



Born July 26, 1969, in Moscow.

Samuel Eilenberg Professor of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027 USA

Education:

- 1993 B.S. in Mathematics, Moscow State University, *summa cum laude*
- 1995 Ph.D. in Mathematics, Moscow State University

Honors:

Sloan Research Fellowship (2000), Packard Fellowship (2001), European Mathematical Society Prize (2004), Fields Medal (2006), Compositio Prize (2009), elected to the US National Academy of Science (2012), American Academy of Arts and Sciences (2016), Royal Swedish Academy of Sciences (2020), Chinese Academy of Sciences (2023), ICM Plenary talk (2018), Frontier of Science Awards (2023,2024).

Service: I am a long-time supporter of the Mathematical Sciences Research Institute in Berkeley, first as a member and then chair of the Scientific Advisory Committee, and then as a member and vice-chair of the Board of Trustees. Further advisory and trustee board memberships include the Clay Mathematics Institute, Packard Foundation, Simons Center for Geometry and Physics, Hamilton Mathematics Institute, and others. I served as the editor of a number of journals, including the Journal of the American Mathematical Society. I organized a large number of conferences, workshops, special programs etc. Resolution #1 of the 2022 General Assembly of the International Mathematics Union expressed the IMU's deep gratitude for my "tireless dedication and commitment" as as a member of the Executive Committee of the International Mathematics Union and the Local Organizing committee of the 2022 International Congress of Mathematicians

Web site: <https://www.math.columbia.edu/~okounkov/>

Graduate students:

Eric Carlsson (2008), Rodolfo Rios-Zertuche (2012), Michael McBreen (2013), Daniel Shenfeld (2013), Andrei Negut (2015), Andrei Smirnov (2016), Changjian Su (2017), Peter Pushkar (2018), Noah Arbesfeld (2018), Anton Osinenko (2019), Ivan Danilenko (2020), Henry Liu (2021), Yakov Kononov (2021), Sam DeHority (2024), Davis Lazowski (current), Che Shen (current)

Postdoctoral supervision:

Alexei Oblomkov, Maria Angelica Cueto, Daniel Halpern-Leistner, Galyna Dobrovolska, Eugene Gorsky, Yi Sun, Sylvain Carpentier, Gus Schrader, Tudor Padurariu

Previous positions:

- Chern-Simons Distinguished Chair in Mathematical Physics, University of California at Berkeley, 2022–2023,
- William S. Tod Professor of Mathematics, Princeton University, 2002–2010,
- Assistant, then Full Professor of Mathematics, UC Berkeley, 1999–2002,
- L. E. Dickson Instructor in Mathematics, University of Chicago, 1996–1999,
- MSRI postdoc, Spring 1997,
- IAS postdoc, Spring 1996,
- Institute for Problems of Information Transmission, Moscow, Research Fellow, then Leading Researcher, 1994–2019.
- Skolkovo Institute for Science and Technology, Professor, 2017-2022

