Chapter 1
Koszul duality I

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Abstract In [39], Bernstein, Gelfand and Gelfand describe a duality between certain parts of the derived categories of the symmetric and exterior algebras. This type of duality, called Koszul duality, has since been observed in various contexts. After introducing the necessary homological algebra, we formulate this phenomenon as a differential graded version of Morita theory, and present Koszul duality for a polynomial ring in one variable. We then present evidence of such a self-duality in category $\mathcal{O}$.

1.1 Introduction

Roughly speaking, a Fourier transform is an operation which collects a signal, and decomposes the signal into its component frequencies; in particular, a Fourier transform exchanges certain frequencies with delta functions. In many important cases, a Fourier transform can reveal hidden structure. It also has the added bonus that applying the Fourier transform twice to a function gives you back the original function up to a constant and a sign.

Fourier transform-like operations appear in geometry and algebra. Roughly speaking, in geometry, such operations often exchange vector bundles with skyscraper sheaves, and in algebra, they exchange projective objects with simple objects. In algebraic settings, such transforms are referred to as instances of Koszul duality; some examples may be found in [36], [38], [40], [41], and [42].
1.1.1 Morita theory

We present Koszul duality as a differential graded version of Morita theory, and hence we begin with a quick overview of Morita theory; one source for this material is [37, Ch. II].

For a projective object $P$ in an abelian category $A$, consider $\langle P \rangle$, the subcategory generated by $P$, i.e., an object $A$ belongs to $\langle P \rangle$ if it has a presentation $P^m \to P^n$. Loosely speaking, Morita theory asserts that the subcategory $\langle P \rangle$ and $\text{End}(P)$ determine each other. It is natural to expect this as objects in $\langle P \rangle$ are given by a presentation, the data of which can be written in terms of endomorphisms of $P$.

For a ring $R$, we let $R-\text{mod}$ denote the category of finitely generated right $R$-modules. Morita theory tells us first, when a given abelian category $A$ is equivalent to a category of the form $R-\text{mod}$, and second, when two categories $R-\text{mod}$ and $R'-\text{mod}$ are equivalent.

To be precise, let $\mathcal{A}$ be an abelian category; we impose the additional assumption that each object of $\mathcal{A}$ has finite length. Recall that a projective object is an object $P \in \mathcal{A}$ for which the functor $\text{Hom}_\mathcal{A}(P, -)$ is exact. A projective object $P$ is called a projective generator if for any object $X \in \mathcal{A}$, there exists a surjection $P^n \to X$ for some $n$.

In $R-\text{mod}$, projective generators can be characterized as finitely generated projective $R$-modules $P$ such that $R$ appears a direct summand of $P^n$ for some $n \in \mathbb{N}$. In particular, $R$ is a compact projective generator of $R-\text{mod}$.

One result of Morita theory is the following characterization of abelian categories with projective generators.

**Theorem 1.1.** If $P \in \mathcal{A}$ is a projective generator, then $\mathcal{A}$ is equivalent to the category $\text{End}_\mathcal{A}(P)-\text{mod}$ via the functor $\text{Hom}_\mathcal{A}(P, -)$.

Another result of Morita theory characterizes rings with equivalent modules categories in terms of projective generators.

**Theorem 1.2.** Let $R$ and $R'$ be two rings. Then, the categories $R-\text{mod}$ and $R'-\text{mod}$ are equivalent if and only if $R'$ is isomorphic to $\text{End}_R(P)$ for some projective generator $P \in R-\text{mod}$.

Two rings $R$ and $R'$ with equivalent module categories are said to be Morita equivalent. Note that the rings $R$ and $\text{End}_R(P)$ have isomorphic centers; in particular, two Morita equivalent commutative rings must also be isomorphic.

