Random tilings

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Lozenge tilings of a hexagon can be viewed as stepped surfaces.
Consider the probability measure on the set of tilings defined by

\[ P(\mathcal{T}) = \frac{\omega(\mathcal{T})}{Z(a, b, c)}, \text{ where } \omega(\mathcal{T}) = \prod_{\diamond \in \mathcal{T}} \omega(\diamond). \]
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- If we set \( \omega(\Diamond) = 1 \) we will obtain the **uniform measure**.
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$$

- If we set $\omega(\diamond) = 1$ we will obtain the uniform measure.
- Let $j$ be the coordinate of $\diamond$. Set $\omega(\diamond) = q^{-j}$, $q > 0$. 
Consider the probability measure on the set of tilings defined by

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$$\omega(\mathcal{T}) = \text{const}(a, b, c) \cdot q^{-\text{volume}}.$$
Limit shape

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Height function
Questions

- Computations on discrete level
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• Central Limit Theorem for the height function
• Description of the frozen boundary
• Different types of local behavior in the bulk and near the boundary
Definition

A dimer covering $\mathcal{D}$ of a graph $\Gamma$ is a subset of edges which covers every vertex exactly once, that is, every vertex is the endpoint of exactly one edge.
Dimer model

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$$\mathcal{P}(\mathcal{D}) = \frac{\omega(\mathcal{D})}{Z(\Gamma)}, \text{ where } \omega(\mathcal{D}) = \prod_{e \in \mathcal{D}} \omega(e).$$
Correspondence

Lozenge tilings $\leftrightarrow$ Dimer coverings
Key statement

Define Kasteleyn matrix $A^k : \mathbb{R}^V(\Gamma) \rightarrow \mathbb{R}^V(\Gamma)$ by

$$A^K_{e=vw} = \begin{cases} 
\omega(e), & v \rightarrow w; \\
-\omega(e), & w \rightarrow v; \\
0, & \text{no edge between } v \text{ and } w.
\end{cases}$$
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Theorem (Kasteleyn ’61, Temperley and Fisher ’61)

$$Z(\Gamma) = \sum_{D \in \Gamma} \prod_{e \in D} \omega(e) = |\text{Pf}(A^K)|.$$ 

$$\text{Pf}(A^{2n \times 2n}) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sign}(\sigma) \prod_{i=1}^n a_{\sigma(2i-1)\sigma(2i)} \quad \text{and} \quad \text{Pf}(A)^2 = \det(A).$$
Corollary

Let $\tau_e(\mathcal{D}) = 1$ if $e \in \mathcal{D}$ and 0 otherwise.

$$
\mathcal{P}(e_1 = v_1 w_1, \ldots, e_k = v_k w_k \in \text{some } \mathcal{D}) = \\
\sum_{\mathcal{D}} \prod_{e \in \mathcal{D}} \omega(e) \tau_{e_1}(\mathcal{D}) \cdots \tau_{e_k}(\mathcal{D}) \\
\frac{1}{Z(\Gamma)} = \\
\text{Pf} \left( ((A^k)^{-1})_I \right), \quad \text{where} \\
I = \{v_1, w_1, \ldots, v_k, w_k\}.
$$
Tiling model
Tiling model

Affine transformation

establishes a bijection between tilings and non-intersecting paths:
Fix a section $t$. Let the coordinates of the nodes be $C(t) = (x_1, \ldots, x_N)$.

**Definition**

The one-interval gap probability function $D_N(s)$ is defined in the following way

$$D_N(s) = \text{Prob}[\max\{x_i\} < s].$$
Orthogonal polynomial ensembles

Let $\mathcal{X}$ be a discrete subset of $\mathbb{R}$. Let $\omega$ be a positive-valued function on $\mathcal{X}$ with finite moments:

$$\sum_{x \in \mathcal{X}} |x|^n \omega(x) < \infty, \quad n = 0, \ldots$$

Fix $k > 0$. The orthogonal polynomial ensemble on $\mathcal{X}$ is a probability measure on the set of all $k$-subsets of $\mathcal{X}$ given by

$$\mathcal{P}(\pi_{x_1}, \ldots, \pi_{x_k}) = \frac{1}{Z} \prod_{1 \leq i < j \leq k} (\pi_{x_i} - \pi_{x_j})^2 \cdot \prod_{i=1}^{k} \omega(x_i),$$

where $Z$ is the normalizing constant.
Fix a section $t$. Let the coordinates of the nodes be $C(t) = (x_1, \ldots, x_N)$. 
q-Hahn ensemble

Theorem (Borodin, Gorin, Rains ’2009)

\[ \text{Prob}\{C(t) = (x_1, \ldots, x_N)\} = \text{const} \cdot \prod_{0 \leq i < j \leq M} (q^{-x_i} - q^{-x_j})^2 \prod_{i=1}^{N} w_t(x_i), \]

where \( w_t(x) \) is the weight function of the q-Hahn polynomial ensemble up to a factor not depending on \( x \).

