmatrix} & t \\ t^2 & \end{pmatrix}$ of slope $\frac{3}{2}$.

(1) Describe $\mathcal{A}_\gamma$.
(2) Show that $f_\gamma^{-1}(a_\gamma)$ is isomorphic to two $\mathbb{P}^1$'s tangent at a point (i.e. has equation $(y - x^2)y = 0$ locally around the intersection point).
(3) Show that the generic fiber of f_γ is isomorphic to the elliptic curve with complex multiplication by $\mathbb{Z}[i]$.

Exercise 3.17. Consider $G = \mathrm{SL}_3$ with homogeneous element $\gamma = \begin{pmatrix} & & 1 \\ t & & \\ & t & \end{pmatrix}$ of slope $\frac{2}{3}$.

(1) Describe $\mathcal{A}_\gamma$.
(2) Show that $f_\gamma^{-1}(a_\gamma)$ is isomorphic to three $\mathbb{P}^1$'s intersect pairwise-transversally at a single point (i.e. has equation $(y - x)xy = 0$ locally at the intersection point).
(3) Show that the generic fiber of f_γ is isomorphic to the elliptic curve with complex multiplication by $\mathbb{Z}[\omega]$.

Exercise 3.18. Consider $G = \mathrm{Sp}_4$ with homogeneous element $\gamma = \begin{pmatrix} & P \\ tQ & \end{pmatrix}$ of slope $\frac{1}{2}$ as in Exercise 2.13. Show that the generic fiber of f_γ is isomorphic to the elliptic curve which is the double cover of the central $\mathbb{P}^1$ ramified at the four points in Exercise 2.13(2).

Exercise 3.19. Repeat Exercise 3.18 for $G = \mathrm{SO}_8$ with homogeneous element of slope $\frac{1}{4}$.

4. Non-abelian Hodge theory

In this section, we study the non-abelian Hodge companions of the Hitchin moduli space $\mathcal{M}_\gamma$.

4.1. **Classical story.** In non-abelian Hodge theory, one is interested in three different moduli spaces $\mathcal{M}_{\mathrm{Dol}}$, $\mathcal{M}_{\mathrm{dR}}$, and $\mathcal{M}_{\mathrm{Bet}}$:

- The moduli stack $\mathcal{M}_{\mathrm{Dol}}$ is the Dolbeaut moduli space (called Hitchin moduli space before), which is the moduli space of Higgs bundles.
- The moduli stack $\mathcal{M}_{\mathrm{dR}}$ is the de Rham moduli space, which is moduli space of vector bundle with connection.
- The moduli stack $\mathcal{M}_{\mathrm{Bet}}$ is the Betti moduli space, which is the moduli space of homomorphisms $\pi_1(X) \to G$.

These moduli spaces are related as follows:

- The stack $\mathcal{M}_{\mathrm{dR}}$ is a deformation of $\mathcal{M}_{\mathrm{Dol}}$. More precisely, there is a (family of) moduli spaces $\lambda : \mathcal{M}_{\mathrm{Hod}} \to \mathbb{A}^1$ called Hodge moduli space, which is the moduli space of λ-connections. It satisfies $\lambda^{-1}(0) \cong \mathcal{M}_{\mathrm{Dol}}$ and $\lambda^{-1}(1) \cong \mathcal{M}_{\mathrm{dR}}$.
- The Riemann-Hilbert correspondence gives a complex analytic isomorphism $\mathrm{RH} : \mathcal{M}_{\mathrm{dR}} \to \mathcal{M}_{\mathrm{Bet}}$.

We now spell out the definition of $\mathcal{M}_{\mathrm{Hod}}$. For $\lambda \in \mathbb{C}$, a λ-connected on a vector bundle $\mathcal{V}$ is a homomorphism of sheaves of abelian groups $\nabla : \mathcal{V} \to \mathcal{V} \otimes \omega_X$ satisfying the λ-Leibnitz rule $\nabla(f \cdot s) = \lambda(df) \cdot s + f \cdot \nabla(s)$. This definition extends to G-bundles by Tannakian formalism. In particular, on a G-bundle, a 0-connection is a G-Higgs field, and a 1-connection is a connection. One defines the Hodge moduli space $\mathcal{M}_{\mathrm{Hod}}$ as the moduli of triples $(\mathcal{E}, \lambda, \nabla)$ in which

- $\mathcal{E} \in \mathrm{Bun}_G$,
- $\lambda \in \mathbb{C}$,
- ∇ is a λ-connection on $\mathcal{E}$.

There is a natural map $\lambda : \mathcal{M}_{\mathrm{Hod}} \to \mathbb{A}^1$ given by $\lambda(\mathcal{E}, \lambda, \nabla) = \lambda$.

The non-abelian Hodge theory shows that (after imposing appropriate stability conditions and taking coarse moduli) $\mathcal{M}_{\mathrm{Dol}}$, $\mathcal{M}_{\mathrm{dR}}$, $\mathcal{M}_{\mathrm{Bet}}$ are differeomorphic to each other. Moreover, they realize different complex structures on the same hyperKähler manifold.

Example 4.1. When $G = \mathrm{GL}_1$, one has $\mathcal{M}_{\mathrm{Dol}} \cong \mathrm{Pic} \times H^0(X, \omega_X)$, $\mathcal{M}_{\mathrm{dR}}$ fits into an exact sequence $0 \to H^0(X, \omega_X) \to \mathcal{M}_{\mathrm{dR}} \to \mathrm{Pic} \to 1$, and $\mathcal{M}_{\mathrm{Bet}} \cong H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{G}_m$. One easily checks that all these spaces are differeomorphic.

