Problems for the 2024 Summer school

1 Grassmannians and Chern classes

1.1

Check that the elementary symmetric function $e_k(x_1, x_2, \dots)$ is the unique symmetric function of degree k such that

$$e_k(x_1, \dots, x_k, 0, 0, 0, \dots) = x_1 x_2 \cdots x_k \,.$$

1.2

Let $c(V, t) = \sum_{k \geqslant 0} t_k c_k(V)$ denote the Chern polynomial of a vector bundle V. Prove that for any subbundle $V' \subset V$ we have

$$c(V, t) = c(V', t)\, c(V/V', t)\,.$$

1.3

Compute the Chern classes of the tangent bundle to $\mathbb{P}^n$. Here and below all projective spaces, Grassmannian, flag varieties etc. denote the corresponding complex manifolds.

1.4

Let $M \subset \mathbb{P}^n$ be a smooth (complex, as always) hypersurface of degree d. Consider the exact sequence of the vector bundles

$$0 \to TM \to T\mathbb{P}^n\Big|_M \to N_{\mathbb{P}^n/M} \to 0$$

on M, where $N_{\mathbb{P}^n/M}$ is the normal bundle. Check that

$$N_{\mathbb{P}^n/M} = \mathscr{O}_{\mathbb{P}^n}(d)\Big|_M$$

and conclude a formula for the Chern classes of M. In particular, when is $c_1(M) = 0$? What is the topological Euler characteristic[1] of M ? What is the genus of the curve M when $n = 2$? Compute $\int_M c_1(M)^{n-1}$.

[1] By the Lefschetz hyperplane section theorem, this is equivalent to knowing the dimension of $H^{\text{middle}}(M)$

1.5

Consider the Grassmannian $X = \mathsf{Gr}(k, n)$. Its points correspond to linear subspaces $V \subset \mathbb{C}^n$ of dimension k. Consider the set of pairs

$$\{(V, v), \text{ such that } v \in V\} \subset \mathsf{Gr}(k, n) \times \mathbb{C}^n \,. \tag{1}$$

Check that the LHS in (1) is a vector bundle of rank k over X, called the tautological subbundle of the trivial bundle in the RHS of (1). By a slight abuse of notation, we denote this tautological bundle by V. Check that

$$TX = V^* \otimes (\mathbb{C}^n/V)\,,$$

where V^* denotes the dual bundle and $\mathbb{C}^n/V$ is the tautological quotient bundle. Express $c_1(TX)$ and $c_2(TX)$ of the tangent bundle TX in terms of the Chern classes of V. If you are familiar with the language of symmetric functions, propose a formula for $c_k(TX)$.

1.6

Consider the product $X \times X$, where $X = \mathsf{Gr}(k, n)$ as in Problem 1.5. On $X \times X$ we have two tautological bundles V_1 and V_2 pulled back from the two factors. Consider the composed map

$$V_1 \to \mathbb{C}^n \to \mathbb{C}^n/V_2\,.$$

Show this section of $(\mathbb{C}^n/V_2) \otimes V_1^*$ vanishes precisely over the diagonal in $X \times X$, in other words, the top Chern class of $(\mathbb{C}^n/V_2) \otimes V_1^*$ is the class

$$\Delta \in H^{2\dim X}(X \times X)$$

of the diagonal. Consider the operation

$$\Phi_\Delta : \gamma \mapsto p_{1,*}(\Delta \cup p_2^*(\gamma))$$

from $H^\bullet(X)$ to $H^\bullet(X)$, where p_1 and p_2 are the projections of $X \times X$ to the respective factors. Show that Φ_Δ is the identity map. Conclude that the cohomology of X is generated by the Chern classes of V.

What is the K-theory analog of these statements ?

1.7

The group $GL(n)$ acts on $X = \mathsf{Gr}(k, n)$ via its defining action on $\mathbb{C}^n$. Describe the orbits of the subgroup $U \subset GL(n)$ formed by lower-triangular matrices with 1s on the diagonal. Show that each orbit contains a unique fixed point for the subgroup

$$\mathsf{A} = \mathrm{diag}(a_1, \dots, a_n)$$

to which all other points are attracted when $a_1/a_2, a_2/a_3, \dots \to \infty$. The orbits are called the Schubert cells and their closures are called Schubert varieties $\mathfrak{S}_\lambda$. They are naturally indexed by partitions λ that fit into $k \times (n-k)$ rectangle. Show they form a basis in integral homology or cohomology of X.

