NOTES FOR "GEOMETRIC LANGLANDS FOR PROJECTIVE CURVES"

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Contents

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1. Lecture 1

1.1. Sheaf theories.
Throughout this course, we fix the following setup:

- An algebraically closed field $k = \overline{k}$, called the *geometric field*.
- A *sheaf theory* with *coefficient field* e such that $\mathsf{char}(\mathsf{e}) = 0$.

Below are some examples of sheaf theories.

Example 1.1.1 (Betti setting)**.** Let $k = \mathbb{C}$. Let $\mathsf{Shv}(Y) = \mathsf{Shv}(Y, \mathsf{e})$ to be the category of sheaves of e-vector spaces on $Y(\mathbb{C})$ equipped with the complex topology, and $\mathsf{Lisse}(Y) \subset \mathsf{Shv}(Y)$ be the full subcategory of finite rank locally constant sheaves.

Example 1.1.2 (ℓ-adic/étale setting)**.** Let $\ell \neq \mathsf{char}(k)$ be a prime number, and $\mathsf{e} = \overline{\mathbb{Q}}_\ell$. Let $\mathsf{Shv}(Y)$ be the category of ℓ-adic sheaves on Y, and $\mathsf{Lisse}(Y) \subset \mathsf{Shv}(Y)$ be the full subcategory of lisse sheaves.

Example 1.1.3 (de Rham setting)**.** Let $\mathsf{char}(k) = 0$, $\mathsf{e} = k$. Let $\mathsf{Shv}(Y) = \mathsf{DMod}(Y)$ be the category of D-modules on Y, and $\mathsf{Lisse}(Y) \subset \mathsf{Shv}(Y)$ be the full subcategory of finite rank vector bundles equipped with a flat connection.

Remark 1.1.4. For $k = \mathbb{C}$, Y a smooth variety, there is an equivalence

$$\mathsf{Lisse}^{\mathsf{dR}}(Y) \xrightarrow{\simeq} \mathsf{Lisse}^{\mathsf{Betti}}(Y, \mathbb{C})$$

sending $(\mathcal{E}, \nabla)$ to the de Rham complex $[\mathcal{E} \to \mathcal{E} \otimes \Omega^1_Y \to \mathcal{E} \otimes \Omega^2_Y \to \cdots]$.

1.2. Geometric class field theory a.k.a. "abelian Geometric Langlands".
Let X be a smooth projective curve over k of genus g. Consider $\mathsf{Bun}_{\mathbb{G}_m}$, the moduli stack of line bundles on X. For any fixed base point $x_0 \in X$, we have an identification

$$\mathsf{Bun}_{\mathbb{G}_m} \simeq \mathsf{Jac}(X) \times \mathbb{Z} \times \mathbb{B}\mathbb{G}_m.$$

In above,

- $\mathsf{Jac}(X)$ is the Jacobian of X, which is an abelian variety prametrizing line bundles on X of degree 0 equipped with a trivialization at x_0.
- The factor $\mathbb{Z}$ corresponds to taking the degree of a line bundle.
- The factor $\mathbb{B}\mathbb{G}_m$ encodes automorphisms of a line bundle, which is $k^\times$.

Goal 1.2.1. Our goal is to conduct the following construction:

- (input) A rank 1 local system (=lisse sheaf) σ on X;
- (output) A lisse sheaf χ_σ on $\mathsf{Bun}_{\mathbb{G}_m}$ equipped with the following data:
  - (Hecke property) For and $\mathcal{L} \in \mathsf{Bun}_{\mathbb{G}_m}$, $x \in X$ and $\mathcal{L}(x) \coloneqq \mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}(x)$, an isomorphism
    $$\chi_\sigma|_{\mathcal{L}(x)} \simeq \chi_\sigma|_{\mathcal{L}} \otimes \sigma_x$$
    depending algebraically on $(\mathcal{L}, x)$.
  - (normalization) An isomorphism
    $$\chi_\sigma|_{\mathcal{O}_X} \simeq \mathsf{e}.$$

Let us give two constructions of χ_σ.

Construction 1.2.2. For simplicity, we work in the Betti setting. Then knowing σ is equivalent to knowing a homomorphism

$$\pi_1(X, x_0) \to \mathsf{e}^\times.$$

We can replace the fundamental group by its abelianization, which is

$$\pi_1(X, x_0)^{\mathsf{ab}} \simeq \mathsf{H}_1(X, \mathbb{Z}) \simeq \mathsf{H}^1(X, \mathbb{Z}),$$

where the last isomorphism is Poincaré duality. Hence σ provides a homomorphism

$$\mathsf{H}^1(X, \mathbb{Z}) \to \mathsf{e}^\times. \tag{1.1}$$

On the other hand, the short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\mathsf{exp}} \mathcal{O}^\times \to 0$$

induces an isomorphism

$$\mathsf{Jac}(X) \simeq \mathsf{H}^1(X, \mathcal{O})/\mathsf{H}^1(X, \mathbb{Z})$$

which implies

$$\pi_1(\mathsf{Jac}(X)) \simeq \mathsf{H}^1(X; \mathbb{Z}).$$

Together with (1.1), we get a homomorphism

$$\pi_1(\mathsf{Jac}(X)) \to \mathsf{e}^\times,$$

which gives a rank 1 local system $\chi_\sigma|_{\mathsf{Jac}(X)}$ on $\mathsf{Jac}(X)$. Now we define χ_σ such that its restiction to $\mathsf{Jac}(X) \times \{d\} \times \mathbb{B}\mathbb{G}_{\mathsf{m}}$ is given by

$$\chi_\sigma|_{\mathsf{Jac}(X)} \boxtimes \sigma_{x_0}^{\otimes d} \boxtimes \mathsf{e}_{\mathbb{B}\mathbb{G}_{\mathsf{m}}}.$$

Exercise 1.2.3. Verify χ_σ satisfies the Hecke property.

Construction 1.2.4 (Deligne)**.** This construction works for any sheaf theory.

For a line bundle $\mathcal{L} \in \mathsf{Bun}_{\mathbb{G}_{\mathsf{m}}}$, a choice of a rational section gives an isomorphism $\mathcal{O}(D) \simeq \mathcal{L}$, where $D = \sum n_i x_i$ is a divisor on X. Our axioms for χ_σ require

$$\chi_\sigma|_{\mathcal{L}} \simeq \bigotimes \sigma_{x_i}^{\otimes n_i}.$$

So we see χ_σ is "overdetermined", and we need to answer the following question:

Why is this vector space independent of the choice of D?

To treat this problem, consider the d-th symmetric power $\mathsf{Sym}^d(X)$ of X, which is the moduli space of effective divisors of degree d on X. Equivalently, it classifies $(\mathcal{L}, s)$ where $\mathcal{L}$ is a degree d line bundle and $s \in \Gamma(X, \mathcal{L})$ is a nonzero section. There is a map

$$\mathsf{AJ}_d : \mathsf{Sym}^d(X) \to \mathsf{Bun}_{\mathbb{G}_{\mathsf{m}}}^d$$

that forgets the section s. This is known as the *Abel–Jacobi map.* The fiber of this map at $\mathcal{L}$ is $\Gamma(X, \mathcal{L}) \smallsetminus 0$.

When $d > 2g - 2$, by Riemann-Roch,

$$\dim \Gamma(X, \mathcal{L}) = d + 1 - g,$$

which implies AJ_d is smooth. Note that the fibers of AJ_d are simply-connected.

Now for the given σ, we can produce its d-th symmetric power, which is a rank 1 local system on $\mathsf{Sym}^d(X)$ given by the formula

$$\sigma^{(d)} := \mathsf{add}_{d,*}(\sigma \boxtimes \cdots \boxtimes \sigma)^{S_d}. \tag{1.2}$$

Here $\mathsf{add}_d : X^d \to \mathsf{Sym}^d(X)$ and S_d is the symmetric group. Our axioms require

$$\chi_\sigma|_{\mathsf{Sym}^d(X)} \simeq \sigma^{(d)}.$$

But by the previous simply-connectedness, for $d >> 0$ this $\sigma^{(d)}$ must descend to a lisse sheaf on $\mathsf{Bun}_{\mathbb{G}_{\mathsf{m}}}^d$. Hence we can define $\chi_\sigma|_{\mathsf{Bun}_{\mathbb{G}_{\mathsf{m}}}^d}$ for $d >> 0$, and then use the Hecke property to extend it to all of $\mathsf{Bun}_{\mathbb{G}_{\mathsf{m}}}$.