4.2. **Hodge moduli space for homogeneous elements.** For $\gamma \in L\mathfrak{g}$ a homogeneous element of slope ν, we define the Hodge moduli space $\mathcal{M}_{\mathrm{Hod},\gamma}$ to be the moduli of tuples $(\lambda, \mathcal{E}, \mathcal{E}_0^B, \mathcal{E}_\infty^{\mathbf{K}_\gamma}, \nabla)$ in which:

- $(\mathcal{E}, \mathcal{E}_0^B, \mathcal{E}_\infty^{\mathbf{K}_\gamma})$ are the same as those date in $\mathcal{M}_\gamma$.
- $\lambda \in \mathbb{C}$, ∇ is a λ-connection on $\mathcal{E}|_{\mathbb{P}^1 \backslash \{0,\infty\}}$.
- At 0, ∇ has a simple pole with $\mathrm{res}_0(\nabla) \in \mathcal{E}_0^B \times^B \mathfrak{n}_B$.
- At ∞, under any trivialization of $(\mathcal{E}, \mathcal{E}_\infty^{\mathbf{K}_\gamma})$, one has $\nabla \in \lambda d + (\gamma + (L_\infty \mathfrak{g})_{\leq 0}) dt/t$.

Note that $\mathcal{M}_{\mathrm{Dol},\gamma} := \lambda^{-1}(0) = \mathcal{M}_\gamma$. Define $\mathcal{M}_{\mathrm{dR},\gamma} = \lambda^{-1}(1)$. We have the following result parallel to Theorem 3.11:

Theorem 4.2. *The following are true:*

- *The map $\lambda : \mathcal{M}_{\mathrm{Hod},\gamma} \to \mathbb{A}^1$ is representable in smooth algebraic spaces, with a canonical relative symplectic structure.*
- *There is a $\mathbb{G}_m$-action on $\mathcal{M}_{\mathrm{Hod},\gamma}$ compatible with the scaling action on $\mathbb{A}^1$, hence, contracting $\mathcal{M}_{\mathrm{Hod},\gamma}$ to Fl_γ.*
- *The restriction maps induce natural isomorphisms*

$$H^*(\mathrm{Fl}_\gamma) \xleftarrow{\sim} H^*(\mathcal{M}_\gamma) \xleftarrow{\sim} H^*(\mathcal{M}_{\mathrm{Hod},\gamma}) \xrightarrow{\sim} H^*(\mathcal{M}_{\mathrm{dR},\gamma}).$$

Remark 4.3. In [JY23], the authors use the Hodge moduli space $\mathcal{M}_{\mathrm{Hod},\gamma}$ to solve particular cases of Deligne-Simpson problem. In their setting, for each pair $(\mathcal{O}, \nu)$ where $\mathcal{O} \subset \mathcal{N}$ is a nilpotent orbit and $\nu \in \mathbb{Q}_{>0}$ is a slope, one asks for existence of G-connections on $\mathbb{P}^1 \backslash \{0, \infty\}$ with regular singularity with monodromy lies in $\mathcal{O}$ at 0 and isoclinic of slope ν at ∞. It was proved in *loc.cit* that such G-connection exists if and only if $[L_\nu(\mathrm{triv}) : E_\mathcal{O}] \neq 0$. Here $L_\nu(\mathrm{triv})$ is a certain representation of a rational Cherednik algebra and $E_\mathcal{O}$ is the representation of W attached to $\mathcal{O}$ via the Springer correspondence.

4.3. **Betti moduli space for homogeneous elements.** Now we come to define the Betti moduli space $\mathcal{M}_{\mathrm{Bet},\gamma}$. It should parametrize topological G-local systems on $\mathbb{P}^1 \backslash \{0, \infty\}$ with Borel reduction at 0 and Stokes data at ∞.

4.3.1. *Stokes data.* We now explain the idea of Stokes data. Choose the local coordinate $\tau = t^{-1}$ at ∞. For a rank n vector bundle $\mathcal{V}$ (which we trivializes at $D_\infty = \mathrm{Spec}\ \mathbb{C}(\!(t^{-1})\!)$) with connection $\nabla = d + A(\tau) d\tau$ with $A(\tau) \in \mathfrak{gl}_n(\mathbb{C}(\tau))$, the flat sections around $D_\infty^\times$ are those $f(\tau) = (f_1(\tau), \cdots, f_n(\tau)) : D_\infty^\times \to \mathbb{C}^n$ satisfying $f'(\tau) = A(\tau) f(\tau)$. On the ray of argument θ starting from ∞, one can canonically identify all the fibers of $\mathcal{V}$ with a single n-dimensional vector space V_θ via parallel transportation. According to the decay rate of flat sections of $(\mathcal{V}, \nabla)$ along the ray, there is a filtration on V_θ. On a general ray, the filtration will give a full flag.

Example 4.4. When $G = \mathrm{GL}_n$ and $\nu = 1$, one takes $A(\tau) = \mathrm{diag}(-a_1 \tau^{-2}, \cdots, -a_n \tau^{-2})$ for $a_i \in \mathbb{C}$. Solving the equation $f_i'(\tau) = -a_i \tau^{-2} f_i(\tau)$, one gets $f_i(\tau) = e^{a_i \tau^{-1}}$. For $\tau = re^{i\theta}$, one has $|e^{a_i \tau^{-1}}| = e^{r^{-1} \mathrm{Re}(a_i e^{-i\theta})}$, whose decay rate is completely modeled by $\mathrm{Re}(a_i e^{-i\theta})$. When the argument θ satisfies that $\mathrm{Re}(a_i e^{-i\theta})$ are distinct, one gets a complete flag $V_{\theta,\bullet}$ in V_θ.

When θ satisfies $\mathrm{Re}((a_i - a_j) e^{-i\theta}) = 0$, it is called a singular direction (or Stokes ray) for (i, j). These rays divide the complex plane into several sectors. The vector space V_θ is equipped with a complex flag on each sector. On each singular direction, the flag is no longer complete. Locally moving around the point $\tau = 0$ and doing parallel transportation allows one to identify vector spaces V_θ for nearby θ. Locally around a singular direction θ_0 for (i, j) (we assume that it is not a singular direction for other pairs (i', j')), let $V = V_\theta$, one gets two filtrations $0 \subset V_1 \subset \cdots \subset V_{k-1} \subset V_k \subset V_{k+1} \subset \cdots \subset V$ and $0 \subset V_1 \subset \cdots V_{k-1} \subset V_k' \subset V_{k+1} \subset \cdots \subset V$ coming from the two sectors near the ray. There two filtrations are different only in a single step, where one has $f_i(\tau) \in V_k$ and $f_j(\tau) \in V_k'$. In this case, we can say these two filtrations has relative position $s_k \in S_n$.