Publications

- [1] *Noncommutative geometry of random surfaces*, *Funct. Anal. Appl.*, **58:1** (2024), 65--79.
- [2] (with M. Moreira, A. Oblomkov, and R. Pandharipande), *Virasoro constraints for stable pairs on toric 3-folds*, *Forum Math. Pi* **10** (2022), Paper No. e20, 62 pp.
- [3] (with A. Smirnov), *Quantum difference equation for Nakajima varieties*, *Invent. Math.* **229** (2022), no. 3, 1203--1299.
- [4] *Inductive construction of stable envelopes*, *Lett. Math. Phys.* **111** (2021), no. 6, 56 pp.
- [5] (with Ya. Kononov and A. Osinenko), *The 2-leg vertex in K-theoretic DT theory*, *Comm. Math. Phys.* **382** (2021), no. 3, 1579--1599.
- [6] (with M. Aganagic), *Elliptic stable envelopes*, *J. Amer. Math. Soc.* **34** (2021), no. 1, 79--133.
- [7] (with A. Oblomkov and R. Pandharipande) *GW/PT descendent correspondence via vertex operators*, *Comm. Math. Phys.* **374** (2020), no. 3, 1321--1359.
- [8] (with D. Maulik), *Quantum groups and quantum cohomology*, *Astérisque* No. 408 (2019), 209 pp.
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- [10] *On the crossroads of enumerative geometry and geometric representation theory*, *Proceedings of the International Congress of Mathematicians, Rio de Janeiro 2018. Vol. I. Plenary lectures*, 839--867, World Sci. Publ., 2018.
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- [12] *Enumerative geometry and geometric representation theory*, *Algebraic geometry: Salt Lake City 2015*, 419--457, *Proc. Sympos. Pure Math.*, **97.1**, Amer. Math. Soc., Providence, RI, 2018.
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- [14] (with M. Aganagic) *Quasimap counts and Bethe eigenfunctions*, *Mosc. Math. J.* **17** (2017), no. 4, 565--600.
- [15] (with N. Nekrasov) *Membranes and Sheaves*, *Algebraic Geometry*, **3** (2016), no. 3, 320--369.
- [16] *Limit shapes, real and imagined*, *Bull. Amer. Math. Soc.(N.S.)* **53** (2016), no. 2, 187--216.
- [17] (with E. Rains) *Noncommutative geometry and Painlevé equations*, *Algebra Number Theory* **9** (2015), no. 6, 1363--1400.
- [18] *Hilbert schemes and multiple q -zeta values* *Funct. Anal. Appl.* **48** (2014), no. 2, 138--144.
- [19] (with E. Carlsson and N. Nekrasov) *Five dimensional gauge theories and vertex operators*, *Mosc. Math. J.* **14** (2014), no. 1, 39--61.
- [20] (with E. Carlsson), *Exts and vertex operators*, *Duke Math. J.* **161** (2012), no. 9, 1797--1815
- [21] (with D. Maulik, A. Oblomkov, and R. Pandharipande) *Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds*, *Invent. Math.* **186** (2011), no. 2, 435--479.
- [22] (with A. Braverman and D. Maulik) *Quantum cohomology of the Springer resolution*, *Adv. Math.* **227** (2011), no. 1, 421--458.
- [23] (with R. Pandharipande) *The quantum differential equation of the Hilbert scheme of points in the plane*, *Transform. Groups* **15** (2010), no. 4, 965--982.
- [24] (with R. Pandharipande) *The local Donaldson-Thomas theory of curves*, *Geom. Topol.* **14** (2010), no. 3, 1503--1567
- [25] (with R. Pandharipande) *Quantum cohomology of the Hilbert scheme of points in the plane*, *Invent. Math.* **179** (2010), no. 3, 523--557

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- [29] (with R. Kenyon) *Limit shapes and the complex Burgers equation*, Acta Math. 199 (2007), no. 2, 263–302.
- [30] (with G. Mikhalkin) *Geometry of planar log-fronts*, Mosc. Math. J. 7 (2007), no. 3, 507–531.
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- [37] (with A. Eskin), *Pillowcases and quasimodular forms*, Algebraic geometry and number theory, 1–25, Progr. Math., 253, Birkhäuser Boston, Boston, MA, 2006.
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- [39] (with R. Kenyon and S. Sheffield) *Dimers and amoebae*, Ann. of Math. (2) 163 (2006), no. 3, 1019–1056.
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Preprints

- [88] (with David Kazhdan) *L-function genera and applications*, arXiv:2311.17747.
- [89] (with David Kazhdan) *On the unramified Eisenstein spectrum*, arXiv:2203.03486.
- [90] *Nonabelian stable envelopes, vertex functions with descendents, and integral solutions of q -difference equations*, arXiv:2010.13217.
- [91] (with R. Bezrukavnikov) *Monodromy and derived equivalences*, in preparation.
- [92] (with H. Nakajima) *Shifted quantum groups and Coulomb branches*, in preparation.
- [93] (with D. Halpern-Leistner and D. Maulik) *Categorical stable envelopes and magic windows*, in preparation.
- [94] (with M. Aganagic) *Duality interfaces in 3-dimensional theories*, in preparation. See the report at String-Math2019 and also this talk at MSRI.

Outreach/popular writings

- [95] *Combinatorial geometry takes the lead*, Proceedings of the ICM 2022, EMS Press, Berlin, 2023, 414–458.
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Research

My research is on the crossroads of mathematical physics, probability theory, representation theory, and algebraic geometry. Below is a very brief description of some of my results in the chronological order. This cannot possibly serve as a survey of the areas of mathematics to which these papers belong¹. I will skip over a very large fraction of my own papers, despite the fact that I have put a lot of thought and work into writing each one of them and I believe none of them deserves obscurity.

In my PhD thesis [79, 86], I completed the program of the classification of admissible representation of the infinite symmetric group $S(\infty)$ initiated by G. Olshanski in [126]. This includes representations with a trace, and hence gives a new proof (in fact, several new proofs) of the classical description of the characters of $S(\infty)$ obtained originally by E. Thoma [131] and later A. Vershik and S. Kerov [133] by different means. So, as a special case, this gives new proofs of the classification of the so-called totally positive, or PF, sequences — a classical result in analysis to which G. Polya, I. Schoenberg, A. Edrei, and others have contributed [103, 107].