Remark 1.2.5. In (1.2), the fiber of $\mathsf{add}_{d,*}(\sigma \boxtimes \cdots \boxtimes \sigma)$ at D is

$$\bigoplus_{D=\sum x_i} \otimes_i \sigma_{x_i},$$

where the direct sum is labelled by all ways of writing D as $\sum x_i$. Now taking invariants for S_d removes the redundancy such that the fiber becomes $\otimes \sigma_{x_i}$.

1.3. **Non-abelian theory.** Now take $G = \mathsf{PGL}_2$. Now our goal becomes:

- (input) An SL_2-local system σ on X. We view σ as a rank 2 local system equipped with a trivialization of $\wedge^2\sigma$.
- (output) A sheaf $\mathcal{F}_\sigma$ on $\mathsf{Bun}_{\mathsf{PGL}_2}$ satisfying a Hecke property (to be explained in future lectures).

However, we no longer require $\mathcal{F}_\sigma$ to be lisse. Rather, we only want it to be preverse. The reason for this will be explained in future lectures.

In above, $\mathsf{Bun}_{\mathsf{PGL}_2}$ is the moduli stack of PGL_2-bundles on X. By definition, a PGL_2-bundle on X is a rank 2 vector bundle modulo ambiguity of tensoring by line bundles. In other words, $\mathcal{E}$ and $\mathcal{E} \otimes \mathcal{L}$ give the same point in $\mathsf{Bun}_{\mathsf{PGL}_2}$. Note that

$$\mathsf{deg}(\mathcal{E} \otimes \mathcal{L}) = \mathsf{deg}(\mathcal{E}) + 2\mathsf{deg}(\mathcal{L}).$$

Hence we have a well-defined map

$$\mathsf{Bun}_{\mathsf{PGL}_2} \xrightarrow{\mathsf{deg}} \mathbb{Z}/2$$

and a decomposition

$$\mathsf{Bun}_{\mathsf{PGL}_2} \simeq \mathsf{Bun}^{\mathsf{even}}_{\mathsf{PGL}_2} \sqcup \mathsf{Bun}^{\mathsf{odd}}_{\mathsf{PGL}_2}$$

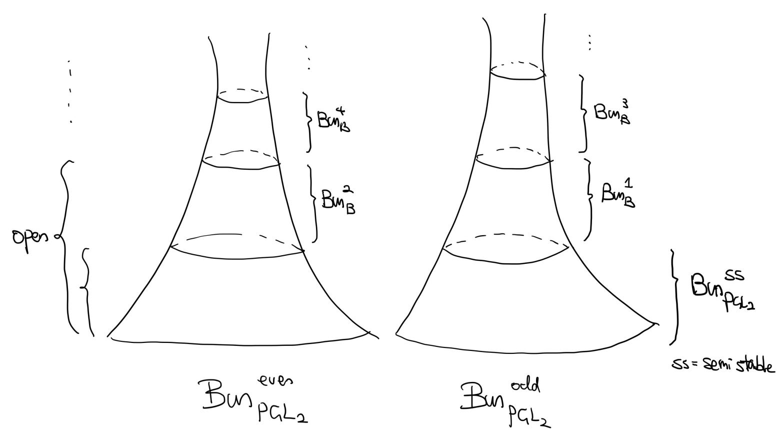

FIGURE 1. A picture of $\mathsf{Bun}_{\mathsf{PGL}_2}$

The above is a picture of $\mathsf{Bun}_{\mathsf{PGL}_2}$. A few explanations are in order:

- $\mathsf{Bun}^{\mathsf{even}}_{\mathsf{PGL}_2}$ and $\mathsf{Bun}^{\mathsf{odd}}_{\mathsf{PGL}_2}$ are both connected.

- The semi-stable locus consists of $\mathcal{E}$ such that for any line subbundle $\mathcal{L} \subset \mathcal{E}$, we have
$$\mathsf{deg}(\mathcal{L}) \le \mathsf{deg}(\mathcal{E})/2.$$
Note that this condition is invariant under twisting by line bundles. This is an open condition and therefore defines an open substack
$$\mathsf{Bun}^{\mathsf{ss}}_{\mathsf{PGL}_2} \subset \mathsf{Bun}_{\mathsf{PGL}_2}.$$
- Let $B \subset \mathsf{PGL}_2$ be the standard Borel subgroup, and Bun_B be the moduli stack of B-bundles on X. By definition, it classifies short exact sequences $[0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{O} \to 0]$. We have a map
$$\mathsf{Bun}_B \xrightarrow{\mathsf{deg}} \mathbb{Z}$$
sending the above sequence to $\mathsf{deg}(\mathcal{L})$. This gives a decomposition
$$\mathsf{Bun}_B \simeq \bigsqcup_{d \in \mathbb{Z}} \mathsf{Bun}^d_B.$$
For $d \ge 1$, the map
$$\mathsf{Bun}^d_B \to \mathsf{Bun}_{\mathsf{PGL}_2}$$
is a locally closed embedding.
- For $n \ge 0$, we have an open substack $U_n \subset \mathsf{Bun}_{\mathsf{PGL}_2}$ which is a disjoint union of $\mathsf{Bun}^{\mathsf{ss}}_{\mathsf{PGL}_2}$ and (the images of) Bun^d_B for $1 \le d \le n$. In particular, $\mathsf{Bun}_{\mathsf{PGL}_2}$ is *not* quasi-compact.

Exercise 1.3.1. Show that $\mathsf{Bun}^{\mathsf{even}}_{\mathsf{PGL}_2}$ and $\mathsf{Bun}^{\mathsf{odd}}_{\mathsf{PGL}_2}$ are connected.

Exercise 1.3.2. Show that for $d \ge 0$, the map $\mathsf{Bun}^d_B \to \mathsf{Bun}_{\mathsf{PGL}_2}$ is injective on k-points.

Exercise 1.3.3. Find the dimension of Bun^d_B and $\mathsf{Bun}_{\mathsf{PGL}_2}$. Deduce that there exists odd semistable PGL_2-bundles.

Exercise 1.3.4. For $d << 0$, show that $\mathsf{Bun}^d_B \to \mathsf{Bun}_{\mathsf{PGL}_2}$ is smooth.

2. Lecture 2

Last time: given an irrducible SL_2 local system on X, we want a corresponding sheaf $\mathcal{F}_\sigma$ on $\mathsf{Bun}_{\mathsf{PGL}_2}$ with some *unspecified* properties.

Let us list a few desired properties for $\mathcal{F}_\sigma$.

- $\mathcal{F}_\sigma$ should be perverse.
- $\mathcal{F}_\sigma|_{\mathsf{Bun}^{\mathsf{even/odd}}_{\mathsf{PGL}_2}}$ should be irreducible as a perverse sheaf.
- $\mathcal{F}_\sigma$ should be *cuspidal*.
- $\cdots$

In above, cuspidality is the counterpart to σ being irreducible, according to Langlands's philosophy.

2.1. Cuspidality.

Definition 2.1.1. Consider the correspondence

$$\mathsf{Bun}_{\mathsf{PGL}_2} \xleftarrow{p} \mathsf{Bun}_B \xrightarrow{q} \mathsf{Bun}_{\mathbb{G}_\mathsf{m}},$$

where p sends a sequence $[0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{O} \to 0] \in \mathsf{Bun}_B$ to $\mathcal{E} \in \mathsf{Bun}_{\mathsf{PGL}_2}$, and q sends it to $\mathcal{L} \in \mathsf{Bun}_{\mathbb{G}_\mathsf{m}}$. We define the *constant term* functor

$$\mathsf{CT}_* := q_* p^! : \mathsf{Shv}(\mathsf{Bun}_{\mathsf{PGL}_2}) \to \mathsf{Shv}(\mathsf{Bun}_{\mathbb{G}_\mathsf{m}}).$$

For each $d \in \mathbb{Z}$, we similarly define a functor

$$\mathsf{CT}^d_* : \mathsf{Shv}(\mathsf{Bun}_{\mathsf{PGL}_2}) \to \mathsf{Shv}(\mathsf{Bun}^d_{\mathbb{G}_\mathsf{m}}).$$

by replacing Bun_B with Bun^d_B.

Definition 2.1.2. We say $\mathcal{F} \in \mathsf{Shv}(\mathsf{Bun}_G)$ is *cuspidal* if $\mathsf{CT}_*(\mathcal{F}) \simeq 0$.

Remark 2.1.3. $\mathcal{F}$ is cuspidal iff $\mathsf{CT}^d_*(\mathcal{F}) \simeq 0$ for any $d \in \mathbb{Z}$.

Remark 2.1.4. For general reductive groups, we use all proper parabolics and their Levi quotients to define cuspidality.