**Example 1.1 (Finite dimensional algebras).** Consider a finite dimensional algebra $A$ over an algebraically closed field $k$. The Krull-Schmidt theorem implies that the indecomposable projective $A$-modules are direct summands of $A$ itself. Let $P_1, \ldots, P_n$ be a set of indecomposable projectives up to isomorphism, and let $P = \bigoplus_{i=1}^n P_i$. Then, $P$ is a projective generator, and hence we have an equivalence between the categories $A-\text{mod}$ and $\text{End}_A(P)-\text{mod}$. It is easy to see that $\text{End}_A(P)$ is a basic algebra.
Any finite dimensional basic algebra is isomorphic to the path algebra of some quiver with relations. Hence, the above application of Morita theory allows us to understand the category $A\text{-mod}$ by studying a corresponding quiver with relations. This latter object can sometimes be easier to compute with.

However, we remark that there is no known explicit quiver algebra description for category $\mathcal{O}$, and no nice description is expected! Instead, our approach will be to form a large projective generator $P$ comprised of Soergel bimodules, and work with the algebra $\text{End}_{\mathcal{O}}(P)$ explicitly.

1.2 dg-algebras

Our next aim is to formulate a version of Morita theory for the derived category of modules over a dg-algebra; in this section we recall the requisite homological algebra.

**Definition 1.1.** A differential graded algebra, or dg-algebra, is a $\mathbb{Z}$-graded algebra $A = \bigoplus_{i \in \mathbb{Z}} A^i$ equipped with a differential $d : A^i \to A^{i+1}$ such that $d(1_A) = 0$ and $d(ab) = d(a)b + (-1)^i ad(b)$ for homogenous elements $a$.

The following construction produces an important family of examples of dg-algebras. Let $\mathcal{A}$ be be an additive category and $C$ be a chain complex in $\mathcal{A}$. Then, form a dg-algebra $\mathcal{E}nd(C) = \bigoplus_{i \in \mathbb{Z}} \mathcal{E}nd^i(C)$ as follows. Set

$$\mathcal{E}nd^i(C) = \prod_{k \in \mathbb{Z}} \text{Hom}(C^k, C^{k+i}),$$

and let multiplication be given by composition of morphisms. We equip $\mathcal{E}nd^i(C)$ with the following differential: for $f = (f^k)_{k \in \mathbb{Z}} \in \mathcal{E}nd^i(C)$, set $df = ((df)^k)_{k \in \mathbb{Z}}$, where

$$(df)^k = d \circ f^k - (-1)^k f^{k+1} \circ d.$$  

Note that even for seemingly innocuous complexes $C$, the dg-algebra $\mathcal{E}nd(C)$ can be unwieldy.

We will be interested in modules over dg-algebras. The appropriate notion of a module in this setting is as follows:

**Definition 1.2.** A (right) dg-module over a dg-algebra $A$ is a graded right $A$-module $M = \bigoplus_{i \in \mathbb{Z}} M^i$ with differential $d : M^i \to M^{i+1}$ such that $d(m \cdot a) = d(m) \cdot a + (-1)^{|m|} m \cdot d(a)$ for homogenous elements $m$.

For example, if $A$ is concentrated in degree zero, then its differential $d$ must be zero. Consequently, a dg-module over such an algebra $A$ is simply a complex of right $A$-modules whose differential is a homomorphism of right $A$-modules.
A more complicated class of examples can be constructed as follows. Given two complexes $C, D \in \mathcal{C}(A)$, we can define $\mathcal{H}om(C, D)$ as above; namely set

$$\mathcal{H}om^i(C, D) = \prod_{k \in \mathbb{Z}} \text{Hom}(C^k, D^{k+i}),$$

and define the differential just as for $\text{End}(C)$ above. Then, given any $D \in \mathcal{C}(A)$, the complex $\mathcal{H}om(C, D)$ is a right dg-module for $\text{End}(C)$; this is a manifestation of the general fact that, in appropriate settings, $\text{Hom}(X, Y)$ is a right $\text{End}(X)$-module.

To construct the derived category of dg-modules, we need the notion of a quasi-isomorphism.

