COHOMOLOGY OF ARITHMETIC GROUPS

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These are my notes for an eponymous graduate student seminar held in Fall 2019 at Columbia University. Let me know if you catch any typos or mistakes.

Contents

1. Overview, 9/17

1.1. **Definition of the cohomology groups.** Let G be a reductive group over $\mathbb{Q}$, e.g. $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_n$. If R is a $\mathbb{Q}$-algebra, e.g. $\mathbb{R}, \mathbb{Q}_p, \mathbb{A}_\mathbb{Q}$, then $G(R)$ forms a group. If R has a topology, then we can topologize $G(R)$ by choosing a closed embedding $G \to \mathbb{A}^n$ into affine space and giving $G(R)$ the subspace topology of $\mathbb{A}^n(R) = R^n$.

Example.

1. $G = \mathrm{SL}_2$. Then $\mathrm{SL}_2 \hookrightarrow M_{2\times 2} \simeq \mathbb{A}^4$ is already closed. So $\mathrm{SL}_2(\mathbb{Q}_p), \mathrm{SL}_2(\mathbb{R}), \mathrm{SL}_2(\mathbb{A}_\mathbb{Q})$ get their topology from $\mathbb{Q}_p^4, \mathbb{R}^4, \mathbb{A}_\mathbb{Q}^4$;
2. $G = \mathrm{GL}_2$. Then $\mathrm{GL}_1 \hookrightarrow M_{1\times 1} = \mathbb{A}^1$ is not closed. Instead $\mathrm{GL}_1 \hookrightarrow \mathbb{A}^2$ given by $x \mapsto (x, x^{-1})$ is closed. Hence $\mathrm{GL}_1(\mathbb{A}_\mathbb{Q}) = \mathbb{I}_\mathbb{Q} \hookrightarrow \mathbb{A}_\mathbb{Q}^2$ gives the correct topology, but $\mathbb{I}_\mathbb{Q} = \mathbb{A}_\mathbb{Q}^\times \subseteq \mathbb{A}_\mathbb{Q}$ does not;
3. $G = \mathrm{GL}_n$ closed, then $G(\mathbb{A}_\mathbb{Q}) = \prod'_{p\leq\infty} G(\mathbb{Q}_p)$ w.r.t. $G(\mathbb{Z}_p)$ gives the correct topology. Here $G(\mathbb{Q}_p)$ gets its topology from $\mathrm{GL}_n(\mathbb{Q}_p) \underset{\text{open}}{\subseteq} M_{n\times n}(\mathbb{Q}_p)$.

Definition 1.1. Let $\Gamma, \Gamma' \subseteq G(\mathbb{Q})$ be subgroups. We say that they are *commensurable* if $[\Gamma : \Gamma \cap \Gamma'], [\Gamma' : \Gamma \cap \Gamma'] < \infty$. This is an equivalence relation.

Definition 1.2. A subgroup $\Gamma \subseteq G(\mathbb{Q})$ is called *arithmetic* if for some (equivalently, every) closed embedding $G \hookrightarrow \mathrm{GL}_n$ we have that Γ and $\mathrm{GL}_n(\mathbb{Z}) \cap G(\mathbb{Q})$ in $\mathrm{GL}_n(\mathbb{Q})$ are commensurable. It is easy to see that such Γ must be discrete in $G(\mathbb{R})$.

Example.

1. $G = \mathrm{SL}_n \hookrightarrow \mathrm{GL}_n$. Then $\mathrm{SL}_n(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z}) = \mathrm{SL}_n(\mathbb{Z})$ is arithmetic;
2. $\Gamma_N = \ker(\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z}))$ is arithmetic and called a *congruence subgroup*;

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(3) $G = \mathrm{SL}_2 \subseteq \mathrm{GL}_2$. Then

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \pmod N \right\}$$
$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \pmod N, a \equiv d \equiv 1 \pmod N \right\}$$
$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \equiv d \equiv 1 \pmod N, b \equiv c \equiv 0 \pmod N \right\}$$

are arithmetic.

Cohomology groups: Let $\Gamma \subseteq G(\mathbb{Q})$ be arithmetic and (ρ, V) be a finite-dimensional representation of G. The cohomology groups we want to study are the groups cohomology groups $H^*(\Gamma, V(\mathbb{C}))$.

Hecke action: Let $\Gamma', \Gamma'' \subseteq \Gamma$ be subgroups of finite index such that there exists an isomorphism $\varphi : \Gamma' \to \Gamma''$. Define $T_\varphi \in \mathrm{End}(H^i(\Gamma, V(\mathbb{C}))$ by

$$H^i(\Gamma, V(\mathbb{C})) \xrightarrow{\mathrm{Res}} H^i(\Gamma'', V(\mathbb{C})) \xrightarrow{\varphi^*} H^i(\Gamma', V(\mathbb{C})) \xrightarrow{\mathrm{Cor}} H^i(\Gamma, V(\mathbb{C})).$$

For example, $G = \mathrm{SL}_2, \Gamma = \mathrm{SL}_2(\mathbb{Z}), \Gamma' = \Gamma_0(p)$,

$$\Gamma'' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid b \equiv 0 \pmod p \right\}$$

and

$$\varphi(\gamma) = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1}.$$

Then $T_\varphi \longleftrightarrow T_p$.