In a sequence [32, 65, 68, 73, 78] of papers with Olshanski we realized the Vershik-Kerov vision of asymptotic spherical function theory for all infinite series of Riemannian symmetric spaces. In fact, we did it for general special functions of Heckmann-Opdam type as their ranks grows to infinity.

A number of new technical ideas from that work had an impact in rather distant areas of mathematics. For instance, convex bodies, now known as Okounkov bodies, were introduced originally in [69, 81] to prove inequalities on multiplicities of irreducible representations. Subsequently, they have become a very popular tool in a wide range of contexts ranging from differential geometry to number theory.

Similarly, multivariate interpolation polynomials introduced by Olshanski and me [73, 78] became a very popular and powerful tool to work with Macdonald polynomials and their generalization. In particular, the general “binomial” formula proven in [74] sheds light on the most difficult properties of Macdonald polynomials, see for instance the short note [54] for a demonstration and [127, 128] for the current state of art in the subject. The binomial formula also plays the key role in the asymptotic analysis.

Also from my graduate years, the inductive approach to constructions of representations of symmetric group that became known as the Okounkov-Vershik approach [80] has been applied in many other situations, and is subject of several monographs.

An important theme in my research are the interactions between probability and representation theory, in which objects like random partitions, random permutations, and random matrices show both their probabilistic and algebraic sides. A good example may be my work on the conjecture of Baik, Deift, and Johansson [105] (BDJ), a work which had a certain resonance in the community and has been highlighted in the popular articles written about my Fields Medal. The conjecture states that the distribution of increasing subsequences in

¹References after [100] refer to the necessarily very abbreviated bibliography at the end of this narrative. Please turn to my papers cited for a comprehensive set of references.

a very large random permutation is, up to normalizations, the same as the distribution of largest eigenvalues of a random Hermitian matrix.

I gave two proofs of the BDJ conjecture. In the first one [63], random permutations are connected to random matrices through an intermediate object, a random branched cover of the sphere, an object which turns out to be closely related to random tessellations of surfaces and random trees.

As one of the ramifications of techniques introduced in [63], R. Pandharipande and I gave a proof [26] of Kontsevich's combinatorial formula for intersection number on the moduli spaces of pointed curves [116]. This formula is the key step in the argument leading to the proof of the Witten conjecture — a result very much celebrated by the community. Witten's conjecture was stimulated by a comparison between different theories of 2-dimensional quantum gravity and computes the intersection numbers in terms of a certain particular solutions to the Korteweg-de Vries hierarchy [134]. It is the point of departure for many interactions between enumerative geometry and integrable systems/representation theory discussed below. Other proofs of the Witten conjecture were subsequently found by M. Mirzakhani [118] and S. Lando and M. Kazaryan [115]. The latter proof very elegantly exhibits the problem as a special case of a more general and richer problem of the Gromov-Witten theory of \mathbb{P}^1 , and of the Toda equations there, proven in various degrees of generality in [41, 61]. A detailed proof of combinatorial formula along the lines originally envisioned by Kontsevich was subsequently given by D. Zvonkine in [135].

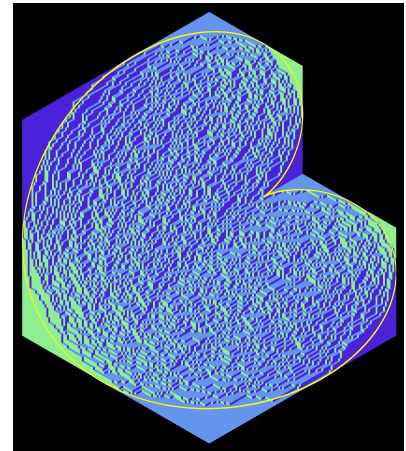
If we replace random covering of the sphere by coverings of the torus, we get into the domain studied by S. Bloch and me. Our formula [57, 64] for the character of the infinite wedge representation may be interpreted, among other things, as an explicit enumeration of branched covers of the torus by their ramification type. This was used by Eskin and me to compute the asymptotics of these numbers and thereby compute the volumes of various moduli spaces of flat surfaces [59]. These computations enter crucially into many formulas of Teichmüller dynamics, which is an area of rapid growth in dynamical systems.