Recall the picture (1.3) from the last lecture. The following result says cuspidality implies vanishing "at ∞".

Proposition 2.1.5. *If $\mathcal{F}$ is cuspidal, then its !-restriction to Bun^d_B is zero for $d > 2g-2$.*

Sketch. Consider the map $q_d : \mathsf{Bun}^d_B \to \mathsf{Bun}^d_{\mathbb{G}_m}$. If $d > 2g-2$, for any point in Bun^d_B, the corresponding sequence $0 \to \mathcal{L} \to \mathcal{B} \to \mathcal{O} \to 0$ splits. It follows that the fiber of q_d at a point $\mathcal{L} \in \mathsf{Bun}^d_{\mathbb{G}_m}$ can be identified with[1] the classifying stack $\mathbb{B}\mathsf{H}^1(X, \mathcal{L})$.

Recall we have an equivalence

$$\pi^* : \mathsf{Vect} \rightleftarrows \mathsf{Shv}(\mathbb{B}\mathbb{G}_\mathsf{a}) : \pi_*$$

because $\mathbb{G}_\mathsf{a}$ is contractible, where $\pi : \mathbb{B}\mathbb{G}_\mathsf{a} \to \mathsf{pt}$ is the projection. Similarly, because $\mathsf{H}^1(X, \mathcal{L})$ is contractible, the functor

$$q_{d,*} : \mathsf{Shv}(\mathsf{Bun}^d_B) \to \mathsf{Shv}(\mathsf{Bun}^d_{\mathbb{G}_\mathsf{m}})$$

is an equivalence for $d > 2g-2$. It follows that $\mathsf{CT}^d_*(\mathcal{F}) \simeq 0$ implies $p^!_d(\mathcal{F}) \simeq 0$ as desired. □

We can also define another version of the constant term functor.

[1]For any d, the fiber can be canonically identified with the vector stack $\mathsf{R}\Gamma(X, \mathcal{L})[1]$.

Definition 2.1.6. We define $\mathsf{CT}_!$ to be the functor

$$q_! p^* : \mathsf{Shv}(\mathsf{Bun}_G) \to \mathsf{Shv}(\mathsf{Bun}_T).$$

Remark 2.1.7. Formally, if $\mathcal{F}$ is constructible, $\mathsf{CT}_*(\mathcal{F}) \simeq \mathbb{D}\mathsf{CT}_!\mathbb{D}(\mathcal{F})$, where $\mathbb{D}$ is the Verdier duality.

Theorem 2.1.8 (Drinfeld–Gaitsgory, "2nd adjointness")**.** *We have a canonical equivalence*

$$\mathsf{CT}_*^d \simeq \mathsf{inv} \circ \mathsf{CT}_!^{-d},$$

where inv *is the functor induced by the isomorphism* $\mathsf{Bun}_{\mathbb{G}_m}^d \simeq \mathsf{Bun}_{\mathbb{G}_m}^{-d}$, $\mathcal{L} \mapsto \mathcal{L}^{-1}$.

Corollary 2.1.9. *If* $\mathcal{F}$ *is constructible and cuspidal, then so is* $\mathbb{D}\mathcal{F}$.

Corollary 2.1.10. *If* $\mathcal{F}$ *is cuspidal, then its* $*$*-restriction to* Bun_B^d *is zero for* $d > 2g-2$

Proof. By Drinfeld–Gaitsgory, $\mathsf{CT}_*^{-d}(\mathcal{F}) \simeq 0$ implies $\mathsf{CT}_!^d(\mathcal{F}) \simeq 0$. As in the proof of Proposition 2.1.5, the latter implies $p_d^*(\mathcal{F}) \simeq 0$ for $d > 2g-2$ □

Corollary 2.1.11. *There exists an open substack* $U_n \subset \mathsf{Bun}_G$*, which is a* finite *union of strata, such that any cuspidal* $\mathcal{F}$ *is both* ! *and* $*$ *extended*[2] *from* $\mathcal{U}$.

Remark 2.1.12. In fact, we can take $n = \max\{0, 2g-2\}$.

Exercise 2.1.13. Verify Theorem 2.1.8 for the constant sheaf on Bun_G.

Exercise 2.1.14. Let B^- be the standard opposite Borel subgroup, and define CT_*^-, $\mathsf{CT}_!^-$ by replacing B with B^-. Show that Theorem 2.1.8 is equivalent to the statement that $\mathsf{CT}_!^- \simeq \mathsf{CT}_*$. Challenge: construct natural transformations $\mathsf{CT}_!^- \to \mathsf{CT}_*$ and $\mathsf{CT}_* \to \mathsf{CT}_!^-$.

Exercise 2.1.15. Challenge: for $g \geq 2$, show that cuspidal sheaves exist.

2.2. **Coefficients.** To state other expectations for the sheaf $\mathcal{F}_\sigma$, we use some motivations from theory of modular forms (a.k.a. automorphic forms for PGL_2).

Recall a (holonomic) *modular form* (of weight k and level 1) is a sum $f = \sum_{n \geq 0} a_n q^n$ converging for $|q| < 1$ such that for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z})$, we have

$$f(g \cdot \tau) = (c\tau + d)^k f(\tau),$$

where $q = \exp(2\pi i \tau)$ with $\mathsf{Im}(\tau) > 0$. The Langlands conjecture say for any (odd) irrducible (unramified) Galois representation[3]

$$\sigma : \text{“}\mathsf{Gal}_{\mathbb{Q}}\text{”} \to \mathsf{SL}_2(\text{“}\mathbb{C}\text{”}),$$

there exists a modular form $f_\sigma = \sum a_n q^n$ such that

- (cuspidality) the *constant term* a_0 is zero;
- $a_p = \mathsf{tr}(\sigma(\mathsf{Fr}_p))$, where $\mathsf{Fr}_p \in \mathsf{Gal}_{\mathbb{Q}}$ is the Frobenious conjugacy class for a prime number p;
- $a_1 = 1$, $a_{nm} = a_n a_m$ if $(n, m) = 1$, and

$$a_{p^{n+1}} = a_p a_{p^n} - p^{k-1} a_{p^{n-1}} \tag{2.1}$$

[2]In other words, $\mathcal{F}$ is *cleanly* extended from $\mathcal{U}$.

[3]The weight k is determined by σ at archemedean place.

Note that the above conditions uniquelly determine f_σ.

Our goal is to provide analogues of a_n, which is the n-th Fourier coefficient of a modular form, for sheaves on $\mathsf{Bun}_{\mathsf{PGL}_2}$.

- We will replace n, which is an effective divisor on $\mathsf{Spec}\mathbb{Z}$, with an effective divisor D on X.
- We will replace the numbder $a_n(f)$ with a vector space $\mathsf{coeff}_D(\mathcal{F}) \in \mathsf{Vect}$, following Geothendieck's philosephy on the sheaf-function correspondence.

Let us first consider the case $D = 0$, which is analogous to $a_1(f)$. Recall

$$a_n(f) = \int_{\mathbb{R}/\mathbb{Z}} f(\tau)\exp(-2\pi i n\tau)d\tau.$$

Here we identify $\mathbb{R}$ with the unipotent radical of the standard Borel of $\mathsf{SL}_2\mathbb{R}$. This motivates us to consider Bun_N, where N is the unipotent radical of the standard Borel of PGL_2. By definition, this is the moduli stack of extensions $[0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O} \to 0]$. For subtle reasons, we need to consider a variant

$$\mathsf{Bun}_N^\Omega := \mathsf{Bun}_B \times_{\mathsf{Bun}_{\mathbb{G}_m}} \{\Omega\},$$

where $\Omega \in \mathsf{Bun}_{\mathbb{G}_m}$ is the point corresponding to the canonical line bundle on X. Note that as a vector stack, we have

$$\mathsf{Bun}_N^\Omega \simeq \mathsf{R}\Gamma(X,\Omega)[1].$$

Now consider the correspondence

$$\mathsf{Bun}_G \xleftarrow{p_N} \mathsf{Bun}_N^\Omega \xrightarrow{\phi} \mathsf{H}^1(X,\Omega) \simeq \mathbb{A}^1,$$

where the identification $\mathsf{H}^1(X,\Omega) \simeq \mathbb{A}^1$ is due to Serre duality.

Definition 2.2.1. We define

$$\mathsf{coeff}_0(\mathcal{F}) := \mathsf{C}^\cdot(\mathsf{Bun}_N^\Omega, p_N^!(\mathcal{F}) \overset{!}{\otimes} \phi^!(\exp)),$$

where $\exp \in \mathsf{Shv}(\mathbb{A}^1)$ is the "exponential sheaf" (explained below).