As one moves around the circle, all the relative positions of the filtrations can be encoded in a positive braid $\beta \in \mathrm{Br}_W^+ = \langle s_i \mid \text{braid relations}\rangle$. For an element $\beta = s_{i_1} \cdots s_{i_N} \in \mathrm{Br}_W^+$ where s_{i_j} are simple reflections, there is an associated braid variety $\mathcal{M}(\beta)$ defined as the moduli space of $(\mathcal{E}_\bullet^B, \alpha_\bullet)$ in which

- $\mathcal{E}_i^B$ are B-bundles over a point for $i = 0, 1, \cdots, N$ and $\mathcal{E}_N^B \cong \mathcal{E}_0^B$
- $\alpha_j : \mathcal{E}_{j-1}^B \times^B G \xrightarrow{\sim} \mathcal{E}_j^B \times^B G$ are isomorphisms of G-bundles such that $\mathcal{E}_{j-1}^B$ and $\mathcal{E}_j^B$ has relative position s_{i_j} for $j = 1, \cdots, N$.

We would like to write the data above as

$$\mathcal{M}(\beta) = \{\mathcal{E}_0^B \dashrightarrow_{s_{i_1}} \cdots \dashrightarrow_{s_{i_N}} \mathcal{E}_N^B \cong \mathcal{E}_N^B\}.$$

The moduli space $\mathcal{M}(\beta)$ is equipped with a map $\mathcal{M}(\beta) \to G/G$ by

$$(\mathcal{E}_\bullet^B, \alpha_\bullet) \mapsto (\alpha_N \circ \cdots \alpha_1 : \mathcal{E}_0^B \times^B G \xrightarrow{\sim} \mathcal{E}_0^B \times^B G).$$

One defines $\widetilde{\mathcal{M}}(\beta) := \mathcal{M}(\beta) \times_{G/G} \widetilde{\mathcal{U}}/G$, in which $\widetilde{\mathcal{U}} \to \mathcal{U}$ is the Springer resolution of the unipotent cone.

We now explain the procedure obtaining a braid β from a homogeneous element γ which generalizes the procedure in Example 4.4. One regard γ as a map $\gamma(\tau) : \mathbb{C}^\times \to \mathfrak{g}^{\mathrm{rs}} \to \mathfrak{g}^{\mathrm{rs}} /\!\!/ G = \mathfrak{h}^{\mathrm{rs}} /\!\!/ W$. Taking the fundamental group, one gets $\mathbb{Z} = \pi_1(\mathbb{C}^\times) \to \pi_1(\mathfrak{h}^{\mathrm{rs}} /\!\!/ W) \cong \mathrm{Br}_W$. This determines the conjugacy class of β. To get a positive braid, one can do the following: For a small enough $\epsilon > 0$, consider the circle $S_\epsilon^1 \subset \mathbb{C}^\times$ of radius ϵ around $0 \in \mathbb{C}$. Inducing from γ, one obtains $\gamma : S_\epsilon^1 \to \mathbb{C}^\times \to \mathfrak{h}^{rs} \to \mathfrak{h}_\mathbb{R}$ in which the last step is taking real part. Take the singular directions (regarded as points on S_ϵ^1) to be the preimage of walls (i.e. root hyperplanes) in $\mathfrak{h}_\mathbb{R}$. Suppose the singular directions are preimages of walls of type $s_{j_1}, \cdots, s_{j_N} \in W$ in order, then one associates $\beta = s_{j_1} \cdots s_{j_N} \in \mathrm{Br}_W^+$. The resulting β is well-defined up to cyclic shift, and only depends on the slope ν.

Remark 4.5. When γ is elliptic, assume $\nu = \frac{d}{m}$ for $\gcd(d, m) = 1$. The braid $\beta \in \mathrm{Br}_W^+$ admits the following easy description: One chooses a regular element $w \in W$ of order m which has minimal length within its conjugacy class. By Exercise 1.17(2), one has $l(w) = \frac{|\Phi|}{m}$. Choose $\widetilde{w} \in \mathrm{Br}_W^+$ to be the (unique) minimal length lift of w in Br_W^+, one has $\beta = \widetilde{w}^d \in \mathrm{Br}_W^+$.

Example 4.6. When $\nu = \frac{d}{h}$, the regular element $w = s_1 \cdots s_r$ is a Coxeter element. Then $\beta = (s_1 \cdots s_r)^d$. In this case, the remark above also works when $d = h$: For $\nu = 1 = \frac{h}{h}$, one gets $\beta = (s_1 \cdots s_r)^h = \widetilde{w}_0^2 \in \mathrm{Br}_W^+$ which is called the full twist. Here w_0 is the longest element in W. In this case, one has $\mathcal{M}(\widetilde{w}_0^2) = \{(\mathcal{E}_0^B \dashrightarrow_{w_0} \mathcal{E}_1^B \dashrightarrow_{w_0} \mathcal{E}_2^B \cong \mathcal{E}_0^B)\} = B^{\mathrm{op}} B / \operatorname{Ad} T$.

4.3.2. *Riemann-Hilbert map.* By the previous discussion, there is a natural map $\mathcal{M}_{\mathrm{dR},\gamma} \to \widetilde{\mathcal{M}}(\beta)$ by taking the associated local system with B-reduction at 0 and Stokes data at ∞. When γ is elliptic, we expect this map to be a finite covering as complex analytic spaces. In general, we further enhance $\widetilde{\mathcal{M}}(\beta)$ to make the map possibly a complex analytic isomorphism: Note that there is a natural map $\mathcal{M}(\beta) \to [T/\operatorname{Ad}_w T]$ by sending

$$(\mathcal{E}_0^B \dashrightarrow_{s_{i_1}} \cdots \dashrightarrow_{s_{i_N}} \mathcal{E}_N^B \cong \mathcal{E}_N^B) \mapsto (\mathcal{E}_0^B \times^B T \xrightarrow{\sim} \mathcal{E}_0^B \times^B T)$$

where Ad_w stands for w-twisted conjugation. Consider the exponential map $\mathfrak{h}^w \cong (L\mathfrak{t}_\gamma)_0 \xrightarrow{\exp} T/\operatorname{Ad}_w T$ in which $T/\operatorname{Ad}_w T$ is quotient by w-twisted conjugation by T. We define $\mathcal{M}_{\mathrm{Bet},\gamma} := \widetilde{\mathcal{M}}(\beta) \times_{T/\operatorname{Ad}_w T} \mathfrak{h}^w$. Then the Riemann-Hilbert map admits a natural lift $\mathrm{RH} : \mathcal{M}_{\mathrm{dR},\gamma} \to \mathcal{M}_{\mathrm{Bet},\gamma}$.