1.2. **Preview of upcoming topics.**

Main idea: $H^*(\Gamma, V(\mathbb{C}))$ can be interpreted as a space of automorphic forms on G of "level Γ".

(1) (Geometric interpretation of $H^*(\Gamma, V(\mathbb{C}))$) Let $K \subseteq G(\mathbb{R})$ be a maximal compact subgroup and $X = G(\mathbb{R})/K$ be a real manifold. Then $X_\Gamma = \Gamma \backslash G(\mathbb{R})/K$ is a real manifold if Γ is torsion-free. One can make a local system $\widetilde{V}$ on X_Γ out of $V(\mathbb{C})$. Then $H^*(\Gamma, V(\mathbb{C})) \simeq H^*(X_\Gamma, \widetilde{V})$.

Example. $G = \mathrm{SL}_2$. Then $K = \mathrm{SO}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \simeq \mathcal{H}$, the upper half-plane. If $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a finite index subgroup, then it acts on $\mathcal{H}$ by Möbius transformations. In general, we can view automorphic forms as functions on certain X_Γ's.

(2) (Matsushima's lemma) Assume that X_Γ is compact (equivalently, $\Gamma \backslash G(\mathbb{R})$ is compact). Then $G(\mathbb{R})$ acts on $L^2(\Gamma \backslash G(\mathbb{R}))$ by right translation.

Theorem 1.3 (Gelfand-Piatetski-Shapiro).

$$L^2(\Gamma \backslash G(\mathbb{R})) = \bigoplus_\pi m(\pi)\pi$$

with $m(\pi) \in \mathbb{Z}_{\geq 0}$ and π-irreducible unitary representations of $G(\mathbb{R})$.

Matsushima's formula says

$$H^*(X_\Gamma, \widetilde{V}) = \bigoplus_\pi m(\pi) H^*(\mathfrak{g}, K, \pi \otimes \xi_V)$$

where ξ_V is the character of $Z(U(\mathfrak{g}))$ on V. These $(\mathfrak{g}, K)$-cohomology groups can be computed.

③ (Eisenstein classes) If $\Gamma\backslash G(\mathbb{R})$ is not compact mod center, then $H^*(X_\Gamma, \widetilde{V})$ has subspaces corresponding to cuspidal automorphic forms, but they don't exhaust the entire $H^*(X_\Gamma, \widetilde{V})$. Other cohomology classes come from cusp forms on Levi's of proper parabolic subgroups. These are the *Eisenstein classes.*

Theorem 1.4 (Franke). *These cuspidal and Eisenstein series exhaust* $H^*(X_\Gamma, \widetilde{V})$.

1.3. Why study automorphic forms this way?

① (The "Chao Li"-style answer) $H^*(\Gamma, V(\mathbb{C}))$ has on obvious rational structure, namely $H^*(\Gamma, V(\mathbb{Q}))$. This leads to rationality results for automorphic forms. For example, we can use this (co)homological structure to prove rationality results for L-values of modular forms.

② (The "Michael Harris"-style answer) X_Γ will often have the structure of not only a complex manifold, but also that of a variety defined over a number field F. If we fix an isomorphism $\overline{\mathbb{Q}_p} \simeq \mathbb{C}$, then $H^*(X_\Gamma, \widetilde{V})$ can be compared with $H^*_{\text{ét}}((X_\Gamma)_{\overline{F}}, \widetilde{V})$.

$$\left.\begin{array}{l}\text{Hecke acts on } H^*(X_\Gamma, \widetilde{V}) \\ \text{Galois acts on } H^*_{\text{ét}}((X_\Gamma)_{\overline{F}}, \widetilde{V})\end{array}\right\} \rightsquigarrow \text{Galois representations attached to automorphic forms.}$$

③ (The "Eric Urban"-style answer) It is hard to interpolate p-adically automorphic forms directly. However, it is much easier to interpolate the V's. Suitably taking cohomology of the interpolated V's is the first step on the way to constructing eigenvarieties.

2. Eichler-Shimura isomorphism, 9/24

THE EICHLER-SHIMURA ISOMORPHISM

ASHWIN IYENGAR

Contents

1. Introduction

We are studying the cohomology of arithmetic groups. Today, we will describe the case where when $G = \mathrm{SL}_2$, and Γ is a congruence subgroup, which is an important case showing up in the theory of modular forms.

Definition 1.1. A subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ is a *congruence subgroup* if for some N,

$$\Gamma(N) = \ker(\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})) \subseteq \Gamma.$$

So fix Γ as above and fix a weight $k \geq 2$. Let $S_k(\Gamma)$ denote the space of cusp forms of level Γ and weight k, and let $\mathcal{E}_k(\Gamma)$ denote the space of Eisenstein series of level Γ and weight k. The goal of today's talk is to prove the following result.