Example 2.2.2 (de Rham setting)**.** Let exp be the D-module $(\mathcal{O}, \nabla = d - dt)$.

Example 2.2.3 (ℓ-adic setting)**.** We have a short exact sequence

$$0 \to \mathbb{F}_p \to \mathbb{G}_a \xrightarrow{\pi} \mathbb{G}_a \simeq 0$$

where π is the Artin–Schreier map $t \mapsto t^p - t$. It follows that $\pi_*(\overline{\mathbb{Q}}_\ell)$ is acted by $\mathbb{F}_p$. For a fixed nontrivial character $\psi : \mathsf{F}_p \to \overline{\mathbb{Q}}_\ell^\times$, we can consider the ψ-component of $\pi_*(\overline{\mathbb{Q}}_\ell)$, which can be shown to be a rank 1 lisse sheaf. We define exp to be this sheaf.

Remark 2.2.4 (Betti setting)**.** In the Betti setting, exp does not exist. But there are tricks to avoid usage of it.

3. Lecture 3

3.1. **Coefficients (continued).** Last time we defined $\mathsf{coeff}_0 : \mathsf{Shv}(\mathsf{Bun}_G) \to \mathsf{Vect}$. Now we define coeff_D for any effective divisor on D.

Consider the stack

$$\mathsf{Bun}_N^{\Omega(-D)} := \mathsf{Bun}_B \underset{\mathsf{Bun}_{\mathbb{G}_m}}{\times} \{\Omega(-D)\}.$$

This is the moduli stack for extensions of $[0 \to \Omega(-D) \to \mathcal{E} \to \mathcal{O} \to 0]$. Let

$$p_{N,D} : \mathsf{Bun}_N^{\Omega(-D)} \to \mathsf{Bun}_G$$

be the map remembering $\mathcal{E}$. Consider the composition

$$\phi_D : \mathsf{Bun}_N^{\Omega(-D)} \to \mathsf{H}^1(X, \Omega(-D)) \to \mathsf{H}^1(X, \Omega) \simeq \mathsf{A}^1,$$

where the last isomorphism is due to the Serre duality.

Definition 3.1.1. We define

$$\mathsf{coeff}_D(\mathcal{F}) := \mathsf{C}^{\cdot}(\mathsf{Bun}_N^{\Omega(-D)}, p_{N,D}^!(\mathcal{F}) \overset{!}{\otimes} \phi_D^!(\exp)).$$

Now we are ready to state our goal.

Goal 3.1.2. Our goal is to conduct the following construction:

- (input) An irreducible SL_2-local system σ on X.
- (output) A cuspidal perverse sheaf $\mathcal{F}_\sigma \in \mathsf{Shv}(\mathsf{Bun}_{\mathsf{PGL}_2})$ that is irreducible on $\mathsf{Bun}_{\mathsf{PGL}_2}^{\mathsf{even/odd}}$ equipped with the following data:
  - For each effective divisor $D = \sum n_i x_i$, an isomorphism

$$\mathsf{coeff}_D(\mathcal{F}_\sigma) \simeq \bigotimes_i \mathsf{Sym}^{n_i}(\sigma_{x_i}). \tag{3.1}$$

Remark 3.1.3. Note that (3.1) is analogous to the multiplicativity and recursion formula (2.1) from last lecture, because

$$\mathsf{Sym}^n(V) \otimes V \simeq \mathsf{Sym}^{n+1}(V) \oplus \mathsf{Sym}^{n-1}(V)$$

for the standard representation $V \in \mathsf{Rep}(\mathsf{SL}_2)$.

In above, the isomorphisms (3.1) should depend algebraically in D. More precisely, for each $d \geq 0$, we can construct a functor

$$\mathsf{coeff}_d : \mathsf{Shv}(\mathsf{Bun}_G) \to \mathsf{Shv}(\mathsf{Sym}^d(X))$$

such that for $D \in \mathsf{Sym}^d(X)$, we can identify coeff_D with the composition

$$\mathsf{Shv}(\mathsf{Bun}_G) \to \mathsf{Shv}(\mathsf{Sym}^d(X)) \xrightarrow{(-)|_D^!} \mathsf{Vect}.$$

Now (3.1) should be upgraded to an isomorphism

$$\mathsf{coeff}_d(\mathcal{F}_\sigma) \simeq \sigma^{(d)},$$

where

$$\sigma^{(d)} := \mathsf{add}_{\mathsf{d},*}(\sigma^{\boxtimes d})^{S_d}$$

is defined as in the rank 1 case. However, $\sigma^{(d)}$ is perverse but not lisse.

Based on the motivation from modular forms, we might want the functor

$$\prod_{d \geq 0} \mathsf{coeff}_d : \mathsf{Shv}(\mathsf{Bun}_G)_{\mathsf{cusp}} \to \prod_{d \geq 0} \mathsf{Shv}(\mathsf{Sym}^d(X))$$

to be fully faithful (so that $\mathcal{F}_\sigma$ is uniquely determined by the previous expectation). However, this is *not* true because the RHS splits onto factors labelled by $\mathbb{Z}^{\geq 0}$, while the LHS only onto $\mathbb{Z}/2$. Nevertheless, we have the following claim:

Proposition 3.1.4. *Let $\mathcal{F}$ be a perverse cuspidal sheaf on $\mathsf{Bun}_{\mathsf{PGL}_2}$ such that its restrictions on $\mathsf{Bun}_{\mathsf{PGL}_2}^{\mathsf{even/odd}}$ are irreducible and have dense supports. Then $\mathcal{F}$ is uniquely determined by $\mathsf{coeff}_d(\mathcal{F})$ and $\mathsf{coeff}_{d+1}(\mathcal{F})$ for any $d > N$, where N is an integer depending only on g.*

Warning 3.1.5. The above claim is special to PGL_2.

3.2. **Fourier transform.** To explain Proposition 3.1.4, we recall the Fourier(–Deligne) transform. For a finite dimensional vector space V and its dual space $V^\vee$, consider the correspondence

$$V \xleftarrow{\mathsf{pr}_1} V \times V^\vee \xrightarrow{\mathsf{pr}_2} V^\vee.$$

We have an equivalence

$$\mathsf{Shv}(V) \xrightarrow{\mathsf{Four}} \mathsf{Shv}(V^\vee),\ \mathcal{F} \mapsto \mathsf{pr}_{2,*}(\mathsf{pr}_1^!(\mathcal{F}) \overset{!}{\otimes} \mathsf{ev}^!(\exp)),$$

where $\mathsf{ev} : V \times V^\vee \to \mathbb{A}^1$ is the pairing map. This can be generalized to vector bundles E and $E^\vee$ over a base S, so we have a canonical equivalence

$$\mathsf{Four} : \mathsf{Shv}(E) \xrightarrow{\simeq} \mathsf{Shv}(E^\vee).$$

Remark 3.2.1. In the Betti setting where exp does not exist, we can do Fourier transform for monodromic sheaves.

We can rewrite coeff_d using Fourier transforms and geometry of bundles as follows. Recall the fiber of the map

$$q_{-d} : \mathsf{Bun}_B^{-d} \to \mathsf{Bun}_{\mathbb{G}_m}^{-d}$$

at $\mathcal{L}$ is identified with $\mathsf{R}\Gamma(X, \mathcal{L})[1]$. For $d > 0$, $\mathsf{H}^0(X, \mathcal{L}) \simeq 0$, hence q_{-d} is a vector bundle over $\mathsf{Bun}_{\mathbb{G}_m}^{-d}$, which we denote by E_d. The fiber of the dual bundle $E_d^\vee$ at $\mathcal{L}$ is

$$\mathsf{H}^1(X, \mathcal{L})^\vee \simeq \mathsf{H}^0(X, \mathcal{L}^\vee \otimes \Omega).$$

It follows that $E_d^\vee$ is the moduli stack classifying $(\mathcal{L}, s : \mathcal{L} \to \Omega)$, where $\mathcal{L} \in \mathsf{Bun}_{\mathbb{G}_m}^{-d}$ and $s : \mathcal{L} \to \Omega$ is a map.

Note that we have an identification

$$\mathsf{Sym}^{d+(2g-2)}(X) \to E_d^\vee \smallsetminus \{0\},\ D \mapsto [\Omega(-D) \to \Omega],$$

where 0 means the zero section of $E_d^\vee$ (which is isomorphic to $\mathsf{Bun}_{\mathbb{G}_m}^{-d}$).