1.8

Show that Schubert cycles $\mathfrak{S}_\lambda^\vee$ for the subgroup U_{opp} of upper-triangular matrices form a basis dual to the basis of Schuber cycles. Translate the equality

$$c_{\mathrm{top}}(V_1^* \otimes (\mathbb{C}^n/V_2)) = \sum_\lambda [\mathfrak{S}_\lambda] \boxtimes [\mathfrak{S}_\lambda^\vee] \in H^\bullet(X \times X)$$

into an identity of symmetric functions.

1.9

Compute the Poincaré polynomial of $\mathsf{Gr}(k,n)$ and compare it with number of points of $\mathsf{Gr}(k,n)$ over a finite field with q elements.

1.10

For $X = \mathbb{P}^{n-1} = \mathsf{Gr}(1,n)$, Schubert classes $\mathfrak{S}_l$, $l = 0, \dots, n-1$, form a chain

$$\mathbb{P}^{n-1} \supset \mathbb{P}^{n-2} \supset \cdots \supset \mathbb{P}^{n-1-l} \supset \dots$$

cut out by the equations $x_1 = \cdots = x_l = 0$. Here x_i are the homogeneous coordinates, or more precisely the components of the natural map

$$(\mathbb{C}^n)^* \otimes \mathscr{O}_X \to \mathscr{O}(1)_X \,.$$

In particular, each individual coordinate x_i is a A-equivariant map

$$\mathscr{O}_X \otimes a_i^{-1} \xrightarrow{x_i} \mathscr{O}(1)_X,$$

and so its zero locus represents the class $\xi + \alpha_i$, where $\xi = c_1(\mathscr{O}(1))$ and $\alpha_i \in H^2_{\mathsf{A}}(\mathrm{pt})$ corresponds to the character a_i. Therefore

$$[\mathfrak{S}_l] = \prod_{i=1}^{l} (\xi + \alpha_i) \in H^\bullet_{\mathsf{A}}(X) \,. \tag{2}$$

1.11

Verify that the polynomial (2) is characterized by the following Newton interpolation properties:

- it has degree l in the variables ξ and α_i, corresponding to the fact that $[\mathfrak{S}_l] \in H^{2l}(\mathsf{Gr})$,
- its restriction to A-fixed points not in $\mathfrak{S}_l$ vanishes ,

- its restriction to the A-fixed point in the Schubert cell equals the Euler class of the normal bundle to the Schubert cell.

Generalize this reasoning to compute the classes of the Schubert cells in the $\mathsf{Gr}(k,n)$. Your answer should look like a Schur function in the Chern roots $\xi_1, \dots, \xi_k$ of the universal bundle, in which the monomials ξ_j^l are replaced by univariate interpolation polynomials of the form (2). Those unfamiliar with Schur functions will discover them for themselves by solving Problem 1.13

1.12

Generalize the results of Problems 1.10 and 1.11 to equivariant K-theory.

1.13

Let G be the group $GL(n,\mathbb{C})$, $B \subset G$ be the subgroup of the upper-triangular matrices and $\chi : B \to \mathbb{C}^\times$ a character. Consider holomorphic, or meromophic, functions $f(g)$ of $g \in G$ which satisfy

$$f(gb) = f(g)\chi(b)\,, \forall b \in B\,.$$

Interpret them as sections of a holomorphic line bundle $\mathscr{L}_\chi$ on flag manifold

$$\mathrm{Flags}_n = G/B = U(n)/\text{diagonal matrices}\,.$$

Compute the Euler characteristic $\chi(\mathscr{L}_\chi)$ by equivariant localization. Compare your result with the Weyl character formula for G and explain[2] this comparison using the Peter-Weyl decomposition

$$\mathbb{C}[G] = \bigoplus_{\text{irreps } V} V^* \boxtimes V\,, \qquad \text{as } G \times G\text{-modules}\,.$$

1.14

Let $\mathscr{L}$ be a complex line bundle with a connection ∇ and corresponding curvature $F \in \Omega^2(X,\mathbb{C})$. Show that, nonequivariantly, the form $\frac{i}{2\pi}F$ represents $c_1(\mathscr{L})$. For an equivariant generalization, see Chapter 7 in [`?Berline-Getzler-Vergne`].