Theorem 4.7. *The map* $\mathrm{RH} : \mathcal{M}_{\mathrm{dR},\gamma} \to \mathcal{M}_{\mathrm{Bet},\gamma}$ *is a complex analytic map.*

Conjecture 4.8. *This is a complex analytic isomorphism.*

Remark 4.9. We also expect that $\mathcal{M}_{\mathrm{Dol},\gamma}$, $\mathcal{M}_{\mathrm{dR},\gamma}$, and $\mathcal{M}_{\mathrm{Bet},\gamma}$ are differeomorphic. But we do not know if one should expect a hyperKähler structure on these spaces.

4.4. **Exercises.**

Exercise 4.10. Consider $G = \mathrm{SL}_n$ with homogeneous element

$$\gamma = \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ t & & & \end{pmatrix}^d$$

of slope $\nu = \frac{d}{n}$ for $d \in \mathbb{Z}_{\geq 1}$ such that $\gcd(d, n) = 1$.

(1) Describe the braid $\beta_\nu \in \mathrm{Br}^+_{S^n}$. Identify the braid closure $\widehat{\beta}_\nu$ which is the link obtained from β_ν by connecting startpoints with endpoints in order [1].

(2) For an algebraic curve $C \subset \mathbb{C}^2$ passing through the origin $0 \in \mathbb{C}^2$, the *algebraic link* associated to C is defined as $C \cap S^3_\epsilon \subset S^3_\epsilon = \{(x, y) \in \mathbb{C}^2 | |x|^2 + |y|^2 = \epsilon\}$ for sufficiently small $\epsilon > 0$. Show that the algebraic link associated to $V(x^n - y^d)$ is equivalent to the link $\widehat{\beta}_\nu$.

Exercise 4.11. Consider $G = \mathrm{SL}_n$ with homogeneous element

$$\gamma = \begin{pmatrix} & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ t & & & & \\ & & & & 0 \end{pmatrix}^d$$

of slope $\nu = \frac{d}{n-1}$ for $d \in \mathbb{Z}_{\geq 1}$ such that $\gcd(d, n-1) = 1$.

(1) Describe the braid $\beta_\nu \in \mathrm{Br}^+_{S^n}$. Identify the braid closure $\widehat{\beta}_\nu$.

(2) Show that the algebraic link associated to $V(x^n - xy^d)$ is equivalent to the link $\widehat{\beta}_\nu$.

Exercise 4.12. This exercise studies the Betti moduli space $\mathcal{M}_{\mathrm{Bet},\gamma}$ with slope $\nu = \frac{1}{m}$ where m is a regular elliptic number of W.

For each elliptic element $w \in W$ with minimal length in its conjugacy class, consider $Z_w = \langle T^w, U_\alpha : w\alpha = \alpha, \alpha \in \Phi\rangle$ and $U_w = U \cap wU^- w^{-1}$. Here U_α is the root subgroup of α, U is the unipotent radical of B, and U^- is the opposite of U. Define the *multiplicative transversal slice* $\Sigma_w = U_w Z_w w$, which satisfies the following properties:

- The map
$$\begin{aligned} U \times \Sigma_w &\to UZ_w wU \\ (u, s) &\mapsto usu^{-1} \end{aligned}$$
is an isomorphism.
- Σ_w is transversal to conjugacy classes in G.

The result above is proved in [HL12]. See [Dua24] for a generalization to non-elliptic case.

(1) Show that $\beta_{1/m} \in \mathrm{Br}^+_W$ is a minimal length representative of a regular elliptic element of order m in W.

(2) Choose $w \in W$ a minimal length representative of a regular elliptic element of order m. Consider $\Sigma^\circ_w = U_w w$. Show that $\mathcal{M}_{\mathrm{Bet},\gamma} \cong \Sigma^\circ_w \times_G \tilde{\mathcal{U}}$. Conclude that $\mathcal{M}_{\mathrm{Bet},\gamma}$ is a classical smooth algebraic variety.

(3) Show that $\mathcal{M}_{\mathrm{Bet},\gamma}$ is a point when $m = h$ is the Coxeter number.

(4) For $G = \mathrm{Sp}_4$ and $\nu = \frac{1}{2}$, show that $\mathcal{M}_{\mathrm{Bet},\gamma}$ can be identified with a resolution of $V(x^2 + (y+z)^2 + xyz) \subset \mathbb{A}^3$. Note that the later has A_3-singularity at the origin as its only singular point.

5. Ramified geometric Langlands

In this section, we formulate a ramified geometric Langlands conjecture for $\mathbb{P}^1$ using the Hitchin moduli space and Betti moduli space, and provide evidence for the conjecture in case $\nu = 1$.

5.1. **Unramified Geometric Langlands.** We first recall the unramified geometric Langlands. For a smooth projective curve X, the geometric Langlands conjecture asks for a relation between $\mathrm{Shv}(\mathrm{Bun}_G)$ and $\mathrm{Coh}(\mathrm{Loc}_{\check{G}})$. Here $\mathrm{Loc}_{\check{G}}$ is the moduli space of local systems on the curve X. The notion Shv and Loc have different meanings in different settings. In the de Rham setting, Shv reads as the category of D-modules and Loc reads as moduli of flat connections over X. In the Betti setting, Shv stands for the category of topological sheaves and Loc stands for the moduli of representations of $\pi_1(X)$. Usually, $\mathrm{Shv}(\mathrm{Bun}_G)$ is called the automorphic side and $\mathrm{Coh}(\mathrm{Loc}_{\check{G}})$ is called the spectral side.