**Definition 1.3.** A quasi-isomorphism of dg-algebras $\phi : A \to A'$ is a homomorphism of dg-algebras such that the induced map on cohomology $H^\bullet(A) \to H^\bullet(A')$ is an isomorphism of graded rings. A quasi-isomorphism of dg-modules is defined analogously.

Let $\text{dg}^-\mathcal{C}(A)$ denote the category of right dg-modules over $A$; by regarding a dg-module as a complex, we may then define the homotopy and derived categories in the usual manner. Namely, define the homotopy category $\text{dg}^-\mathcal{K}(A)$ to be the category obtained by identifying homotopic maps of dg-modules, and define the derived category $\text{dg}^-\mathcal{D}(A)$ to be the category obtained from $\text{dg}^-\mathcal{K}(A)$ by inverting quasi-isomorphisms. Both $\text{dg}^-\mathcal{K}(A)$ and $\text{dg}^-\mathcal{D}(A)$ are triangulated categories.

Given quasi-isomorphic dg-algebras, one has the following relationship between their derived dg-module categories.

**Proposition 1.1.** A quasi-isomorphism $\phi : A \to A'$ induces an equivalence of derived categories

$$\text{dg}^-\mathcal{D}(A') \xrightarrow{\sim} \text{dg}^-\mathcal{D}(A).$$

### 1.3 dg-Morita theory

Recall that Morita theory loosely states that for a projective object $P$ in an abelian category $\mathcal{A}$, the subcategories $\langle P \rangle$ and $\text{End}(P)$ determine each other. Following [42, Section 1.3], we present a derived version of such a statement.

Given an abelian category $\mathcal{A}$, and a bounded above complex of projectives $C$, a naive hope would be that the full triangulated subcategory $\langle C \rangle_{\triangle}$ generated by $C$ inside the derived category $D(\mathcal{A})$ may be recovered from the algebra

$$\text{End}^\bullet(C) = \oplus_{m \in \mathbb{Z}} \text{Hom}_{\mathcal{D}(\mathcal{A})}(C, C[m]);$$

here, generation is in a triangulated category, so shifts of $C$ are allowed. In words, one would hope that knowing maps from $C$ to all of its shifts is enough to recover the
entire subcategory generated by $C$. In general, however, more information is needed; such information can sometimes be contained in $\mathcal{E}nd(C)$. Namely, for certain complexes $C$, one may formulate a differential graded analog of Morita equivalence using the homological algebra introduced above.

**Definition 1.4.** A complex $C$ is called *end-acyclic* if, for all $m \in \mathbb{Z}$, the natural homomorphism

\[
\text{Hom}_{\mathcal{K}(\mathcal{A})}(C, C[m]) \to \text{Hom}_{\mathcal{D}(\mathcal{A})}(C, C[m])
\]

is an isomorphism.

For example, bounded above complexes of projectives are end-acyclic. This condition allows us to pass from the derived category to the homotopy category, which can be easier to compute with.

Let $C$ be such a complex, and let us set $A$ to be the dg-algebra $\mathcal{E}nd^*(C)$. Then, we have a functor

\[
\mathcal{H}om(C, -) : C(\mathcal{A}) \to \text{dg-}C(A)
\]

sending $C \mapsto A$. This functor descends to a functor between homotopy categories

\[
\mathcal{H}om(C, -) : \mathcal{K}(\mathcal{A}) \to \text{dg-}\mathcal{K}(A),
\]

yielding the following diagram of functors

\[
\mathcal{D}(\mathcal{A}) \leftarrow \mathcal{K}(\mathcal{A}) \xrightarrow{\mathcal{H}om(C, -)} \text{dg-}\mathcal{K}(A) \to \text{dg-}\mathcal{D}(A).
\]

Then, dg-Morita theory describes how the category $\langle C \rangle_\triangle \subset \mathcal{K}(\mathcal{A})$ behaves under these functors.