Let \( \mathcal{X} = \{\pi_i = q^{-i} : i = 0, \ldots, N\} \). Set

\[ \omega_{Hahn}^q(x) = (\alpha \beta q)^{-x} \frac{(\alpha q, q^{-N}; q)_x}{(q, \beta^{-1} q^{-N}; q)_x}, \]

where \( (y_1, \ldots, y_i; q)_k = (y_1; q)_k \cdots (y_i; q)_k \), and \( (y; q)_k = (1-y) \cdots (1-yq^{k-1}) \).
Theorem [K. ’16], q-Volume case

The gap probability $D_N(s)$ can be computed recursively

$$D_N(s) = \frac{f(r_s, t_s, r_{s-1}, t_{s-1})D_N^2(s-2)}{D_N(s-1)}$$
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where the sequence $(r_s, t_s)$ satisfies recursion

$$(r_st_{s-1} + 1)(r_{s-1}t_{s-1} + 1) =$$

$$\frac{a_1^s a_2^s (t_{s-1} - a_3)(t_{s-1} - a_4)(t_{s-1} - a_5)(t_{s-1} - a_6)}{a_3 a_4 a_5 a_6 (qt_{s-1} - a_1^s)(qt_{s-1} - a_2^s)},$$

$$(r_st_s + 1)(r_st_{s-1} + 1) =$$

$$\frac{uv(a_1^s a_2^s)^2 (r_sa_3 + 1)(r_sa_4 + 1)(r_sa_5 + 1)(r_sa_6 + 1)}{(r_sw_s - va_1^s a_2^s)(qr_s w_s - ua_1^s a_2^s)}.$$ 

The parameters $v, w, a_1, \ldots, a_6$ and the initial conditions are explicitly computed.
\[ f(r_s, t_s, r_{s-1}, t_{s-1}) = \]
\[ = \frac{(r_{s-1}w_s - qva_1^s a_2^s)(r_sw_s - qua_1^s a_2^s)(t_{s-1} - qa_1^s)(t_{s-1} - qa_2^s)}{uva_1^s a_2^s (qa_1^s - a_3)(qa_1^s - a_5)(qa_2^s - a_4)(qa_2^s - a_6)}. \]
Domino tilings: Aztec diamond
Rectangular Aztec diamonds

Figure: Six particles in the boundary row in positions 1, 2, 3, 7, 8, 9.
Rectangular Aztec diamonds

Rectangular Aztec Diamond $\mathcal{R} \rightarrow \lambda = (\lambda_N > \lambda_{N-1} > \cdots > \lambda_1)$, where $\lambda_i$ is the position of the $i$-th particle in the boundary row.

$N$-tuple $\lambda \rightarrow$ probability measure $m[\lambda] = \frac{1}{N} \sum_{i=1}^{N} \delta \left( \frac{\lambda_i}{N} \right)$.
Formation of the limit shape

Denote the set of domino tilings of the domain $\mathcal{R}(N, \lambda)$ with $N$ particles in the boundary row in positions $\lambda$ by $\mathcal{D}(N, \lambda)$. We consider uniform measure on $\mathcal{D}(N, \lambda)$.

Figure: A limit shape simulation.
A height function $h_D$ (where $D$ is a tiling) is an integer-valued function on the vertices of the lattice squares of the domain which satisfies the following properties:

- if an edge $(u, v)$ does not cross any domino in $D$ then $h(v) = h(u) + 1$ if $(u, v)$ has a dark square on the left, and $h(v) = h(u) - 1$ otherwise.

- if an edge $(u, v)$ crosses a domino in $D$ then $h(v) = h(u) + 3$ if $(u, v)$ has the dark square on the left, and $h(v) = h(u) - 3$ otherwise.
Particle configuration

We can encode any domino tiling $D \in \mathcal{D}(N, \lambda)$ pictorially.
Corresponding sequence of Young diagrams
Sequence of signatures

Lemma
Let $\mathcal{R}(N, \lambda)$ be a rectangular Aztec diamond. Let $\omega$ be a signature corresponding to its boundary row. The construction above defines a map from the set of tilings $\mathcal{D}(N, \alpha) \mapsto$

$$\{(\omega = \mu^{(N)}, \nu^{(N)}, \ldots, \mu^{(1)}, \nu^{(1)}) \text{ where } \nu^{(i+1)} \setminus \mu^{(i)} \text{ is a horizontal strip and } \nu^{(i)} \setminus \mu^{(i)} \text{ is a vertical strip } \}.$$  

This map is bijective.
Assume that the sequence of $N$-tuples $\lambda(N)$ corresponding to the boundary is regular and $\lim_{N \to \infty} m[\lambda(N)] = \eta_\lambda$.