A different proof of the BDJ conjecture was subsequently given by me in collaboration with Borodin and Olshanski in [62]. The route taken in that paper is very different. It goes through seeing both increasing subsequences in random permutations and eigenvalues of a random matrix as two different instances of the same general notion of a determinantal point process. Determinantal and pfaffian point processes encompass many other objects of central importance in probability and mathematical physics, such as perfect matching or dimer configurations on planar graphs. The ideas of [62] were subsequently expanded in my papers [45, 53, 57]. These papers introduce the notion of a Schur measure and a Schur process, respectively, and give general formulas for the n -point correlation functions of these particle systems/processes, both exact and asymptotic. Statements like the BDJ conjecture, or its process version, become simple corollaries of these techniques.

Dimer models on periodic planar graphs, another instance of a determinantal process, were the subject of a sequence of papers [29, 38, 39] of R. Kenyon and I, also joint with S. Sheffield in the first issue. There are two principal results in this sequence of papers. The first concerns the surface tension for the dimer models (which may be interpreted as random surface models via their associated height functions). In [39], we identify these functions, and their singularities (which are, from the mathematical physics point of view, some of their most

important characteristics) in terms of the geometry of certain real algebraic curves, the “spectral curves” of the model. Interestingly, the singularities persist outside of a small resonant sets of parameters and this corresponds to spectral curves being very special real algebraic curves known as Harnack curves. Here, the information flows both ways between probability theory and real algebraic geometry. For example, R. Kenyon and I showed in [38] that the set of Harnack curves is connected (and, in fact, contractible), which was an open question in real algebraic geometry.

The second main result of Kenyon and I is an explicit construction of surface tension minimizers (and thus limit shapes, that is, law of large numbers configurations) in terms of algebraic geometry [29]. The essence of this result may be illustrated by the rather popular image from [47]. It shows a random stepped surface spanning a given frame of 8 segments. One can really see the cardioid, both its real points as the frozen boundary and its complex points as the disordered region. The picture appears e.g. as the Wikipedia image for the Notices of the AMS.



The idea of a formation of a limit shape in a counting problem, and its determination in terms of real algebraic geometry, played the key role in my paper [44], joint with N. Nekrasov. In the 90s, Seiberg and Witten had a remarkable physical insight that allowed them to guess the low-energy description of certain supersymmetric gauge theories in terms of periods of a particular family of hyperelliptic curves [130]. Nekrasov proposed a direct gauge-theoretic derivation of this result through what can be described as the thermodynamics of a gas of instantons [124]. By localization, this may be also seen as a random partition problem, which was completely solved in [44]. The curve of Seiberg and Witten appears very directly as the limit shape in the random partition problem, just like the cardioid in the above image. Google Scholar thinks this is my most influential work since it has counted 1142 citations for it the last time I checked.

A quantization of these limit shapes, which involves a noncommutative deformation of the curve responsible for the limit shape, allows to describe the fluctuations around the limit shape in all orders of the expansion in the inverse size of the system [1]. Quasiperiodic phenomena in this expansion were connected by E. Rains and me to remarkable dynamical systems [17] which are vast generalizations of the Sakai’s elliptic Painleve equations and thus also of the ubiquitous differential equations from Painleve’s list.

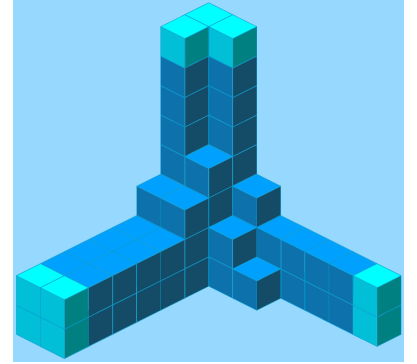
Counting branched covers of the sphere and of the torus, proof of Kontsevich’s formula, etc. above can all be seen as special cases of enumerative geometry of maps from one Riemann surface to another, known as the Gromov-Witten (GW) theory of curves (which, in turn, is a very special case of the more general enumerative theories of curves to be discussed below). Here, and in several instances below, the word “theory” is traditionally used in the theoretical physics sense of the word, that is, to denote a framework in which something could be, in principle, computed and well-defined questions may be asked. In other words, translated to the usual language of mathematics the word “theory” should read “an infinite set of challenging questions”.

A complete answer to all question in the Gromov-Witten theory of curves has been obtained

in a sequence [40–42] of papers by R. Pandharipande and me. This includes a proof of the 2-Toda equations in the equivariant Gromov-Witten theory of \mathbb{P}^1 , which is the natural generalization of KdV equations of Witten, previously conjectured based on the insights of T. Eguchi and his collaborators [108, 109]. It also includes the proof of (a generalization) of the Virasoro constraints in the Gromov-Witten theory of curves. An important feature of our description is that it is effective and is given in the standard operator language of mathematical physics. For instance, the 2-Toda equation follows from such description, and the work of the Kyoto school [119], in a very transparent way. The explicit nature of our answers made it a basis of many further computations.