For a rank r vector bundle V, the curvature form is matrix-valued, that is, $F \in \Omega^2(X, \operatorname{End} V)$. Show that

$$\sum_k t^k c_k(V) = \det\left(1 + \frac{it}{2\pi}F\right)\,,$$

nonequivariantly.

[2] For a simple proof of fact that at most one cohomology group of $\mathscr{L}_\chi$ is nonvanishing see [`?Demazure`]

2 Elliptic functions and elliptic curves

2.1

Consider holomorphic functions $f(z)$ of $z \in \mathbb{C}^\times$ solving the q-difference equation

$$f(qz) = cz^{-d} f(z)\,, \tag{3}$$

where q is a fixed complex number such that $|q| < 1$. Compute the dimension of the space of solutions as a function of d (and c, for $d = 0$) in two ways: first by analyzing the Laurent series expansion of f, and then by using the Riemann-Roch formula for the complex elliptic curve $E = \mathbb{C}^\times/q^{\mathbb{Z}}$.

2.2

Consider the function[3]

$$\theta(z) = \prod_{n>0}(1-q^n z)\prod_{n\geqslant 0}(1-q^n z^{-1})$$

and check that is solves the equation

$$\theta(qz) = -q^{-1}z^{-1}\theta(z)\,.$$

Prove that a general solution of (3) has the form

$$f(z) = \text{const}\prod_{i=1}^{d}\theta(z/w_i)\,, \quad \prod w_i = (-q)^d c\,.$$

Interpret this result as saying that two divisors

$$\sum_{i=1}^{d} w_i, \sum_{i=1}^{d} w_i' \in S^d E$$

are linearly equivalent if and only if $\prod w_i = \prod w_i'$ in E, that is, modulo $q^{\mathbb{Z}}$. In other words, the natural map

$$S^d E \to \operatorname{Pic}_d E \cong E$$

is from divisors to line bundles of the same degree is given by the multiplication in the group E. Its fibers are projective spaces for $d > 0$.

[3] In many, many contexts, it is more convenient to use a different normalization of the theta function, namely

$$\vartheta(z) = z^{1/2}\theta(z) = (z^{1/2} - z^{-1/2})\prod_{n>0}(1-q^n z)(1-q^n z^{-1})\,.$$

It has a series expansion in half-integer powers of z, that is, satisfies $\vartheta(e^{2\pi i}z) = -\vartheta(z)$. It is still the unique, up to multiple, section of the line bundle $\mathscr{O}(e)$, where $e = 1 \in E = \mathbb{C}^\times/q^{\mathbb{Z}}$ is the identity, just for a different trivialization of the pullback of this line bundle to $\mathbb{C}^\times$. The extra convenience of using $\vartheta(z)$ is due to its anti-symmetry $\vartheta(z^{-1}) = -\vartheta(z)$.

2.3

Let $f(z)$ be a meromorphic function on E, equivalently a rational function on the algebraic variety E. Show that is has the form

$$f(z) = \text{const} \prod_{i=1}^{d} \frac{\theta(z/a_i)}{\theta(z/b_i)}$$

for some values of a_i and b_j, where $\prod a_i = \prod b_i$ in E, that is, modulo $q^{\mathbb{Z}}$.

2.4

By cutting partitions λ along the diagonal, prove that

$$\sum_{\lambda} q^{\lambda} = \text{coefficient of } z^0 \text{ in } \prod_{n>0} (1 + q^{n-1/2} z)(1 + q^{n-1/2} z^{-1}) .$$

Deduce that

$$\sum_{n \in \mathbb{Z}} q^{n^2/2} z^n = \prod_{n>0} (1 - q^n)(1 + q^{n-1/2} z)(1 + q^{n-1/2} z^{-1}) ,$$

which is one of the equivalent forms of the Jacobi triple product identity, and of the Macdonald identity for the Lie algebra $\mathfrak{sl}(2)$. Note this means

$$\sum_{n \in \mathbb{Z}} (-1)^n q^{\binom{n+1}{2}} z^n = \theta(z) \prod_{n>0} (1 - q^n)$$

and compare the rate of convergence of two sides.