Exercise 3.2.2. The functor $\mathsf{coeff}_{d+(2g-2)}$ can be identified with the composition

$$\mathsf{Shv}(\mathsf{Bun}_G) \xrightarrow{p_{-d}^!} \mathsf{Shv}(\mathsf{Bun}_B^{-d}) \xrightarrow{\mathsf{Four}} \mathsf{Shv}(E_d^\vee) \xrightarrow{\text{restriction}} \mathsf{Shv}(\mathsf{Sym}^{d+(2g-2)}(X))$$

Fact 3.2.3. *If $f : Y \to Z$ is a smooth map with connected fibers, then the pullback functor on perverse sheaves is fully faithful.*

Fact 3.2.4. *If $f : Y \to Z$ is a smooth map with connected or empty fibers, and Y is nonempty, then the pullback functor on irreducible perverse sheaves with dense support is fully faithful.*

Exercise 3.2.5. For any open substack $U_n \subset \mathsf{Bun}_{\mathsf{PGL}_2}$, there exists an integer N depending only on n and g such that the fibers of $p_{-d} : \mathsf{Bun}_B^{-d} \to \mathsf{Bun}_G$ over points in U_n are connected or empty for any $d > N$.

Proof of Proposition 3.1.4. Recall $\mathcal{F}$ is determined by its restriction on $U_n \subset \mathsf{Bun}_{\mathsf{PGL}_2}$ where n is an integer depending only on g (Corollary 2.1.11). Hence by the above fact and exercise, there exists an integer N depending only on g such that the functor $p_{-d}^!$ does not lose information about $\mathcal{F}|_{\mathsf{Bun}_{\mathsf{PGL}_2}^{d+2\mathbb{Z}/2\mathbb{Z}}}$ for any $d > N$. Hence to prove

Proposition 3.1.4, we only need to recover $\mathsf{Four}(p_{-d}^{!}\mathcal{F})$ from its restriction along the open embedding

$$E_d^\vee \smallsetminus 0 \subset E_d^\vee.$$

To see this, we note that the !-restriction of $\mathsf{Four}(p_{-d}^{!}\mathcal{F})$ on the zero section can be identified with $\mathsf{CT}_*^{-d}(\mathcal{F})$, which is zero because $\mathcal{F}$ is assumed to be cuspidal.

□[Proposition 3.1.4]

3.3. **Geometric Langlands Correspondence.** We have already seen $\mathcal{F}_\sigma$ is uniquely determined by σ if exists. The following result is known as geometric Langlands correspondence for PGL_2.

Theorem 3.3.1 (Drinfeld, Laumon, Frenkel–Gaitsgory–Vilonen)**.** *For any irreducible SL_2-local system σ, there exists $\mathcal{F}_\sigma \in \mathsf{Shv}(\mathsf{Bun}_{\mathsf{PGL}_2})$ that is perverse and irreducible on each connected component of $\mathsf{Bun}_{\mathsf{PGL}_2}$, such that*

$$\mathsf{coeff}_d(\mathcal{F}_\sigma) \simeq \sigma^{(d)}.$$

Moreover, $\mathcal{F}_\sigma$ is a *Hecke eigensheaf.* To explain this notion, consider the moduli stack Hecke classifying $(x \in X, \mathcal{E} \subset \mathcal{E}' \subset \mathcal{E}(x))$, where (up to tensoring with a line bundle) $\mathcal{E}$ and $\mathcal{E}'$ are rank 2 bundles on X, and the inclusions $\mathcal{E} \subset \mathcal{E}' \subset \mathcal{E}(x)$ are strict. We have a correspondence

$$\mathsf{Bun}_G \xleftarrow{\overleftarrow{h}} \mathsf{Hecke} \xrightarrow{\overrightarrow{h}} \mathsf{Bun}_G \times X$$

such that $\overleftarrow{h}$ sends $(x \in X, \mathcal{E} \subset \mathcal{E}' \subset \mathcal{E}(x))$ to $\mathcal{E}$, and $\overrightarrow{h}$ sends it to $(\mathcal{E}', x)$. Consider the functor

$$\mathsf{H}_{\mathsf{std}} := \overrightarrow{h}_* \overleftarrow{h}^{!}. \tag{3.2}$$

Now the Hecke eigenproperty says

$$\mathsf{H}_{\mathsf{std}}(\mathcal{F}_\sigma) \simeq \mathcal{F}_\sigma \boxtimes \sigma. \tag{3.3}$$

4. Lecture 4

4.1. **Hecke functors.** Let us define Hecke property for general reductive group G (split and connected over k). Let $\check{G}$ be the dual reductive group over e.

We have the notion of G-bundles.

Example 4.1.1. G-bundles mean the following:

- For $G = \mathsf{GL}_n$, rank n vector bundles $\mathcal{E}$;
- For $G = \mathsf{SL}_n$, rank n vector bundles $\mathcal{E}$ equipped with $\det\mathcal{E} \simeq \mathcal{O}$;
- For $G = \mathsf{PGL}_n$, rank n vector bundles $\mathcal{E}$ up to tensoring with line bundles;
- For $G = \mathsf{O}_n$, rank n vector bundles $\mathcal{E}$ equipped with a symmetric non-degenerate form $\mathcal{E} \otimes \mathcal{E} \to \mathcal{O}$;
- For $G = \mathsf{Sp}_n$, rank n vector bundles $\mathcal{E}$ equipped with an alternating non-degenerate form $\mathcal{E} \otimes \mathcal{E} \to \mathcal{O}$;
- $\cdots$

For any G-bundle $\mathcal{P}_G$ on a scheme S, we have a right t-exact[4] symmetric monoidal functor

$$\mathsf{Rep}(G) \to \mathsf{QCoh}(S),\ V \mapsto V_{\mathcal{P}_G}.$$

[4]For derived schemes S, the functor below is not t-exact.

If V is finite dimensional, $V_{\mathcal{P}_G}$ is a vector bundle of finite rank.

We also have the notion of $\check{G}$-local systems, which can be *defined* as right t-exact symmetric monoidal functors

$$\mathsf{Rep}(G) \to \mathsf{Shv}(X),\ V \mapsto V_\sigma$$

If $V \in \mathsf{Rep}(G)$ is finite dimensional, V_σ is lisse.

Now for any $V \in \mathsf{Rep}(\check{G})$, we can define a functor

$$\mathsf{H}_V : \mathsf{Shv}(\mathsf{Bun}_G) \to \mathsf{Shv}(\mathsf{Bun}_G \times X),$$

which generalizes the Hecke functor defined in the last lecture (which is the case $\check{G} = \mathsf{SL}_2$ and V being the standard representation). The ingredient for the definition is the following equivalence.

Theorem 4.1.2 (Geometric Satake)**.** *For any $x \in X$, there is a canonical monoidal functor*

$$\mathsf{Rep}(\check{G}) \to \mathsf{Shv}(\mathsf{L}_x^+ G \backslash \mathsf{L}_x G / \mathsf{L}_x^+ G),\ V \mapsto \mathcal{S}_V$$

where $\mathsf{L}_x(G) \coloneqq G(k((t_x)))$, $\mathsf{L}_x^+(G) \coloneqq G(k[[t_x]])$ and t_x is a local coordinate near x.

Remark 4.1.3. One can check that $\mathcal{S}_V$ does not depend on the choice of t_x.

The double quotient

$$\mathsf{Hecke}_x^{\mathsf{loc}} \coloneqq \mathsf{L}_x^+ G \backslash \mathsf{L}_x G / \mathsf{L}_x^+ G$$

classifies two G-bundles $\mathcal{P}_G$, $\mathcal{P}'_G$ on $\mathcal{D}_x \coloneqq \mathsf{Spec} k[[t_x]]$ equipped with an isomorphism between their restrictions on $\overset{\circ}{\mathcal{D}}_x$. We can also consider a global version of this stack

$$\mathsf{Hecke}_x$$

which classifies two G-bundles on X equipped with an isomorphism between their restrictions on $\overset{\circ}{X} \coloneqq X \smallsetminus x$. We have the following diagram

$$\begin{array}{ccccc} \mathsf{Bun}_G & \xleftarrow{\overleftarrow{h}} & \mathsf{Hecke}_x & \xrightarrow{\overrightarrow{h}} & \mathsf{Bun}_G \\ & & \downarrow{\scriptstyle \pi} & & \\ & & \mathsf{Hecke}_x^{\mathsf{loc}}. & & \end{array}$$

Definition 4.1.4. We define a functor

$$\mathsf{H}_{V,x} : \mathsf{Shv}(\mathsf{Bun}_G) \to \mathsf{Shv}(\mathsf{Bun}_G),\ \mathcal{F} \mapsto \overrightarrow{h}_*(\overleftarrow{h}^!(\mathcal{F}) \overset{!}{\otimes} \pi^!(\mathcal{S}_V)).$$

We can also vary $x \in X$ and similarly define a functor

$$\mathsf{H}_V : \mathsf{Shv}(\mathsf{Bun}_G) \to \mathsf{Shv}(\mathsf{Bun}_G \times X)$$

such that $\mathsf{H}_V(\mathcal{F})|_{\mathsf{Bun}_G \times x} \simeq \mathsf{H}_{V,x}(\mathcal{F})$. We call them the *Hecke functors*.