[1]You may find more background on links and link invariants in [GKS21].

The precise relation between two sides is a categorical equivalence $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \cong \mathrm{Ind}\,\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}})$, which is now a Theorem due to [GR24a][ABC+24a][CCF+24][ABC+24b][GR24b] when X is over a characteristic zero field.

One the automorphic side, $\mathrm{Nilp} := T^*\mathrm{Bun}_G \times_{\mathcal{A}} \{0\} \subset T^*\mathrm{Bun}_G$ is the global nilpotent cone in the Hitchin moduli stack. For each bounded object $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G)$, there is an attached singular support which is a conical subset $\mathrm{SS}(\mathcal{F}) \subset T^*\mathrm{Bun}_G$. The category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is the full subcategory of $\mathrm{Shv}(\mathrm{Bun}_G)$ consisting of sheaves with its cohomologies having singular support contained in $\mathrm{Nilp} \subset T^*\mathrm{Bun}_G$. One can regard the inclusion $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G)$ as a global analogue of the inclusion $\mathrm{CS}(G) \subset \mathrm{Shv}(G/G)$ in which $\mathrm{CS}(G)$ stands for the category of character sheaves, which consists of $\mathcal{F} \in \mathrm{Shv}(G/G)$ with $\mathrm{SS}(\mathcal{F}) \subset (G \times \mathcal{N}^*)/G \cap T^*(G/G) \subset T^*(G/G)$.

The ind-completion and singular support condition on the spectral side aims to match the compact objects on both sides, which have a very different flavor from the singular support condition on the automorphic side.

5.2. **Geometric Langlands for homogeneous elements.** In our case, we consider $X = \mathbb{P}^1$ but allow ramifications at 0 and ∞. We specialize to the Betti setting.

Fix a homogeneous element $\gamma \in L\mathfrak{g}$. On the automorphic side, we consider the moduli space $\mathcal{M}_\gamma \cong T^*\mathrm{Bun}_G(\mathbf{I}_0, \mathbf{J}'_\gamma) /\!\!/_\gamma \mathbb{G}_{\mathrm{a}}$, which is something related the moduli stack $T^*\mathrm{Bun}_G(\mathbf{I}_0, \mathbf{J}_\gamma)$. On the spectral side, we consider the moduli stack $\widetilde{\mathcal{M}}(\beta)$ where $\beta \in \mathrm{Br}_W^+$ is attached to γ.

Since $\mathcal{M}_\gamma$ is not of cotangent type, defining a correct notion of category of sheaves attached to it is more subtle comparing to the unramified case. In this case, the analogue of $\mathrm{Nilp} \subset T^*\mathrm{Bun}_G$ in our case is $\mathrm{Fl}_\gamma \subset \mathcal{M}_\gamma$. We can consider $\mu\mathrm{Sh}_{\mathrm{Fl}_\gamma}(\mathcal{M}_\gamma)$ which is the category of microlocal sheaves on $\mathcal{M}_\gamma$ with singular support contained in Fl_γ.

The category of microlocal sheaves with prescribed singular support $\mu\mathrm{Sh}_\Lambda(\mathcal{M})$ is defined in [KS02] for any conical Lagrangian inside a symplectic manifold $\Lambda \subset \mathcal{M}$. When $\mathcal{M} = T^*S$ and $\Lambda = \cup_\alpha T^*S_\alpha \subset T^*S$ for a stratification $S = \cup_\alpha S_\alpha$ with the scaling $\mathbb{G}_m$-action on cotangent fibers, one has $\mu\mathrm{Sh}_\Lambda(\mathcal{M}) = \mathrm{Shv}^b_{\{S_\alpha\}}(S)$ consisting of bounded complexes which is locally constant with respect to the stratification $\{S_\alpha\}$.

In our case, the space $\mathcal{M}_\gamma$ is not of cotangent type but is close to, which gives us a more concrete sheaf theory which we are going to explain now. Recall that $\mathcal{M}_\gamma = T^*\mathrm{Bun}_G(\mathbf{I}_0, \mathbf{J}'_\gamma) /\!\!/_\gamma \mathbb{G}_{\mathrm{a}}$. We would like to consider more generally the case $\mathcal{M} = T^*\widetilde{S} /\!\!/_1 \mathbb{G}_{\mathrm{a}}$ in which $1 \in (\mathrm{Lie}\,\mathbb{G}_{\mathrm{a}})^*$ is a non-zero element and $\widetilde{S}$ is a $\mathbb{G}_{\mathrm{a}}$-torsor over S.

We would like a sheaf theory microlocalizes to $\mathcal{M}$. In the l-adic setting over characteristic p, $\mathrm{Shv}(S)$ microlocalizes to T^*S while $\mathrm{Shv}(\widetilde{S}/(\mathbb{G}_{\mathrm{a}}, \mathrm{AS}_\psi))$ microlocalizes to $\mathcal{M}$. Here $\mathrm{Shv}(\widetilde{S}/(\mathbb{G}_{\mathrm{a}}, \mathrm{AS}_\psi))$ is the category of $(\mathbb{G}_{\mathrm{a}}, \mathrm{AS}_\psi)$-equivariant sheaves on $\widetilde{S}$. The sheaf $\mathrm{AS}_\psi \in \mathrm{Shv}(\mathbb{G}_{\mathrm{a}})$ is the Artin-Schreier sheaf defined as follows: Consider $\alpha : \mathbb{G}_{\mathrm{a}} \to \mathbb{G}_{\mathrm{a}}$ defined by $x \mapsto x - x^p$, then $\alpha_*\overline{\mathbb{Q}}_\ell = \bigoplus_{\psi:\mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times} \mathrm{AS}_\psi$. We take any non-trivial character $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$. The sheaf AS_ψ is a character sheaf on $\mathbb{G}_{\mathrm{a}}$ (i.e. it is equipped with a structure $\mathrm{add}^*\mathrm{AS}_\psi \cong \mathrm{AS}_\psi \boxtimes \mathrm{AS}_\psi$ for the addition map $\mathrm{add} : \mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{a}} \to \mathbb{G}_{\mathrm{a}}$), this gives us a notion of $(\mathbb{G}_{\mathrm{a}}, \mathrm{AS}_\psi)$-equivariant sheaves on $\widetilde{S}$ defined as the category of sheaves $\mathcal{F} \in \mathrm{Shv}(\widetilde{S})$ equipped with $a^*\mathcal{F} \cong \mathrm{AS}_\psi \boxtimes \mathcal{F}$ in which $a : \mathbb{G}_{\mathrm{a}} \times \widetilde{S} \to \widetilde{S}$ is the action map.