2.5

Let $\Gamma \subset \mathbb{C}$ be a lattice and $x \in \mathbb{C}$ be a complex number. Form the following product

$$\sigma(x) = x \prod_{0 \neq \gamma \in \Gamma} \left(1 - \frac{x}{\gamma}\right) \exp\left(\frac{x}{\gamma} + \frac{x^2}{2\gamma^2}\right) ,$$

known as the Weierstrass σ-funciton, and show that it represents an odd entire function of x. Consider a vector $\gamma \in \Gamma \backslash 2\Gamma$, which means

$$\sigma(\gamma/2) = -\sigma(\gamma/2) \neq 0 .$$

For such vector γ, prove that

$$\frac{\sigma(x + \gamma)}{\sigma(x)} = -\exp(\eta_\gamma (x + \gamma/2))$$

for some constant $\eta_\gamma \in \mathbb{C}$. Express the function $\sigma(x)$ in terms of the theta function of the elliptic curve $E = \mathbb{C}/\Gamma$.

2.6

Given a lattice Γ as in Problem 2.5, its holomorphic Eisenstein series are defined by

$$\mathrm{Eisenstein}(\Gamma, n) = \sum_{0 \neq \gamma \in \Gamma} \gamma^{-n} \,,$$

which converges for $n > 2$ and vanishes for n odd. Relate these series to $\ln \sigma(x)$ and express them in terms of the parameter q in the isomorphism $E = \mathbb{C}/\Gamma \cong \mathbb{C}^\times / q^{\mathbb{Z}}$.

2.7

The *theorem of the cube* refers to various generalizations of the following basic statement. Let X and Y be complete algebraic varieties over a field $\Bbbk$, and let Z be an arbitrary variety over $\Bbbk$. Let $x \in X$, $y \in Y$, $z \in Z$ be $\Bbbk$-points. Let $\mathscr{L}$ be a line bundle over $X \times Y \times X$. If $\mathscr{L}$ is trivial when restricted to $\{x\} \times Y \times Z$, $X \times \{y\} \times Z$, and $X \times Y \times \{z\}$, then $\mathscr{L}$ is trivial. Find or read [4] a proof of this or any related statement. In the example $X = Y = E$, check that the corresponding statement for two factors is false.

2.8

Let A be an abelian variety, which for our purposes we will always assume to be of the form $A = E^n$. Let $\mathrm{Pic}(A)$ denote the Picard group of line bundles on A and let $\mathrm{Pic}_0(A)$ be the subgroup of line bundles that are algebraically equivalent to zero [5] Consider the map

$$\phi : \mathrm{Pic}(A) \times A \to \mathrm{Pic}_0(A)$$

that takes

$$(\mathscr{L}, a) \mapsto (\text{translation by } a)^* \mathscr{L} \otimes \mathscr{L}^{-1} \,.$$

Prove this is a group homomorphism, which is one of the forms of the *theorem of the square*.

[4]there are many sources for reading about this result, which goes back to A. Weil, with a classical exposition by Mumford. Among online resources, `https://www.math.ru.nl/personal/bmoonen/BookAV/LineBund.pdf` may be recommended.

[5]Two line bundles $\mathscr{L}_1$ and $\mathscr{L}_2$ are algebraically equivalent $\mathscr{L}_1 \sim \mathscr{L}_2$ if there is a line bundle $\widetilde{\mathscr{L}}$ on $A \times B$, where B connected, such that

$$\mathscr{L}|_{A \times \{b_1\}} = \mathscr{L}_1 \,, \quad \mathscr{L}|_{A \times \{b_2\}} = \mathscr{L}_2 \,,$$

for some $b_1, b_2 \in B$. You should check that this is an equivalence relation and

$$\mathscr{L}_1 \sim \mathscr{L}_1', \mathscr{L}_2 \sim \mathscr{L}_2' \quad \Rightarrow \quad \mathscr{L}_1 \otimes \mathscr{L}_2 \sim \mathscr{L}_1' \otimes \mathscr{L}_2' \,.$$