Exercise 4.1.5. For $G = \mathsf{PGL}_2$ and $\check{G} = \mathsf{SL}_2$, consider the standard representation $\mathsf{std} \in \mathsf{Rep}(\check{G})$. Verify that the Hecke functor $\mathsf{H}_{\mathsf{std}}$ defined above is equivalent to the functor (3.2).

The Hecke functors commute in the following sense. For V_1, $V_2 \in \mathsf{Rep}(\check{G})$, let H_{V_1,V_2} be the composition

$$\mathsf{Shv}(\mathsf{Bun}_G) \xrightarrow{\mathsf{H}_{V_2}} \mathsf{Shv}(\mathsf{Bun}_G \times X) \xrightarrow{\mathsf{H}_{V_1}} \mathsf{Shv}(\mathsf{Bun}_G \times X \times X).$$

Then we have a canonical identification

$$\mathsf{H}_{V_1,V_2} \simeq \mathsf{swap} \circ \mathsf{H}_{V_2,V_1},$$

where swap is the involution on $\mathsf{Shv}(\mathsf{Bun}_G \times X \times X)$ induced by swaping the two factors X. More generally, given a finite set I and $\check{G}$-representations $\underline{V} = \{V_i\}_{i\in I}$ indexed by I, we have a well-defined functor

$$\mathsf{H}_{\underline{V}} : \mathsf{Shv}(\mathsf{Bun}_G) \to \mathsf{Shv}(\mathsf{Bun}_G \times X^I)$$

given by composing of H_{V_i}'s in any order.

Exercise 4.1.6. Verify that

$$\mathsf{H}_{V_1,x} \circ \mathsf{H}_{V_2,x} \simeq \mathsf{H}_{V_1\otimes V_2,x}.$$

Question 4.1.7. What acts on $\mathsf{Shv}(\mathsf{Bun}_G)$?

For each $x \in X$, we have an endo-functor $\mathsf{H}_{V,x}$ on $\mathsf{Shv}(\mathsf{Bun}_G)$, but the functors $\mathsf{H}_{\underline{V}}$ is *not* an endo-functor. There is a formal way to produce an endo-functor out of these Hecke functors as follows.

For a finite set I, $V = \boxtimes_{i\in I} V_i \in \mathsf{Rep}(\check{G})^{\otimes I}$ and $\mathcal{G} \in \mathsf{Shv}(X^I)$, consider the composition

$$\mathsf{Shv}(\mathsf{Bun}_G) \xrightarrow{\mathsf{H}_{\underline{V}}} \mathsf{Shv}(\mathsf{Bun}_G \times X^I) \xrightarrow{-\otimes^! \mathsf{pr}_2^!(\mathcal{G})} \mathsf{Shv}(\mathsf{Bun}_G \times X^I) \xrightarrow{\mathsf{pr}_{1,*}} \mathsf{Shv}(\mathsf{Bun}_G). \tag{4.1}$$

This gives us a functor

$$\mathsf{Rep}(\check{G})^{\otimes I} \otimes \mathsf{Shv}(X^I) \to \mathsf{End}(\mathsf{Shv}(\mathsf{Bun}_G))$$

sending $V \boxtimes \mathcal{G}$ to the above endo-functor.

Remark 4.1.8. When $I = \{1,2\}$, the endo-functor (4.1) is the "integration" of the functors $\mathsf{H}_{V_1,x_1} \circ \mathsf{H}_{V_2,x_2}$ (when $x_1 \neq x_2$) and $\mathsf{H}_{V_1\otimes V_2,x}$ (when $x_1 = x_2 = x$), against the "mearsure" $\mathcal{G}$ on X^2.

For any map $\alpha : I \to J$ between finite sets, we have a functor

$$\mathsf{Rep}(\check{G})^{\otimes I} \xrightarrow{\mathsf{mult}_\alpha} \mathsf{Rep}(\check{G})^{\otimes J}$$

provided by the symmetric monoidal structure on $\mathsf{Rep}(\check{G})$, and a functor

$$\Delta^\alpha_* : \mathsf{Shv}(X^J) \to \mathsf{Shv}(X^I)$$

induced by the map $\Delta^\alpha : X^J \to X^I$. One can check the following diagram commutes:

$$\begin{array}{ccc}
\mathsf{Rep}(\check{G})^{\otimes I} \otimes \mathsf{Shv}(X^J) & \xrightarrow{\mathsf{mult}_\alpha\otimes \mathsf{id}} & \mathsf{Rep}(\check{G})^{\otimes J} \otimes \mathsf{Shv}(X^J) \\
\downarrow{\scriptstyle \mathsf{id}\otimes\Delta^\alpha_*} & & \downarrow \\
\mathsf{Rep}(\check{G})^{\otimes I} \otimes \mathsf{Shv}(X^I) & \longrightarrow & \mathsf{End}(\mathsf{Shv}(\mathsf{Bun}_G))
\end{array}$$

It follows that we have a functor

$$\mathsf{Rep}(\check{G})_{\mathsf{Ran}} := \operatorname*{colim}_{\alpha:I\to J} \mathsf{Rep}(\check{G})^{\otimes I} \otimes \mathsf{Shv}(X^J) \to \mathsf{End}(\mathsf{Shv}(\mathsf{Bun}_G)), \tag{4.2}$$

where the colimit is indexed by maps $\alpha : I \to J$, and is contravariant in I while covariant in J.

Note that $\mathsf{Rep}(\check{G})_{\mathsf{Ran}}$ has a (symmetric) monoidal structure given by tensoring the $I \xrightarrow{\alpha} J$ and $I' \xrightarrow{\alpha'} J'$ entries into the $I \sqcup I' \xrightarrow{(\alpha,\alpha')} J \sqcup J'$ entry. One can check that (4.2) is compatible with the monoidal structures on both sides.

In summary, we have a monoidal functor

$$\mathsf{Rep}(\check{G})_{\mathsf{Ran}} \to \mathsf{End}(\mathsf{Shv}(\mathsf{Bun}_G))$$

obtained by putting all the Hecke functors together.

4.2. **Hecke property.**

Question 4.2.1. What can $\mathsf{Rep}(\check{G})_{\mathsf{Ran}}$ do for us?

We give two answers:

- It gives all the Fourier coefficients into one home.
- It gives a full and correct definition of eigensheaves.

Let us first explain the second point. Fix a $\check{G}$-local system σ on X, we have a symmetric monoidal functor

$$\mathsf{ev}_\sigma : \mathsf{Rep}(\check{G})_{\mathsf{Ran}} \to \mathsf{Vect}$$

whose restriction on the $I \xrightarrow{=} I$ entry is

$$\mathsf{Rep}(\check{G})^{\otimes I} \otimes \mathsf{Shv}(X^I) \xrightarrow{\sigma^{\boxtimes I} \otimes \mathsf{id}} \mathsf{Shv}(X^I) \otimes \mathsf{Shv}(X^I) \xrightarrow{- \otimes^! -} \mathsf{Shv}(X^I) \xrightarrow{\Gamma} \mathsf{Vect},$$

where recall the $\check{G}$-local system $\sigma^{\boxtimes I}$ on X^I is viewed as a symmetric monoidal functor

$$\mathsf{Rep}(\check{G})^{\otimes I} \to \mathsf{Shv}(X^I).$$

Definition 4.2.2. A Hecke eigensheaf with eigenvalue σ is a $\mathsf{Rep}(\check{G})_{\mathsf{Ran}}$-linear functor

$$\mathsf{Vect} \to \mathsf{Shv}(\mathsf{Bun}_G),\ \mathsf{e} \mapsto \mathcal{F},$$

where $\mathsf{Rep}(\check{G})_{\mathsf{Ran}}$ acts on Vect via ev_σ. We often abuse notation and write $\mathcal{F}$ for the Hecke eigensheaf.

Exercise 4.2.3. Show that σ is uniquely determined by the symmetric monoidal functor ev_σ.

Exercise 4.2.4. Let $\mathcal{F}$ be a Hecke eigensheaf with eigenvalue σ. Show that

$$\mathsf{H}_V(\mathcal{F}) \simeq \mathcal{F} \boxtimes V_\sigma \tag{4.3}$$

Remark 4.2.5. Note that (4.3) generalizes (3.3).