For other sheaf theories, one can use the *Kirillov model* defined by Gaitsgory in [Gai21], which works uniformly in de Rham, étale, and Betti settings, but requiring an extra $\mathbb{G}_m$-action. Suppose the action of $\mathbb{G}_{\mathrm{a}}$ on $\widetilde{S}$ can be extended to an action of $\mathrm{Aff} = \mathbb{G}_{\mathrm{a}} \rtimes \mathbb{G}_m$, one defines the *Kirillov category* as the Verdier quotient $\mathrm{Kir}(\widetilde{S}) := \mathrm{Shv}(\widetilde{S}/\mathbb{G}_m)/\mathrm{Shv}(\widetilde{S}/\mathrm{Aff})$. When we are in the l-adic setting over characteristic p, the averaging functor with respect to the Artin-Schreier sheaf induces an equivalence $\mathrm{Av}_{(\mathbb{G}_{\mathrm{a}}, \mathrm{AS}_\psi)} : \mathrm{Kir}(\widetilde{S}) \xrightarrow{\sim} \mathrm{Shv}(\widetilde{S}/(\mathbb{G}_{\mathrm{a}}, \mathrm{AS}_\psi))$ for any non-trivial character $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$.

The category $\mathrm{Kir}(\widetilde{S})$ microlocalizes on $\mathcal{M}$, which means for any $\mathcal{F} \in \mathrm{Kir}(\widetilde{S})$ one can associate a conical subset $\mathrm{SS}(\mathcal{F}) \subset \mathcal{M}$ behaves as the singular support of $\mathcal{F}$. To see this, note that $\mathrm{Shv}(\widetilde{S})$ microlocalizes over $T^*(\widetilde{S}/\mathbb{G}_m) = \mu^{-1}_{\mathbb{G}_m}(0)/\mathbb{G}_m$ and $\mathrm{Shv}(\widetilde{S}/\mathrm{Aff})$ microlocalizes on $\mu^{-1}_{\mathrm{Aff}}(0)/\mathrm{Aff}$. Here $\mu_?$ is the moment map for ?-action on $T^*\widetilde{S}$. We know that $\mathrm{Kir}(\widetilde{S})$ microlocalizes on $\mu^{-1}_{\mathrm{Aff}}((\mathrm{Lie}\,\mathbb{G}_{\mathrm{a}}\backslash\{0\}) \times \{0\})/\mathbb{G}_m$. Then our claim is justified by the following exercise:

Exercise 5.1. Show that $\mu^{-1}_{\mathrm{Aff}}((\mathrm{Lie}\,\mathbb{G}_{\mathrm{a}}\backslash\{0\}) \times \{0\})/\mathbb{G}_m \cong \mathcal{M}$.

Given this, for any conical subset $\Lambda \subset \mathcal{M}$, one can define the full subcategory category $\mathrm{Kir}_\Lambda(\widetilde{S}) \subset \mathrm{Kir}(\widetilde{S})$. There is a natural functor $\mathrm{Kir}^b_\Lambda(\widetilde{S}) \to \mu\,\mathrm{Sh}_\Lambda(\mathcal{M})$, which is a equivalence sometimes but not in general.

In our case, take $\widetilde{S} = \mathrm{Bun}_G(\mathbf{I}_0, \mathbf{J}'_\gamma)$ equipped with the natural action of Aff, we have a category $\mathrm{Kir}_{\mathrm{Fl}_\gamma}(\mathrm{Bun}_G(\mathbf{I}_0, \mathbf{J}'_\gamma))$. This the category we consider on the automorphic side.

On the spectral side, we would like to find a version of $\mathrm{Loc}_{\check{G}}$ with level structures at 0 and ∞. For this, we use $\widetilde{\mathcal{M}}_{\check{G}}(\beta)$. Here we use the subscript $\check{G}$ to indicate that we are working with $\check{G}$ rather than G, and we regard β as a braid for $\check{G}$ since it has the same braid semigroup as G.

Conjecture 5.2. *There exists a fully faithful functor* $\Psi : \mathrm{Kir}_{\mathrm{Fl}_\gamma}(\mathrm{Bun}_G(\mathbf{I}_0, \mathbf{J}'_\gamma)) \hookrightarrow \mathrm{IndCoh}(\widetilde{\mathcal{M}}_{\check{G}}(\beta))$.

5.2.1. *Compatibilities with Hecke actions.* The functor in Conjecture 5.2 should be compatible with various symmetries.

For any point $x \in \mathbb{P}^1 \backslash \{0, \infty\}$, the geometric Satake equivalence gives an action of $\mathrm{Rep}(\check{G})$ on the category $\mathrm{Kir}_{\mathrm{Fl}_\gamma}(\mathrm{Bun}_G(\mathbf{I}_0, \mathbf{J}'_\gamma))$. One the spectral side, one has a natural map $\widetilde{\mathcal{M}}_{\check{G}}(\beta) \to [*/\check{G}]$, hence, a natural action of $\mathrm{Rep}(\check{G})$ on $\mathrm{IndCoh}(\widetilde{\mathcal{M}}_{\check{G}}(\beta))$ via pull-back and tensoring. The functor Ψ should intertwine these two actions.