2.9

Let $\mathscr{L} \in \operatorname{Pic}(A)$ be homogeneous, that is, $\phi(\mathscr{L}, a) = 0$ for all $a \in A$. Let

$$p_1, p_2, m : A^2 \to A$$

be the two projections and the multiplication map. Prove that

$$p_1^* \mathscr{L} \otimes p_2^* \mathscr{L} = m^* \mathscr{L} ,$$

and conclude

$$H^\bullet(\mathscr{L}) \otimes H^\bullet(\mathscr{L}) = H^\bullet(\mathscr{L}) \otimes H^\bullet(\mathscr{O}_A) .$$

Since $\dim H^0(\mathscr{O}_A) = 1$, it follows that

$$\dim H^0(\mathscr{L}) \in \{0, 1\} .$$

Show that in first case $H^i(\mathscr{L}) = 0$ for all i, while in the second case $\mathscr{L}$ is trivial.

2.10

Show that any $\mathscr{L} \in \operatorname{Pic}_0(A)$ is homogeneous. We will see a converse to this statement below in Problem 2.14

2.11

For an elliptic curve E, check the exact sequence

$$0 \to \operatorname{Pic}_0(E) \to \operatorname{Pic}(E) \xrightarrow{\deg} \mathbb{Z} \to 0$$

and identify $\operatorname{Pic}_1(E)$, and hence $\operatorname{Pic}_0(E)$ with E itself.

Prove that for any B and any line bundle $\widetilde{\mathscr{L}}$ on $E \times B$ whose restrictions to the E-fibers has degree 0 and whose restriction to $0 \times B$ is trivial, there is a map $f : B \to E$ such that

$$\mathscr{L} = (\mathrm{id} \times f)^* \mathscr{P}$$

where $\mathscr{P}$ is the following line bundle on

$$\mathscr{P} = \mathscr{O}(\operatorname{diag} E - E \times \{0\} - \{0\} \times E)$$

on $E \times E$. This realizes E as the *dual* abelian variety $E^\vee = E$, and $\mathscr{P}$ as the Poincaré line bundle on $E \times E^\vee$.

2.12

For an abelian variety of the form $A = E^n$ prove that

$$A^\vee \cong A\,,$$

and construct the Poincare line bundle.

2.13

For elliptic curve E, define the Fourier-Mukai transform

$$\Phi : D^b \operatorname{Coh} E \to D^b \operatorname{Coh} E$$

by the equality

$$\mathscr{F} \mapsto p_{1,*}\left(\mathscr{P} \otimes p_2^*(\mathscr{F})\right)$$

where $p_1, p_2 : E^2 \to E$ are the two projections and the pushforward $p_{1,*}$ is derived. Show that

$$\Phi^2 = (\text{pullback by } a \mapsto -a)\,[-1]\,,$$

where $[-1]$ denotes the cohomological shift of a complex by one step to the right. In particular, Φ is an equivalence. Generalize to an abelian variety of the form $A = E^n$.

2.14

Let $\mathscr{L}$ be a homogeneous line bundle on $A = E^n$ which is not in $\operatorname{Pic}_0(A)$. Using the results of problem 2.9, show that $\Phi(\mathscr{L}) = 0$ and derive a contradiction. This proves that

$$\mathscr{L} \in \operatorname{Pic}_0(A) \quad \Leftrightarrow \quad \phi(\mathscr{L}, \text{—}) = 0\,.$$

Prove the exact sequence

$$0 \to \operatorname{Pic}_0(A) \to \operatorname{Pic}(A) \xrightarrow{\phi(\mathscr{L},\text{—})} \operatorname{Hom}_{\text{symmetric}}(A, A^\vee) \to 0\,, \tag{4}$$

where symmetric means $\phi : A \to A^\vee$ is equal to the pullback map $\phi^\vee : \operatorname{Pic}_0(A^\vee) = A \to \operatorname{Pic}_0(A) = A^\vee$.

2.15

The sequence (4) is true for all abelian varieties, not just those of the form $A = E^n$, but the proof is more involved. It shows that the map $\phi(\mathscr{L}, \text{—})$ is the correct multivariate generalization of the degree.