**Theorem 1.3.** Let $\mathcal{A}$ be an abelian category, and let $C \in C(\mathcal{A})$ be an end-acyclic complex. Then, there is an equivalence of triangulated categories

\[
\mathcal{D}(\mathcal{A}) \supset \langle C \rangle_\triangle \cong \langle A \rangle_\triangle \subseteq \text{dg-}\mathcal{D}(A).
\]

A more general version of this statement is presented in [42, Section 1.3], where $C$ is replaced by a finite family of complexes that pairwise satisfy the condition for end-acyclicity above.

Loosely, dg-Morita theory can be summarized as the statement that triangulated subcategories (in algebraic settings) are described by dg-algebras.

In practice, it is difficult to work with all of $\mathcal{E}nd(C)$. The following condition will enable us to pass from $\mathcal{E}nd(C)$ to its cohomology, which will be a simpler object.

**Definition 1.5.** A dg-algebra $A$ is said to be *formal* if $A$ is quasi-isomorphic to its cohomology $H^*(A)$.

For a general dg-algebra $A$, the cohomology $H^*(A)$ is a graded ring, and can be viewed as a dg-algebra with trivial differential. The following exercise relates the
cohomology ring of the dg-algebra $\mathcal{E}nd(C)$ with maps in the homotopy category between $C$ and its shifts.

**Exercise 1.1.** For any complex $C \in \mathcal{C}(\mathcal{A})$, one has

\[ H^\bullet(\mathcal{E}nd(C)) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{A}}(C, C[i]). \]

In the particular case where $C$ is end-acyclic, we obtain

\[ H^\bullet(\mathcal{E}nd(C)) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{A}}(C, C[i]) = \text{End}^\bullet(C). \]

As a quasi-isomorphism of dg-algebras induces an equivalence of derived categories, we deduce the following corollary.

**Corollary 1.1.** If $C$ is end-acyclic, and $\mathcal{E}nd(C)$ is formal, we obtain an equivalence of derived categories

\[ \mathcal{D}(\mathcal{E}nd(C)) \sim \rightarrow \mathcal{D}(\text{End}^\bullet(C)), \]

where $\text{End}^\bullet(C)$ is viewed as a dg-algebra with trivial differential.

### 1.4 Koszul duality for polynomial rings

Koszul duality is an equivalence of derived categories of graded modules over certain types of graded rings. We illustrate this phenomenon with the following extended example.

Let $V$ be a 1-dimensional vector space over $k$, and let $\Lambda = AV = k \oplus V$ be its exterior algebra, considered as a graded algebra with $V$ in degree $-1$, and $k$ in degree 0. Let $S = \text{Sym}(V^*) = k[x]$, be the symmetric algebra of its dual, considered a graded algebra with $V^*$ in degree 1.

Consider the trivial module $k$ in the category $\Lambda$-gr.mod, i.e., the category of graded $\Lambda$-modules. The graded module consisting of $k$ placed in zero degree generates $\mathcal{D}^b(\Lambda - \text{gr.mod})$ as a triangulated category.

The following equivalence will be our first example of Koszul duality.

**Theorem 1.4.** There is an equivalence of derived categories

\[ \kappa : \langle k \rangle \triangle \sim \rightarrow \langle S \rangle \triangle \subset \mathcal{D}^b(S - \text{gr.mod}). \]

The equivalence $\kappa$ sends the simple module $k$ to the projective module $S$ and sends the projective, injective module $\Lambda$ to the simple module $k$. The functor $\kappa$ is a composition of two equivalences.
The first such equivalence is dg-Morita theory. In order to apply Theorem 1.3, we collect a few observations. First, note that the module $k$ admits a projective resolution

$$
\cdots \rightarrow \Lambda(2) \rightarrow \Lambda(1) \rightarrow \Lambda \rightarrow k.
$$

(1.1)

Using this resolution, one can show the following.