4.3. **Geometric Langlands correspondence.**

Conj 4.3.1. *If σ is a irreducible $\check{G}$-local system*[5]*, then there exists a unique Hecke eigensheaf $\mathcal{F}_\sigma$ equipped with $\mathsf{coeff}_0(\mathcal{F}_\sigma) \simeq \mathsf{e}$.*

Theorem 4.3.2 (Arinkin–Beraldo–Campbell–Chen–Faergeman–Gaitsgory–Lin–R.–Rozenblyum)**.** *This conjecture is true if $\mathsf{char}(k) = 0$.*

Moreover, we know

- $\mathcal{F}_\sigma$ is perverse;
- $\mathcal{F}_\sigma$ is cuspidal;

[5]This means it does not come from a proper parabolic subgroup

- $\mathcal{F}_\sigma$ is semisimple. More precisely, let S_σ be the automorphism group of σ, then we have a decomposition
$$\mathcal{F}_\sigma \simeq \bigoplus_{\rho\in \mathsf{Irr} S_\sigma} \mathcal{F}_{\sigma,\rho}^{\oplus \dim \rho}$$
such that $\mathcal{F}_{\sigma,\rho}$ is simple perverse.
- The characteristic cycle of $\mathcal{F}_\sigma$ is $[\mathsf{Nilp}]$, wher $\mathsf{Nilp} \subset \mathsf{T}^*\mathsf{Bun}_G$ is the global nilpotent cone.
- For $g > 1$, the generic rank of $\mathcal{F}_\sigma$ is
$$\prod d_i^{(2d_i-1)(g-1)}$$
where d_i's are the exponents of G.

Remark 4.3.3. The connected components of Bun_G can be identified with the set $\mathsf{Irr}(Z_{\check{G}})$ of irreducible representations of the center of $\check{G}$. The perverse sheaf $\mathcal{F}_{\sigma,\rho}$ is supported on the connected component labelled by $\rho|_{Z_{\check{G}}}$.

5. Lecture 5

5.1. **Whittaker coefficients.** We have the following motto:

$\mathsf{Rep}(\check{G})_{\mathsf{Ran}}$ *lets us "glue" all the Whittaker coefficients of a sheaf on* Bun_G.

To explain it, let us first define Whittaker coefficients for general G. Consider
$$\mathsf{Bun}_N^\Omega := \mathsf{Bun}_B \underset{\mathsf{Bun}_T}{\times} \{\check{\rho}(\Omega)\},$$
where $\check{\rho}(\Omega) := 2\check{\rho}(\Omega^{1/2})$ and $\Omega^{1/2}$ is a fixed square root of the line bundle Ω. For each positive simple root α_i, we have a map $N \to \mathbb{G}_a$ which induces a map
$$\phi_i : \mathsf{Bun}_N^\Omega \to \mathsf{Bun}_{\mathbb{G}_a}^\Omega \to \mathsf{H}^1(X,\Omega) \simeq \mathbb{A}^1$$
Define
$$\phi := \sum_{\alpha_i} \phi_i : \mathsf{Bun}_N^\Omega \to \mathbb{A}^1$$
and consider the correspondence
$$\mathsf{Bun}_G \xleftarrow{p_N} \mathsf{Bun}_N^\Omega \xrightarrow{\phi} \mathbb{A}^1.$$
As in the PGL_2-case, we make the following definition:

Definition 5.1.1. Let $\mathsf{coeff}_0 : \mathsf{Shv}(\mathsf{Bun}_G) \to \mathsf{Vect}$ be the functor given by the formula
$$\mathsf{coeff}_0(\mathcal{F}) := \mathsf{C}^\cdot(\mathsf{Bun}_N^\Omega, p_N^!(\mathcal{F}) \overset{!}{\otimes} \phi^!(\exp)).$$
Similarly, for each $D = \sum \check{\lambda}_i x_i$ with $\lambda_i \in \check{\Lambda}^+$, we use $\check{\rho}(\Omega)(-D) \in \mathsf{Bun}_T$ to produce a functor
$$\mathsf{coeff}_D : \mathsf{Shv}(\mathsf{Bun}_G) \to \mathsf{Vect}.$$
We call them the *Whittaker coefficient* functors.

Example 5.1.2. For $G = \mathbb{G}_m$, D can be an arbitrary divisor, and the functor
$$\mathsf{coeff}_D : \mathsf{Shv}(\mathsf{Bun}_{\mathbb{G}_m}) \to \mathsf{Vect}$$
is taking the !-fiber at $\mathcal{O}(-D)$.

Example 5.1.3. For $G = \mathsf{PGL}_2$, these functors recover those denoted by the same notations in the previous lectures.

The following result is known as the geometric Casselman–Shalika formula:

Theorem 5.1.4 (Frenkel–Gaitsgory–Vilonen)**.** *For $D = \sum \check{\lambda}_i x_i$ with $\lambda_i \in \check{\Lambda}^+$, we have*

$$\mathsf{coeff}_D(\mathcal{F}) \simeq \mathsf{coeff}_0(\mathsf{H}_{\boxtimes V^{\check{\lambda}_i},\underline{x}}(\mathcal{F})),$$

where recall that $\mathsf{H}_{\boxtimes V^{\check{\lambda}_i},\underline{x}}$ is the composition of $\mathsf{H}_{V^{\check{\lambda}_i},x_i}$ in any order.

Note that we have an object

$$\boxtimes_i (V^{\check{\lambda}_i} \otimes \delta_{x_i}) \in \mathsf{Rep}(\check{G})^{\otimes I} \otimes \mathsf{Shv}(X^I),$$

which gives an object in $\mathsf{Rep}(\check{G})_{\mathsf{Ran}}$. Now the above theorem says coeff_D is equivalent to the Hecke action of this object followed by coeff_0.

Definition 5.1.5. Let $\mathcal{F}_\sigma$ be a Hecke eigensheaf with eigenvalue σ. We say $\mathcal{F}$ is *normalized* if it is equipped with an isomorphism

$$\mathsf{coeff}_0(\mathcal{F}) \simeq \mathsf{e}.$$

Let $\mathcal{F}_\sigma$ be a normalized Hecke eigensheaf $\mathcal{F}_\sigma$. By Exercise 4.2.4, we have

$$\mathsf{H}_{\boxtimes V^{\check{\lambda}_i},\underline{x}}(\mathcal{F}_\sigma) \simeq \mathcal{F}_\sigma \bigotimes \underset{i}{\otimes} (V_\sigma^{\check{\lambda}_i})_{x_i}.$$

It follows from the theorem that this condition implies

$$\mathsf{coeff}_D(\mathcal{F}_\sigma) \simeq \mathsf{coeff}_0(\mathcal{F}_\sigma) \bigotimes \underset{i}{\otimes} (V_\sigma^{\check{\lambda}_i})_{x_i} \simeq \underset{i}{\otimes} (V_\sigma^{\check{\lambda}_i})_{x_i},$$

where the last isomorphism is because $\mathcal{F}_\sigma$ is normalized. Note that this generalizes (3.1).

Now consider the following functor

$$\mathsf{Rep}(\check{G})_{\mathsf{Ran}} \otimes \mathsf{Shv}(\mathsf{Bun}_G) \xrightarrow{\text{Hecke action}} \mathsf{Shv}(\mathsf{Bun}_G) \xrightarrow{\mathsf{coeff}_0} \mathsf{Vect}. \tag{5.1}$$

The previous theorem implies this functor can be viewed as assembling all the Whittaker coefficients in a family. The category $\mathsf{Rep}(\check{G})_{\mathsf{Ran}}$ is self dual, with pairing functor given by

$$\mathsf{Rep}(\check{G})_{\mathsf{Ran}} \otimes \mathsf{Rep}(\check{G})_{\mathsf{Ran}} \xrightarrow{-\star-} \mathsf{Rep}(\check{G})_{\mathsf{Ran}} \xrightarrow{\Gamma} \mathsf{Vect},$$

where $-\star-$ is the symmetric monoidal structure on $\mathsf{Rep}(\check{G})_{\mathsf{Ran}}$ and $\Gamma := \mathsf{Hom}(\mathsf{triv}, -)$. Hence the functor (5.1) can be rewritten as a functor

$$\mathsf{coeff}^{\mathsf{ult}} : \mathsf{Shv}(\mathsf{Bun}_G) \to \mathsf{Rep}(\check{G})_{\mathsf{Ran}}$$

characterized by the formula

$$\Gamma(\mathsf{coeff}^{\mathsf{ult}}(\mathcal{F}) \star \mathcal{G}) \simeq \mathsf{coeff}_0(\mathcal{G} \cdot \mathcal{F}),\ \mathcal{F} \in \mathsf{Shv}(\mathsf{Bun}_G),\ \mathcal{G} \in \mathsf{Rep}(\check{G})_{\mathsf{Ran}}.$$

Here $\mathcal{G} \cdot \mathcal{F}$ is the action of $\mathcal{G}$ on $\mathcal{F}$.