General question in Gromov-Witten theory concern the enumerative geometry of maps from a variable source curve C (or Riemann surface) to a fixed complex manifold X . This is closely related to the study of the supersymmetric sigma-model with target X coupled with the 2-dimensional gravity in the domain C , and thus has received a very significant influx of ideas from modern theoretical physics.

One particular such idea, known as the “topological vertex” was proposed by Aganagic, Klemm, Marino, and Vafa based on conjectural connections between curve counts in a Calabi-Yau 3-fold X and knot invariants [102]. Reshetikhin, Vafa, and I linked this conjecture with the earlier work [53] of Reshetikhin and I, and with the geometry of the Hilbert schemes of curves in X . Concretely, in [43] we showed that the topological vertex enumerates 3-dimensional partitions with infinite legs ending on three given 2-dimensional partitions. We also matched that data to torus-fixed points in the Hilbert scheme of curves in \mathbb{C}^3 .



This was a very important input into the sequence of papers [35, 36] by Maulik, Nekrasov, Pandharipande, and me, usually abbreviated MNOP. In these papers, in which the term “Donaldson-Thomas” theory (DT) was coined, a very general conjectural correspondence was proposed between fully equivariant Gromov-Witten counts for a quasi-projective 3-fold X (which need not be Calabi-Yau, or otherwise restricted) and enumerative geometry of sheaves on X (such as 1-dimensional ideal sheaves) of the kind studied by Richard Thomas in [132] as an extension of Donaldson’s theory for Kähler surfaces. The correspondence is very powerful but not direct: it equates generating functions after an exponential change of the variable, thus producing e.g. all-genera Gromov-Witten counts in any given degree. It would be fair to say that the MNOP conjectures shaped a lot of subsequent research in enumerative geometry.

Gromov-Witten theory strongly depends on the dimension of the target X and exhibits a certain periodicity by which the counts in dimensions 1 may be directly seen as a special case of counts in dimension 3. Our earlier results with Pandharipande may be thus seen as an organic part of the $GW=DT$ framework. Extending these earlier results, R. Pandharipande and I completed the proof the correspondence in the case when X is a rank 2 bundle over a curve in [24].

A key technical step for this proof was the determination of the quantum cohomology of the Hilbert scheme of points of \mathbb{C}^2 in [25]. Quantum cohomology is a deformation of the prod-

uct structure on the cohomology of X that takes into account the enumerative geometry of rational curves in X . Enumerative theory of sheaves on any fibration is related to the maps from the base to the moduli spaces of sheaves on the fiber, and this relation becomes partially tight if the base is 1-dimensional and the fiber is a symplectic surface such as \mathbb{C}^2 . Equivariant quantum cohomology of the Hilbert scheme of point of \mathbb{C}^2 was completely and effectively described in [25], and this served as key input in the DT theory of fibrations and into the subsequently developed (by D. Maulik and me) theory of quantum cohomology of Nakajima varieties [121], of which $\text{Hilb}(\mathbb{C}^2)$ is an example.

A further development of these ideas lead Maulik, Oblomkov, Pandharipande, and me to the proof of the GW=DT correspondence for arbitrary toric 3-folds [21]. As a very particular special case, this gave a proof of the topological vertex formula of [102]. Extending that formula, our results imply, among other things, that the above partition counts, taken with natural weights, reproduce general triple Hodge integrals over the moduli spaces of curves — a vast generalization of the generating functions that satisfy the KdV and Toda equations. The exact quantum group solution of the DT theory to be discussed below gives one an excellent control over all such counts.

The direction of research started in my work with Pandharipande on the quantum cohomology of $\text{Hilb}(\mathbb{C}^2)$, was continued in our work with D. Maulik on quantum cohomology of general Nakajima varieties X . The motivations for this continuation were multifold. Very importantly, in the context of supersymmetric gauge theories, Nekrasov and Shatashvili had a pioneering insight that connected the quantum cohomology of certain moduli spaces of vacua (of which Nakajima varieties are examples) with Bethe Ansatz and other fundamental structures of quantum integrable system.

On the other hand, Bezrukavnikov and I conjectured a precise link between the quantum differential equation for a symplectic resolution X , and its monodromy, with the derived equivalences of X constructed by Bezrukavnikov and Kaledin in their study of quantizations of equivariant symplectic resolutions². Since my work with Braverman and Maulik [22] on quantum cohomology of T^*G/B , it has become clear that quantum cohomology of symplectic resolutions has certain general powerful features that make the methods of geometric representation theory directly applicable to enumerative problems.