**Exercise 1.2.** There is an isomorphism of algebras $\text{End}^* (k)_{\Lambda - \text{gr.mod}} \simto S$.

Moreover, if $P$ denotes the projective resolution above, one can show that $\mathcal{E nd}(P,P)$ is formal. Applying dg-Morita theory, we obtain an equivalence

$$
\mathcal{D}^b(\Lambda - \text{gr.mod}) \supseteq \langle k \rangle_\Delta \simto \langle S \rangle_\Delta \subseteq \mathcal{D}^b(S - \text{gr.mod}).
$$

A second equivalence is needed to pass from the dg-derived category to the ordinary derived category; this equivalence is a shearing operation that intertwines the internal grading (i.e. the grading on the module) with the cohomological grading. We briefly indicate this process. First, note that a dg-module $M = \bigoplus M_k$ over the dg-algebra $S$ consists of a complex of vector spaces, equipped with a chain map $x: M_i \rightarrow M_{i+1}$. If such a module $M$ admits a grading, i.e. if $M \in \langle S \rangle_\Delta \subseteq \mathcal{D}^b(S - \text{gr.mod})$, we may further decompose $M$ as $\bigoplus_{i,j} M_{i,j}$, where

$$
d(M_{i,j}) \subset M_{i+1,j}, \ x(M_{i,j}) \subset M_{i+1,j+1},
$$

as depicted below.

$$
\begin{array}{ccc}
M_{0,2} & \xrightarrow{d} & M_{1,2} & \xrightarrow{d} & M_{2,2} \\
& \uparrow{x} & & \uparrow{x} \\
M_{0,1} & \xrightarrow{d} & M_{1,1} & \xrightarrow{d} & M_{2,1} \\
& \uparrow{x} & & \uparrow{x} \\
M_{0,0} & \xrightarrow{d} & M_{1,0} & \xrightarrow{d} & M_{2,0}
\end{array}
$$

Redrawing the picture, we can view the dg-module $M$ as a complex of graded modules over $S$. 
Here the vertical columns can be viewed as graded $S$-modules. Hence, after a shift in grading, $M$ may be regarded as a complex of graded $S$-modules, whose differentials are the horizontal maps in the picture. This grading shift is called shearing, and furnishes an equivalence

$$dg - D^b(S - \text{gr}.\mod) \supseteq \langle S \rangle \triangleleft \langle S \rangle \subseteq D^b(S - \text{gr}.\mod)$$

Combining dg-Morita theory and shearing, we obtain the equivalence $\kappa$. In particular, if $(\cdot)$ denotes the internal grading and $[\cdot]$ denotes the cohomological grading, for $M \in D^b(\Lambda - \text{gr}.\mod)$, we have

$$\kappa(M(i)) = \kappa(M)(-i)[-i].$$

A similar equivalence holds when $V$ is replaced by any finite dimensional vector space $V$; in this case, the so-called Koszul resolution plays the role of the resolution (1.1). The general class of graded rings for which Koszul duality can be formulated are called Koszul rings; the definition may be found, for example, in [38, Def. 1.2.1].

### 1.5 Evidence of Koszul duality in category $\mathcal{O}$

Let $\mathfrak{g}$ be a semisimple lie algebra. In order to define a duality functor on $\mathcal{O}$, we must first define an anti-automorphism $\tau$ on $\mathfrak{g}$. Let $\{e_\alpha\}_{\alpha \in \Delta^+}$ and $\{f_\alpha\}_{\alpha \in \Delta^+}$ denote bases for $n^+$ and $n^-$ such that $e_\alpha$ and $f_\alpha$ lie in weight spaces $\pm \alpha$, respectively. The anti-automorphism $\tau$ exchanges $e_\alpha$ and $f_\alpha$, and fixes $\mathfrak{h}$.