The Hecke action at 0 gives us an action of $\mathrm{Shv}(\mathbf{I}\backslash LG/\mathbf{I})$ on $\mathrm{Kir}_{\mathrm{Fl}_\gamma}(\mathrm{Bun}_G(\mathbf{I}_0, \mathbf{J}'_\gamma))$. Via the Bezrukavnikov's equivalence $\mathrm{Shv}(\mathbf{I}\backslash LG/\mathbf{I}) \cong \mathrm{Ind\,Coh}((\widetilde{\check{\mathcal{U}}} \times_{\check{G}} \widetilde{\check{\mathcal{U}}})/\check{G})$ proved in [Bez21], the same category acts on the spectral side $\mathrm{IndCoh}(\widetilde{\mathcal{M}}_{\check{G}}(\beta))$. The functor should intertwine these actions.

There are also symmetries from ∞: There is an action of $L_\infty T_\gamma/(L_\infty T_\gamma)_{<0}$ on $\mathcal{M}_\gamma$ and Fl_γ. Consider subgroup $\Lambda_\gamma = \pi_0(L_\infty T_\gamma) \cong X_*(T)_w \subset L_\infty T_\gamma/(L_\infty T_\gamma)_{<0}$, it gets an induced action on $\mathrm{Kir}_{\mathrm{Fl}_\gamma}(\mathrm{Bun}_G(\mathbf{I}_0, \mathbf{J}'_\gamma))$. On the spectral side, consider the map $\widetilde{\mathcal{M}}_{\check{G}}(\beta) \to \mathcal{M}_{\check{G}}(\beta) \to \check{T}/\mathrm{Ad}_w \check{T} \to [*/\check{T}^w]$. It gives rise to an action of $\mathrm{Rep}(\check{T}^w)$ on $\mathrm{Ind\,Coh}(\widetilde{\mathcal{M}}_{\check{G}}(\beta))$, hence, inducing an action of Λ_γ on $\mathrm{Ind\,Coh}(\widetilde{\mathcal{M}}_{\check{G}}(\beta))$.

5.3. **Slope one case.** Now we give evidence for the conjecture in case $\nu = 1$. We take $\gamma = \gamma_0 \cdot t$ for $\gamma_0 \in \mathfrak{h}^{\mathrm{rs}}$. Regard $\gamma : \mathbf{K}_1 \to \mathbf{K}_1/\mathbf{K}_2 \cong \mathfrak{g} \xrightarrow{\langle -,\gamma\rangle} \mathbb{G}_{\mathrm{a}}$. Define $\mathbf{K}'_1 = \mathrm{Ker}(\mathbf{K}_1 \to \mathbb{G}_{\mathrm{a}})$. We have $\mathbf{J}_\gamma = \mathbf{K}_1$ and $\mathbf{J}'_\gamma = \mathbf{K}'_1$.

In this case, $\mathrm{Bun}_G(\mathbf{I}_0, \mathbf{J}_\gamma) = \mathrm{Bun}_G(\mathbf{I}_0, \mathbf{K}_1)$. The braid $\beta = \widetilde{w}_0^2$. We have $\widetilde{\mathcal{M}}_{\check{G}}(\widetilde{w}_0^2) = \widetilde{\check{B}^{\mathrm{op}}\check{B} \cap \check{\mathcal{U}}}/\,\mathrm{Ad}\,\check{T}$. Here $\widetilde{\check{B}^{\mathrm{op}}\check{B} \cap \check{\mathcal{U}}} = (\check{B}^{\mathrm{op}}\check{B} \cap \check{\mathcal{U}}) \times_{\check{G}} \widetilde{\check{\mathcal{U}}}$.

We first consider a variant of Conjecture 5.2 without the Iwahori level structure at 0:

Theorem 5.3. *There is an equivalence of categories* $\mathrm{Kir}(\mathrm{Bun}_G(\mathbf{K}'_1)) \xrightarrow{\sim} \mathrm{Rep}(\check{T})$.

The category $\mathrm{Kir}(\mathrm{Bun}_G(\mathbf{K}'_1))$ microlocalizes on $\overline{\mathcal{M}}_\gamma$ which has the same description as $\mathcal{M}_\gamma$ except lacking the Iwahori level structure at 0. It admits a Hitchin map $f_\gamma : \overline{\mathcal{M}}_\gamma \to \mathcal{A}_\gamma$ with the same Hitchin base as before. One has $\overline{\mathcal{M}}_{\gamma,\mathrm{red}} = X_*(T)$. There is a natural map $\mathcal{M}_\gamma \to \overline{\mathcal{M}}_\gamma$ (and also a map $p : \mathrm{Bun}_G(\mathbf{I}_0, \mathbf{K}'_1) \to \mathrm{Bun}_G(\mathbf{K}'_1)$). This gives us a pull-back functor $p^* : \mathrm{Kir}(\mathrm{Bun}_G(\mathbf{K}'_1)) \to \mathrm{Kir}_{\mathrm{Fl}_\gamma}(\mathrm{Bun}_G(\mathbf{I}_0, \mathbf{K}'_1))$.

One defines the category $D_\gamma \subset \mathrm{Kir}_{\mathrm{Fl}_\gamma}(\mathrm{Bun}_G(\mathbf{I}_0, \mathbf{K}'_1))$ to be the full subcategory generated by the essential image of p^* and the Hecke action at 0 by the category $\mathrm{Shv}(\mathbf{I}\backslash LG/\mathbf{I})$.

Theorem 5.4. *There is an equivalence of categories* $D_\gamma \xrightarrow{\sim} \mathrm{Coh}^{\check{T}}(\widetilde{\check{\mathcal{N}}})_{\check{\mathcal{B}}}$. *Here* $\mathrm{Coh}^{\check{T}}(\widetilde{\check{\mathcal{N}}})_{\check{\mathcal{B}}}$ *is the category of* $\check{T}$*-equivariant coherent sheaves on the Springer resolution* $\widetilde{\check{\mathcal{N}}}$ *supported on the zero section* $\check{\mathcal{B}} \subset T^*\check{\mathcal{B}} \subset \widetilde{\check{\mathcal{N}}}$.