By design [121,122], Nakajima varieties provide a geometric realization of representations of certain quantum groups, which are precisely the objects at the heart of quantum integrable systems. However, to describe the quantum cohomology of an arbitrary quiver variety, these quantum group actions were insufficient, and Maulik and I had to rethink the whole paradigm of geometric representation theory to get away from actions defined by an explicit assignment of correspondences to generators. Our theory of stable envelopes provides a geometric construction of a tensor structure on equivariant cohomology and equivariant K-theory of Nakajima varieties, thus turning them geometrically into modules over a certain quantum group. That Maulik-Okounkov quantum group is typically much larger than the Kac-Moody construction of Nakajima. For these larger quantum groups, Maulik and I give a represen-

²The quantum differential equation, or the Dubrovin connection, is a remarkable flat connection with base $H^2(X)$ and fiber $H^*(X)$, which packages the enumerative data of rational curves in X . Equivariant symplectic resolutions form a very special class of algebraic varieties which is rapidly gaining importance in algebraic geometry and geometric representation theory. They include e.g. cotangent bundles T^*G/P of homogeneous projective varieties and all Nakajima quiver varieties.

tation theoretic description of quantum cohomology of the precise shape conjectured by Nekrasov and Shatashvili from one side, and Bezrukavnikov and his collaborators — from the other. The technical constructions and steps that make the proof work take over 200 pages of the book [8] by D. Maulik and me.

As one of the many applications of the techniques developed in [8], we prove several conjectures proposed by Alday, Gaiotto, and Tachikawa (AGT) in [101], which is a paper that had a significant resonance in both mathematics and physics literature. Note the AGT conjectures concern the usual, classical cohomology of the moduli spaces of instantons, and not its quantum cohomology which has also been sorted out completely in [8].

Among the AGT conjectures, the one that identifies the Ext-operator as a vertex operator for certain W -algebras presented a particular challenge. This operator is defined by the kernel $\text{Ext}^*(\mathcal{F}_1, \mathcal{F}_2)$ on the product of two moduli spaces of sheaves. It is often called the Carlsson-Okounkov operator, because it was introduced and computed in [20] for rank 1 sheaves \mathcal{F}_i on an algebraic surface. In [19], we extended the results of [20] to K-theory and also gave a very useful general factorization of the Ext-operator in terms of the operator corresponding to the universal sheaf. A decisive general progress on the Ext-operator has been obtained by A. Neguț, see in particular [123] for the geometric representation theory description of the universal sheaf operator for sheaves of arbitrary rank on an arbitrary surface.

Stable envelopes turned out to be an exceptionally powerful and versatile tools for enumerative geometry and geometric representation theory. In addition to equivariant cohomology and equivariant K-theory, they may be defined in equivariant elliptic cohomology, as in the work [6] of M. Aganagic and I, and in the derived category of coherent sheaves [93]. Many of their properties are surveyed in my PCMI lectures [13]. These lectures also contain many new results.

Using the K-theoretic version of stable envelopes, and the associated quantum group actions, A. Smirnov and I were able to compute the quantum difference equation in the K-theory of Nakajima varieties [3]. K-theoretic quantum computations are much more delicate and mysterious than quantum cohomology, but there is also a great geometric interest and reward in those computation. For example, the Higgs/Coulomb duality of symplectic resolutions, known as the symplectic duality in mathematics, and as the 3-dimensional mirror symmetry in physics [114], is expected to exchange equivariant quantum difference equations of dual varieties in a Langlands-like fashion.

The quantum difference equations for Hilbert schemes of points hold the key to K-theoretic Donaldson-Thomas theory of 3-folds in the exact same way as the quantum cohomology of the Hilbert scheme yields key computations in the traditional, cohomological, Donaldson-Thomas theory. One can say that we have obtained a quantum group solution of the K-theoretic Donaldson-Thomas theory in the same sense in which quantum groups solve the Chern-Simons (CS) theory of real 3-folds, the theory that served as the main inspiration for the early work on the DT theory. One important difference is that CS deals with ordinary quantum groups, that is, deformations of $\mathcal{U}(\mathfrak{g})$, where \mathfrak{g} is a finite-dimensional Lie algebra. Here, by contrast, we work with object of the size of double loop groups, that is, deformations of $\mathcal{U}(\widehat{\mathfrak{g}})$.