Our results:

- **Law of Large numbers** for the height function + explicit computation of the limit height function
- **Explicit computation of the frozen boundary**

Previous results for the rhombus tilings: [Petrov '12], [Duse–Metcalfe '14].
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- **Central Limit Theorem**: convergence of the global fluctuations of the height function to the GFF

Previous results: [Kenyon’01], for the rhombus tilings [Petrov’13], [Duse–Metcalfe’14].
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- **Central Limit Theorem**: convergence of the global fluctuations of the height function to the GFF
- **Local fluctuations**: discrete sine kernel (for the rhombus tilings [Gorin’16]).
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**Main tool**: Schur generating functions and Moment method ([Bufetov–Gorin’13], [Bufetov–Gorin’16]).
Theorem [Kenyon-Okounkov ’05]
For a simply-connected tileable polygonal domain $\Omega$ with $3d$ sides in coordinate directions the frozen boundary is a rational algebraic curve of rank $d$. Moreover, it is uniquely determined by the geometry of the domain.
Let us introduce some notation:

Denote $A^{(s)} = (A_1, A_2, \ldots, A_s)$ and $B^{(s)} = (B_1, B_2, \ldots, B_s)$, where

$$\sum_{i=1}^{s} (B_i - A_i + 1) = N.$$
Frozen boundary

Theorem (Bufetov, K. ’16)

Consider the following asymptotic regime of the growth of Rectangular Aztec diamonds

\[ A_i(N) = [a_i N], \quad m(N) = [\mu N], \quad B_i(N) = [b_i N], \]

where \( a_1 < b_1 < \cdots < a_s < b_s \) and \( \sum_{i=1}^{s} (b_i - a_i) = 1 \).

Then the frozen boundary is an algebraic curve \( C \) of genus zero. Its dual \( C^\vee \) is given by

\[
C^\vee = (\theta, \frac{2\theta \Pi_s(\theta)}{(\Pi_s(\theta) - 1)(\Pi_s(\theta) + 1)}), \quad \Pi_s(\theta) = \frac{(1 - a_1 \theta) \cdots (1 - a_s \theta)}{(1 - b_1 \theta) \cdots (1 - b_s \theta)}.
\]
Frozen boundary
For $p \in (0; 1)$ a discrete sine kernel is defined by the formula

$$K_p(y_1, y_2) = \frac{\sin(p \cdot \pi(y_1 - y_2))}{\pi(y_1 - y_2)}, \quad y_1, y_2 \in \mathbb{Z}.$$
Let $k(N)/N \to \kappa$ as $N \to \infty$.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N - \lfloor k - 1 \rfloor})$ be a random signature distributed according to $\rho^{k(N)}$. For $x_1, \ldots, x_m \in \mathbb{Z}$ denote by $\tilde{\theta}^{(m)}(x_1, \ldots, x_m)$ the probability that

$$\{x_1, \ldots, x_m\} \subset \{\lambda_i + N - i\}_{i=1,\ldots,N-\lfloor k-1 \rfloor}.$$
Let $k(N)/N \to \kappa$ as $N \to \infty$.
Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N-[k-1]/2})$ be a random signature
distributed according to $\rho^k(N)$. For $x_1, \ldots, x_m \in \mathbb{Z}$ denote by
$\tilde{\theta}^{(m)}(x_1, \ldots, x_m)$ the probability that
$$\{x_1, \ldots, x_m\} \subset \{\lambda_i + N - i\}_{i=1,\ldots,N-[k-1]/2}.$$  

**Lemma**

*Let $x \in \mathbb{R}$ and $x(N)$ be a sequence of integers such that
$x(N)/N \to x$, as $N \to \infty$. For $m \in \mathbb{N}$ let $x_1(N), \ldots, x_m(N)$ be
sequences of integers such that $x_i(N) - x(N)$ does not depend on $N$, $i = 1, \ldots, m$. Then

$$\lim_{N \to \infty} \tilde{\theta}^{(m)}(x_1(N), \ldots, x_m(N)) = \det_{i,j=1}^{m} \left[ K_{d\eta^\kappa(x)}(x_i(N), x_j(N)) \right].$$*
To each row with number \([2\kappa N]\) of \(\mathcal{R}(N, \lambda)\) we assigned a Young diagram given by a \([(1 - \kappa)N]\)-tuple
\[
\mu_i^{\left([(1-\kappa)N]\right)} = (\mu_1, \ldots, \mu_{[(1-\kappa)N]}).
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\]
Define the moment function as 
\[
p_j^\kappa := \sum_{i=1}^{[(1 - \kappa)N]} \left( \mu_i^{([(1 - \kappa)N])} + [(1 - \kappa)N] - i \right)^j.
\]
CLT

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Define the moment function as 
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p_j^{\kappa} := \sum_{i=1}^{[(1-\kappa)N]} \left( \mu_i^{[(1-\kappa)N]} + [(1 - \kappa)N] - i \right)^j.
\]

Theorem

In the notations above, the collection of random functions 
\[
\{ N^{-\kappa} (p_j^{\kappa} - E p_j^{\kappa}) \}_{0 < \kappa \leq 1, j \in \mathbb{N}}
\]
is asymptotically Gaussian and the limit covariance can be computed explicitly.