Remark 5.5. This is a part of the conjectural equivalence in 5.2. On the automorphic side, note that $D_\gamma \subset \mathrm{Kir}_{\mathrm{Fl}_\gamma}(\mathrm{Bun}_G(\mathbf{I}_0, \mathbf{K}'_1))$. On the spectral side, $\mathrm{Coh}^{\check{T}}(\widetilde{\check{\mathcal{N}}})_{\check{\mathcal{B}}} \xrightarrow{\sim} \mathrm{Coh}^{\check{T}}(\widetilde{\check{B}^{\mathrm{op}}\check{B} \cap \mathcal{U}})_{\check{\mathcal{B}}} \subset \mathrm{Ind\,Coh}(\widetilde{\mathcal{M}}_{\check{G}}(\beta))$.

This equivalence of categories is compatible with symmetries introduced in 5.2.1. Moreover, one has richer symmetry from ∞.

Pretend we are in the l-adic setting over characteristic p. In this case, one has

$$\mathrm{Kir}_{\mathrm{Fl}_\gamma}(\mathrm{Bun}_G(\mathbf{I}_0, \mathbf{K}'_1)) \cong \mathrm{Shv}_{\mathrm{Fl}_\gamma}(\mathrm{Bun}_G(\mathbf{I}_0, \mathbf{K}'_1)/(\mathbb{G}_{\mathrm{a}}, \mathrm{AS}_\psi)).$$

Consider the Hecke category $\mathcal{H}_\infty = \mathrm{Shv}((\mathbf{K}_1, \gamma^*\mathrm{AS}_\psi)\backslash L_\infty G/(\mathbf{K}_1, \gamma^*\mathrm{AS}_\psi))$ which acts on the category above.

Fact 5.6. *There is an equivalence of categories* $\mathcal{H}_\infty \xrightarrow{\sim} \mathrm{Rep}(\check{T}) \otimes \mathrm{Shv}(T)$.

Note that $\mathrm{Rep}(\check{T}) \otimes \mathrm{Shv}(T) \cong \mathrm{Shv}(X_*(T) \times T)$. We have a functor $\mathrm{Rep}(\check{T}) \to \mathcal{H}_\infty$ by sending the character $\lambda \in X^*(\check{T})$ to the universal local system on $\{\lambda\} \times T$. This gives rise to an action of $\mathrm{Rep}(\check{T})$ on $\mathrm{Shv}_{\mathrm{Fl}_\gamma}(\mathrm{Bun}_G(\mathbf{I}_0, \mathbf{K}'_1)/(\mathbb{G}_\mathrm{a}, \mathrm{AS}_\psi))$, which preserves the subcategory D_ψ. This action is compatible with the action of $\mathrm{Rep}(\check{T})$ on $\mathrm{Coh}^{\check{T}}(\widetilde{\check{\mathcal{N}}})_{\check{\mathcal{B}}}$.

Remark 5.7. When $\nu = \frac{1}{m}$, there are many cases we can formulate and prove analogues of Theorem 5.4.

References

[ABC+24a] D. Arinkin, D. Beraldo, J. Campbell, L. Chen, J. Faergeman, D. Gaitsgory, K. Lin, S. Raskin, and N. Rozenblyum, *Proof of the geometric langlands conjecture ii: Kac-moody localization and the fle*, 2024.

[ABC+24b] D. Arinkin, D. Beraldo, L. Chen, J. Faergeman, D. Gaitsgory, K. Lin, S. Raskin, and N. Rozenblyum, *Proof of the geometric langlands conjecture iv: ambidexterity*, 2024.

[Bez96] Roman Bezrukavnikov, *The dimension of the fixed point set on affine flag manifolds*, Mathematical Research Letters **3** (1996), no. 2, 185–189.

[Bez21] Roman Bezrukavnikov, *On two geometric realizations of an affine hecke algebra*, 2021.

[CCF+24] Justin Campbell, Lin Chen, Joakim Faergeman, Dennis Gaitsgory, Kevin Lin, Sam Raskin, and Nick Rozenblyum, *Proof of the geometric langlands conjecture iii: compatibility with parabolic induction*, 2024.

[Dua24] Chengze Duan, *Good position braid representatives and transversal slices of unipotent orbits*, 2024.

[Gai21] Dennis Gaitsgory, *The local and global versions of the whittaker category*, 2021.

[GKS21] Eugene Gorsky, Oscar Kivinen, and José Simental, *Algebra and geometry of link homology*, 2021.

[GR24a] Dennis Gaitsgory and Sam Raskin, *Proof of the geometric langlands conjecture i: construction of the functor*, 2024.

[GR24b] ———, *Proof of the geometric langlands conjecture v: the multiplicity one theorem*, 2024.

[HL12] Xuhua He and George Lusztig, *A generalization of steinberg's cross section*, Journal of the American Mathematical Society **25** (2012), no. 3, 739–757.

[JY23] Konstantin Jakob and Zhiwei Yun, *A deligne-simpson problem for irregular g-connections over* $\mathbb{P}^1$, 2023.

[KL88] David Kazhdan and George Lusztig, *Fixed point varieties on affine flag manifolds*, Israel Journal of Mathematics **62** (1988), 129–168.

[KS02] M. Kashiwara and P. Schapira, *Sheaves on manifolds: With a short history. les débuts de la théorie des faisceaux. by christian houzel*, Grundlehren der mathematischen Wissenschaften, Springer Berlin Heidelberg, 2002.

[OY16] Alexei Oblomkov and Zhiwei Yun, *Geometric representations of graded and rational cherednik algebras*, Advances in Mathematics **292** (2016), 601–706.

[RY14] Mark Reeder and Jiu-Kang Yu, *Epipelagic representations and invariant theory*, Journal of the American Mathematical Society **27** (2014), no. 2, 437–477.

[Spr74] Tonny Albert Springer, *Regular elements of finite reflection groups*, Inventiones mathematicae **25** (1974), no. 2, 159–198.

[Ste65] Robert Steinberg, *Regular elements of semi-simple algebraic groups*, Publications Mathématiques de l'IHÉS **25** (1965), 49–80.

[Ste75] ———, *Torsion in reductive groups*, Advances in Mathematics **15** (1975), no. 1, 63–92.