In an attempt to give a mathematical description of supersymmetric membranes of M-theory [104], Nekrasov and I proposed a set of challenging conjectures relating K-theoretic DT theory of an arbitrary 3-fold to enumerative geometry of immersed algebraic curves in an associated Calabi-Yau 5-fold [15]. (It is very important for this conjecture to work with in equivariant K-theory and not in ordinary cohomology because the Kähler, that is, degree-counting variables are traded for equivariant variables in our $DT=M$ correspondence). Now at least one side of this conjectural equation is under excellent control. Perhaps the hardest challenge on the 5-fold side is to find what corresponds to cutting and gluing of 3-folds, which need not be Calabi-Yau as I already stressed previously.

Elliptic quantum groups [110] form a very important and, traditionally, very technical chapter of representation theory. The elliptic stable envelopes of M. Aganagic and I [6] provide a geometric construction and a geometric understanding of new and old elliptic quantum groups in the same way as [8] dealt with Yangians etc. One of the main applications of elliptic stable envelopes is to the computation of the monodromy of the quantum difference equations. These difference equations include, as a special case, many important difference equations of mathematical physics such as the quantum Knizhnik-Zamolodchikov equations of [112]. Their monodromy is thus of great theoretical and practical interest, see e.g. [111]. By specialization, this also yields the monodromy of the quantum differential equation that has been the subject of conjectures by Bezrukavnikov and me, see more on this topic below.

More precisely, the monodromy of the quantum difference equations in *equivariant* variables was computed in [6] in terms of the elliptic R-matrices constructed *ibidem*. The monodromy in the *Kähler* variables has the exact same form, but now involving elliptic stable envelopes for a different group. In place of a group that acts on X , we take the group G used in the GIT quotient construction of X . This group is a general connected reductive group, not necessarily a torus. The corresponding theory of elliptic stable envelopes has been developed in [4, 90]. In those papers, I revisit the foundations of the elliptic stable envelopes theory and prove their existence in uniqueness in a much broader context than the context of the Nakajima quiver varieties treated in [6]. In addition to the computations of the monodromy of the quantum difference equations, the results of [90] also give an integral formula for the fundamental solution of the quantum difference equation.

It is a classical and much studied problem in mathematical physics to find integral solutions of equations of, broadly speaking, Knizhnik-Zamolodchikov type. This subject is directly linked with Bethe ansatz and other core issues in the analysis of exactly solvable models. In [14], M. Aganagic and I revisit this problem from the point of enumerative geometry and geometric representation theory. By a geometric argument, we prove very general formulas for integral solutions. We also show, very generally, that stable envelopes give the off-shell Bethe eigenvectors, that is, they become eigenvectors when the Bethe equations are satisfied. This results e.g. in a general R-matrix formula for these eigenvectors. This revolutionizes the technical foundations in the theory of Quantum Integrable Systems (QIS), vastly extends the previously known results, see [129], and completes the vision of Nekrasov and Shatashvili who first found a geometric interpretation of the Bethe eigenvalues [125] (as opposed to eigenvectors).

The main result of [90] explicitly solves the fundamental problems on the both the enumerative side and the QIS side in terms of the theory nonabelian stable envelopes developed *ibidem*. These nonabelian stable envelopes are object of classical (as opposed to quantum,

curve-counting) topology, but due to the features of elliptic cohomology they have a manifest and nontrivial dependence on variables in $\text{Pic}(X) \otimes \mathbb{C}^\times$ that are used to keep track of degrees in curve counts. As a result, nonabelian stable envelopes explicitly and effectively sum of those curves counts. On the QIS side, they have the role of specifying the measure of integration for integral solutions of equations of Knizhnik-Zamolodchikov type.

Also, integral representations play a central role in [11], where M. Aganagic, E. Frenkel, and I conjecture and prove a certain two-fold deformation of the key statement of the geometric Langlands correspondence. We identify, with a concrete match of parameters, conformal blocks of deformed W-algebras of Frenkel and Reshetikhin with conformal blocks of the quantum affine Kac-Moody Lie algebras. The identification is on an arbitrary level and it specializes to the usual statement of the geometric Langlands correspondence when the deformation is turned off on both sides of the correspondence and the level is taken to the classical/critical limit on the respective sides.

Categorical lifts of stable envelopes are the subject of a number of papers currently in various stages of completion. In [93], D. Halpern-Leistner, D. Maulik, and I give a construction of stable envelopes in $D^b\text{Coh}(X)$. The braid relations that they satisfy give, in particular, a categorical analog of the Yang-Baxter equation and should provide a basis for a theory of categorical actions of quantum groups that is not based on explicit assignment of functors to generators.