What is the degree of the Poincaré bundle on $E \times E$? What is the degree of the line bundle on $A = E^n$ whose section $s(z)$ is given by

$$s(z) = \prod \theta(c_\mu z^\mu)^{m_\mu}$$

where $z^\mu = \prod_{i=1}^n z_i^{\mu_i}$, $c_\mu \in \mathbb{C}^\times$, and $m_\mu \in \mathbb{Z}$. When does such expression give a rational function on A ?

3 Krichever's proof of rigidity of the elliptic genus

3.1

We begin with a discussion of how to read and interpret localization formulas. Let V be an equivariant vector bundle on X. Define

$$\Lambda_t^\bullet V = \sum_n (-t)^n \Lambda^n V \,, \quad S_t^\bullet = \sum_n t^n S^n V \,,$$

where we interpret the second expression as an element of $K_G(X)[[t]]$. Check that[6]

$$\Lambda_t^\bullet V \otimes S_t^\bullet V = 1 \,.$$

3.2

Let $V \in K(X)$ be a vector bundle and assume that $\dim_{\mathbb{Q}} K(X) \otimes \mathbb{Q}$ is a finite-dimensional vector space over $\mathbb{Q}$. Prove that all eigenvalues of the operator of tensor product by V in $K(X) \otimes \mathbb{Q}$ are equal to $\operatorname{rk} V$. Moreover, these operators commute for different bundles V_1, V_2. Conclude that all eigenvalues of the operator $\otimes \Lambda_t^\bullet V$ are equal to $(1-t)^{\operatorname{rk} V}$, and hence this operator is invertible as a rational function in t with a pole at $t = 1$. As $t \to 0, \infty$, we have

$$(\Lambda_t^\bullet V)^{-1} \sim 1, \quad t \to 0, \qquad (\Lambda_t^\bullet V)^{-1} \sim t^{-\operatorname{rk} V} \Lambda^{\mathrm{top}} V^*, \quad t \to \infty \,.$$

What does this say about series of the form

$$\chi(X, \mathscr{F} \otimes S_t^\bullet V) \in \mathbb{Z}[[t]]$$

where $\mathscr{F} \in K(X)$ is arbitrary ?

3.3

What is the equivariant analog of the results in problem 3.2 ?

3.4

We abbreviate

$$\phi(z) = \prod_{n>0} (1 - q^n z)$$

[6] you may want to interpret this equality in terms of the Koszul resolution of structure sheaf $\mathscr{O}_0$ of the zero section of V^*

and define Krichever genus by

$$\mathscr{E}_y(X) = \chi\left(X, \frac{\theta(y \otimes TX)}{\phi(TX)\phi(T^*X)}\right) \quad y \in \mathbb{C}^\times .$$

Here X is compact complex or, more generally, a stably almost complex manifold. Assuming there is S^1-action on X, write make the equivariant localization formula for $\mathscr{E}_y(X)$ explicit. Determine the possible singularities of $\mathscr{E}_y(X)$ as a function on the complexification $\mathbb{C}^\times$ of the group S^1.

3.5

Consider the canonical bundle $\mathscr{K}_X = \Lambda^{\mathrm{top}} T^*X$. Assume that $\mathscr{K}_X$ admits, equivariantly, a root of order N and that $y^N = 1$. (This includes the case when $\mathscr{K}_X$ is trivial and y is arbitrary.) These are the assumptions in the rigidity theorem for $\mathscr{E}_y(X)$. Show that, with these assumptions, $\mathscr{E}_y(X)$ is invariant under $t \mapsto qt$, that is, a function on $E = \mathbb{C}^\times / q^{\mathbb{Z}}$.

3.6

For any $n = 1, 2, \ldots$, let $\mu_n \in S^1$ be the group of elements of order n, and let X_n be the fixed locus of μ_n. Show that it is a smooth and (stably, almost) complex. Denote by N_n the normal bundle of X_n in X. Check that

$$\mathscr{E}_y(X) = \chi\left(X^{(n)}, \frac{\theta(y \otimes TX_n)}{\phi(TX_n)\phi(T^*X_n)} \frac{\theta(y \otimes N_n)}{\theta(N_n)}\right) .$$

Conclude that $\mathscr{E}_y(X)$ is regular at all points of order n in E and, hence, a constant.

Solutions and hints