For an object $M \in \mathcal{O}$, we set $\mathbb{D}(M) = \bigoplus_{\lambda \in \mathfrak{h}} \mathbb{D}(M)_\lambda$, where $\mathbb{D}(M)_\lambda = M^*_\lambda$. For $x \in \mathfrak{g}$, and $f \in M^*_\lambda$, we define $(x \cdot f)(v) = f(\tau(x) \cdot v)$; in this way $\mathbb{D}(M)$ carries the structure of a representation of $\mathfrak{g}$. It turns out that $\mathbb{D}$ is a contravariant functor on $\mathcal{O}$, and this is often referred to as duality. The duality functor preserves the principal block $\mathcal{O}_0$.

Recall that in $\mathcal{O}_0$, for each $x \in W$, we have a Verma module $\Delta_x$, a simple module $L_x$, and an indecomposable projective cover $P_x$ of $L_x$. Using the duality functor, we
define dual Verma modules to be $\nabla_x = \mathbb{D}(\Delta_x)$.

The following exercises describe the behavior of $L_x$ and $P_x$ under $\mathbb{D}$.

**Exercise 1.3.** Show that $\mathbb{D}(L_x) = L_x$.

**Exercise 1.4.** Show that $I_x := \mathbb{D}(P_x)$ is the indecomposable injective envelope of $L_x$.

The character of a module $M \in \mathcal{O}_0$ is preserved under duality. As a module’s character determines its class in the Grothendieck group $[\mathcal{O}_0]$ we get that $[M] = [\mathbb{D}(M)]$.

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The Grothendieck group $[\mathcal{O}_0]$ carries a bilinear pairing $\langle - ,- \rangle$, called the *Euler form*, defined by $\langle [M],[N] \rangle = \sum_i (-1)^i \dim \text{Ext}^i(M,N)$.

As $\mathcal{O}$ has finite homological dimension, this pairing is well-defined. The following proposition implies that $\{ \Delta_x \}$ and $\{ \nabla_y \}$ are dual bases.

**Proposition 1.2 (Fundamental vanishing).** We have $\dim \text{Ext}^i(\Delta_x, \nabla_y) = \delta_{i,0} \delta_{x,y}$.

Moreover, $\{ P_x \}$ and $\{ L_y \}$ are dual bases; so too are $\{ I_x \}$ and $\{ \nabla_y \}$.

We can reformulate the Kazhdan-Lusztig conjecture in terms of the Grothendieck group $[\mathcal{O}_0]$.

**Theorem 1.5 (Kazhdan-Lusztig conjecture).** We have

$$\left[ P_x \right] = \sum_{y \in W} h_{x,y} \left[ \Delta_y \right].$$

Kazhdan and Lusztig also gave an interesting inversion formula, which can be seen as evidence of Koszul duality.

**Theorem 1.6 (Kazhdan-Lusztig inversion formula).** For $x,y \in W$, we have

$$\sum_{z \in W} (-1)^{l(x)+l(y)} h_{x,z} h_{z,w_0,y} = \delta_{x,y}.$$

Now, let $\phi : [\mathcal{O}_0] \to [\mathcal{O}_0]$ be the unique homomorphism such that $\phi([\nabla_x]) = (-1)^{l(x)}[\Delta_{w_0}]$. The following exercise may be deduced from the inversion formula.

**Exercise 1.5.** Show that $\phi([P_x]) = (-1)^{l(x)}[L_{w_0}]$.

As recognized by Beilinson and Ginzburg, the above exercise suggests that $D^b(\mathcal{O}_0)$ is in some sense Koszul self-dual. The derived category admits a cohomological grading, but Koszul duality is a statement about derived categories of *graded* modules! An extra “internal” grading is therefore needed to formulate Koszul duality for category $\mathcal{O}$. Such a grading is not immediately evident; however, there is a
mysterious grading on Kazhdan-Lusztig polynomials which hints towards an extra grading on the derived category. Our approach, in the following chapter, will to be use Soergel bimodules to construct a graded category containing lifts of the important objects of $O_0$, including the simple, injective, projective and Verma modules; it is this graded variant of $\mathcal{D}(O_0)$ that will be Koszul self-dual.

References