In [91], R. Bezrukavnikov and I prove our *monodromy=derived equivalences* conjecture for the majority of known symplectic resolutions, including all Nakajima varieties. The key technical result there is that the categorical stable envelope becomes the parabolic induction functor under the Bezrukavnikov-Kaledin equivalence [106] between $D^b\text{Coh}(X)$ and modules over a quantization of X (this also gives an independent construction of categorical stable envelopes for symplectic resolutions).

Natural next steps in this direction will be addressed in forthcoming papers. For instance, the Bezrukavnikov-Kaledin story is about quantizing X in characteristic $p \gg 0$. From the point of view of both theoretical physics and enumerative geometry, it is much more natural to quantize a multiplicative analog of X as a quantum group at a root of unity. The order of the root of unity here need not be large and it need not be prime for there to be a direct link with the enumerative story³. Both technically and conceptually, this line of research is closely interwoven with the work [92].

There is an aspect in which the 3-dimensional mirror symmetry of Intriligator and Seiberg [114] is different from the more familiar 2-dimensional mirror symmetry, and also from e.g. the geometric Langlands correspondence. For a dual pair of varieties X and X^\vee , it relates objects that, while different, are naturally compatible. I thus believe it is given by an accessible Fourier-Mukai kernel on $\text{Loop}(X) \times \text{Loop}(X^\vee)$, where tilde denotes the universal cover. This kernel is the subject of [94]. In particular, in the Grothendieck group of the corresponding category, it reduces to a remarkable elliptic cohomology class on $X \times X^\vee$, which we call the duality interface. It has the property that restricted to the torus-fixed points in X it gives the stable envelopes of the dual points of X^\vee , and vice versa. We prove the existence of this class for many dual pairs of varieties. This implies the striking numerical predictions

³See e.g. my talk at the CMI at 20 conference for a preliminary report on this.

of the 3d mirror symmetry, such as the exchange of the quantum difference equations in Kähler and equivariant variables between X and X^\vee . In the case $X = T^*G/B$, this reduces to statements like the label-argument symmetry of Macdonald polynomials, but already for $X = \text{Hilb}(\mathbb{C}^2, n)$ this leads to rather dramatic new identities.

Starting with the paper [89] joint with David Kazhdan, my work and interests have branched out in a new direction, taking aim at some fundamental questions in the theory of automorphic forms, and Langlands' vision linking them to the Galois representation. Our methods appear to be absolutely new and really powerful. Remarkably, while fundamentally geometric, they work equally well for both *number* fields and function fields. Already in the first installment [89] we were able to solve spectral problems that were waiting for a comprehensive solution since Langlands pioneering work on Eisenstein series back in 1964, see [117]. The Eisenstein series outputs an automorphic forms on a group G over a global field \mathbb{F} starting from an automorphic form on a Levi subgroup $M \subset G$. Langlands computed the spectrum of invariant differential operators and Hecke operators acting on the Eisenstein series in terms of residues in certain integrals of L-functions. For instance, focusing on the most basic case when M is the maximal torus and its automorphic representation is trivial, this is a certain $\text{rank}(G)$ -fold integral involving the ζ -function of \mathbb{F} . Langlands did these integrals explicitly for groups of rank 2 in [117] and found a remarkable new phenomenon for the exceptional group G_2 . Classical groups were sorted out by C. Mœglin and J.-L. Waldspurger, see [120]. As to the exceptional groups, the program to study the corresponding integrals was initiated by Volker Heiermann, Marcelo de Martino, and Eric Opdam in [113]⁴. That remarkable program analyzes the integrals using a variety of different theoretical tools, including also computer computations in the case E_8 .

Our approach with Kazhdan is very different, works uniformly for all groups, and outputs directly the objects appearing in the multitude of influential conjectures in the subject made by Landlands, Arthur, and probably others. The key step is to interpret Langlands' Riemann integral is a equivariant pushforward (in the sense of distributions) of a certain cohomology or K-theory class on the cotangent bundle to a certain flag variety on the Galois side (this can all be phrased in a very powerful abstract language, but effectively reduces to this). The nilpotent elements in the Langlands dual group that are supposed to control the spectrum of Eisenstein series indeed appear absolutely directly in our computation as we push forward the class in question along the Springer map. The results obtained in [89] have a direct generalization to most general Eisenstein series. This is the subject of our current ongoing work, see [88] for the next installment in this series.

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⁴Over the years, there have been many updates to [113], and the most recent one that appeared in July of 2022 explains the completion the program started in back in 2015.

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