Remark 5.1.6. The functor $\mathsf{coeff}^{\mathsf{ult}}$ should be viewed as the best version of "q-expansion" for automorphic sheaves.

Theorem 5.1.7 (Beraldo, Frenkel–Gaitsgory–Vilonon)**.** *For $G = \mathsf{GL}_n$ or PGL_n, the functor $\mathsf{coeff}^{\mathsf{ult}}$ is fully faithful on $\mathsf{Shv}(\mathsf{Bun}_G)_{\mathsf{cusp}}$.*

For PGL_2, the proof imitates that of Fourier inversion. For GL_n, it imitates the works by Piatetski–Shapiro and Shalika.

5.2. **Categorical geometric Langlands equivalence.** Now we work with the de Rham setting. Consider $\mathsf{LS}_{\check{G}} = \mathsf{LS}^{\mathsf{dR}}_{\check{G}}$, the moduli stack of (de Rham) $\check{G}$-local systems on X. There exists a canonical symmetric monoidal functor

$$\mathsf{Loc} : \mathsf{Rep}(\check{G})_{\mathsf{Ran}} \to \mathsf{QCoh}(\mathsf{LS}_{\check{G}})$$

which sends $V \otimes \delta_x \in \mathsf{Rep}(\check{G}) \otimes \mathsf{DMod}(X)$ to $\mathsf{ev}_x^*(V)$. Here

$$\mathsf{ev}_x : \mathsf{LS}_{\check{G}} \to \mathsf{LS}_{\check{G}}(\mathcal{D}_x) \simeq \mathbb{B}\check{G}$$

is the evaluation map, and we identify $\mathsf{QCoh}(\mathbb{B}\check{G})$ with $\mathsf{Rep}(\check{G})$.

Exercise 5.2.1. For $\sigma \in \mathsf{LS}_{\check{G}}$, identify the functor

$$\mathsf{ev}_\sigma : \mathsf{Rep}(\check{G})_{\mathsf{Ran}} \to \mathsf{Vect}$$

with the composition

$$\mathsf{Rep}(\check{G})_{\mathsf{Ran}} \xrightarrow{\mathsf{Loc}} \mathsf{QCoh}(\mathsf{LS}_{\check{G}}) \xrightarrow{(-)!_\sigma} \mathsf{Vect}.$$

Theorem 5.2.2 (Lurie, Gaitsgory–Rozenblyum)**.** *The functor* Loc *has a fully faithful right adjoint*

$$\mathsf{QCoh}(\mathsf{LS}_{\check{G}}) \xrightarrow{\subset} \mathsf{Rep}(\check{G})_{\mathsf{Ran}}$$

Theorem 5.2.3 (Drinfeld–Gaitsgory)**.** *The action of* $\mathsf{Rep}(\check{G})_{\mathsf{Ran}}$ *on* $\mathsf{DMod}(\mathsf{Bun}_G)$ *factors through* Loc*.*

Corollary 5.2.4. *The functor* $\mathsf{coeff}^{\mathsf{ult}}$ *factors as*

$$\begin{array}{ccc} \mathsf{DMod}(\mathsf{Bun}_G) & \xrightarrow{\mathbb{L}_{G,\mathsf{coarse}}} & \mathsf{QCoh}(\mathsf{LS}_{\check{G}}) \\ & \searrow^{\mathsf{coeff}^{\mathsf{ult}}} & \downarrow \subset \\ & & \mathsf{Rep}(\check{G})_{\mathsf{Ran}} \end{array}$$

Idea of categorical geometric Langlands: the functor $\mathbb{L}_{G,\mathsf{coarse}}$ is "almost" an equivalence. Note that it cannot be an equivalence because for nonabelian G, it sends the constant sheaf to 0.

Main Theorem 5.2.5 (Categorical Geometric Langlands, 1st version)**.** *The functor* $\mathbb{L}_{G,\mathsf{coarse}}$ *induces an equivalence*

$$\mathsf{DMod}(\mathsf{Bun}_G)_{\mathsf{cusp}} \simeq \mathsf{QCoh}(\mathsf{LS}^{\mathsf{irred}}_{\check{G}}),$$

where $\mathsf{LS}^{\mathsf{irred}}_{\check{G}} \subset \mathsf{LS}_{\check{G}}$ *is the locus of irreducible* $\check{G}$*-local systems.*

Corollary 5.2.6. $\mathsf{coeff}^{\mathsf{ult}}$ *is fully faithful on the cuspidal subcategory.*

There is also a correction of $\mathbb{L}_{G,\mathsf{coarse}}$ due to Arinkin–Gaitsgory. Recall for nice enough stack Y, we have

$$\mathsf{QCoh}(Y) \simeq \mathsf{Ind}(\mathsf{Perf}(Y)),$$

where $\mathsf{Perf}(Y)$ is the subcategory of locally bounded cmoplexes of finite rank projective modules. We can also consider

$$\mathsf{IndCoh}(Y) := \mathsf{Ind}(\mathsf{Coh}(Y)),$$

where $\mathsf{Coh}(Y)$ is the category of locally bounded complexes of finite generated modules. We have an inclusion

$$\mathsf{Perf}(Y) \subset \mathsf{Coh}(Y),$$

which is strict if Y has singularities.

Now we can define a subcategory

$$\mathsf{Coh}_{\mathsf{Nilp}}(\mathsf{LS}_{\check{G}}) \subset \mathsf{Coh}(\mathsf{LS}_{\check{G}})$$

such that it is generated by the images of

$$\mathsf{Perf}(\mathsf{LS}_{\check{M}}) \xrightarrow{q^*} \mathsf{Perf}(\mathsf{LS}_{\check{P}}) \xrightarrow{p^*} \mathsf{Coh}(\mathsf{LS}_{\check{G}})$$

for all parabolic subgroups $\check{P}$ and their Levi quotients $\check{M}$. Here the map $q : \mathsf{LS}_{\check{P}} \to \mathsf{LS}_{\check{M}}$ is a map of finite Tor amplitude, and the map $p : \mathsf{LS}_{\check{P}} \to \mathsf{LS}_{\check{G}}$ is proper, so that the above functors are well-defined.

Remark 5.2.7. The subcategory $\mathsf{Coh}_{\mathsf{Nilp}}(\mathsf{LS}_{\check{G}}) \subset \mathsf{Coh}(\mathsf{LS}_{\check{G}})$ can be defined in an intrinsic way using the theory of singular supports of coherent sheaves.

Theorem 5.2.8. *There exists a unique functor*

$$\mathbb{L}_G : \mathsf{DMod}(\mathsf{Bun}_G) \to \mathsf{IndCoh}_{\mathsf{Nilp}}(\mathsf{LS}_{\check{G}})$$

subject to a technical condition[6] *such that the following diagram commutes*

$$\begin{array}{ccc} \mathsf{DMod}(\mathsf{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathsf{IndCoh}_{\mathsf{Nilp}}(\mathsf{LS}_{\check{G}}) \\ & \searrow^{\mathbb{L}_{G,\mathsf{coarse}}} & \downarrow \Psi \\ & & \mathsf{QCoh}(\mathsf{LS}_{\check{G}}), \end{array}$$

where Ψ is the ind-extension of the embedding $\mathsf{Coh}_{\mathsf{Nilp}} \subset \mathsf{Coh} \subset \mathsf{QCoh}$.

Main Theorem 5.2.9 (Categorical Geometric Langlands, ultimate version)**.** *The functor $\mathbb{L}_G$ is an equivalence.*

Remark 5.2.10. The proof uses particularities of the de Rham setting, such as using Kac–Moody localization at the critical level to prove "geometric" statements about $\mathsf{DMod}(\mathsf{Bun}_G)$.

More seriously, we use the fact that the de Rham moduli stack $\mathsf{LS}_{\check{G}}^{\mathsf{dR}}$ has few global (algebraic) functions.

Still, we can obtain the Betti version (and ℓ-adic versions when $\mathsf{char}(k) = 0$) by "Riemann–Hilbert" in some sense.

Remark 5.2.11. The "concrete" theorem 4.3.2 in the last lecture really use all the categorical assertions.

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[6] It should send compact objects in the source to objects bounded from below in the t-structure of the target.