Theorem 1.2 (Eichler-Shimura)**.** *There is a Hecke-equivariant isomorphism*

$$S_k(\Gamma) \oplus \overline{S_k(\Gamma)} \oplus \mathcal{E}_k(\Gamma) \xrightarrow{\sim} H^i(\Gamma, \mathrm{Sym}^{k-2}(\mathbf{C}^2))$$

where Γ *acts on* $\mathbf{C}^2$ *via* $\Gamma \hookrightarrow \mathrm{GL}_2(\mathbf{C})$.

Here $\overline{S_k(\Gamma)}$ denotes the space of anti-holomorphic cusp forms, which in this case is actually isomorphic to $S_k(\Gamma)$. We will explain what "Hecke-equivariant" means later on in the talk.

2. Modular Symbols

Modular symbols (which basically amount to homology classes) turn out to be a nice way of computationally accessing the link between spaces of modular forms and cohomology, so we will use them as our tool to construct the Eichler-Shimura map. They also turn out to be a nice way to construct p-adic L-functions, which is one of their primary uses.

Definition 2.1. Let Δ denote the group of divisors on $\mathbf{P}^1(\mathbf{Q})$. In other words, these are finite sums $\sum_i n_i[r_i]$ with $r_i \in \mathbf{Q} \cup \{\infty\}$. Let

$$\Delta_0 = \left\{ D = \sum_i n_i[r_i] \in \Delta : \sum_i n_i = 0 \right\}$$

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Note Δ_0 admits a left Γ-action via Möbius transformations. In other words if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have an action on $\mathbf{P}^1(\mathbf{Q})$ given by

$$\gamma \cdot [r] = \left[\frac{ar+b}{cr+d}\right]$$

and this extends to all of Δ_0.

Remark 2.2. Note that the set of Γ-equivalence classes of $\mathbf{P}^1(\mathbf{Q})$ is exactly the set of cusps in the compactification of the modular curve. When we see the definition of a modular symbol associated to a modular form, we will need the interpretation of Δ_0 as the space of (finite sums of) equivalence classes of spaces of paths between cusps.

Definition 2.3. Let V be a group with a left Γ-action. We let

$$\mathrm{Symb}_\Gamma(V) = \mathrm{Hom}_\Gamma(\Delta^0, V)$$

and call $\mathrm{Symb}_\Gamma(V)$ the space of *modular symbols of level* Γ with values in V.

Note that we can put a left Γ-action on $\mathrm{Hom}(\Delta_0, V)$ which takes

$$\phi \mapsto [\gamma \cdot \phi : D \mapsto \gamma^{-1} \cdot \phi(\gamma \cdot D)]$$

Then we visibly have $\mathrm{Hom}(\Delta_0, V)^\Gamma = \mathrm{Hom}_\Gamma(\Delta_0, V)$.

3. Cohomology

We now write down a cohomological interpretation of these modular symbols. First we make the technical assumption that Γ acts freely on $\mathcal{H}$ (this is satisfied, for instance, when $\Gamma(3) \subseteq \Gamma$). Note $Y_\Gamma = \mathcal{H}/\Gamma$ is a classifying space for Γ (basically because Γ acts freely and discontinuously on $\mathcal{H}$, and $\mathcal{H}$ is contractible), which (by general manipulations) implies that for V any left Γ-module,

$$H^i(\Gamma, V) \cong H^i(Y_\Gamma, \widetilde{V})$$

where $\widetilde{V}$ is the local system associated to V, and where on the right we take singular cohomology. By abuse of notation, we let $H^i_c(\Gamma, V)$ denote the compactly supported cohomology of Y_Γ with $\widetilde{V}$-coefficients.

Proposition 3.1. *There is a canonical and functorial isomorphism*

$$\mathrm{Symb}_\Gamma(V) \xrightarrow{\sim} H^i_c(\Gamma, V)$$

Proof. Let $\overline{\mathcal{H}} = \mathcal{H} \cup \mathbf{P}^1(\mathbf{Q})$ denote the compactification of $\mathcal{H}$ given by adding cusps. Note $\mathcal{H}$ is not compact, so $H^0_c(\overline{\mathcal{H}}, \widetilde{V}) = 0$. Note further that $\overline{\mathcal{H}}$ is contractible, so $H^1(\overline{\mathcal{H}}, \widetilde{V}) = 0$. Then if we consider the long exact sequence for a closed subset for compactly supported cohomology, and if we take Homs into V for the split short exact sequence $0 \to \Delta_0 \to \Delta \to \mathbf{Z} \to 0$, we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\overline{\mathcal{H}}, \widetilde{V}) & \longrightarrow & H^0(\mathbf{P}^1(\mathbf{Q}), \widetilde{V}) & \longrightarrow & H^1_c(\mathcal{H}, V) & \longrightarrow & 0 \\ & & \| & & \| & & & & \\ 0 & \longrightarrow & V & \longrightarrow & \mathrm{Hom}(\Delta, V) & \longrightarrow & \mathrm{Hom}(\Delta_0, V) & \longrightarrow & 0 \end{array}$$

The left vertical equality follows from the fact that $\mathcal{H}$ is connected, and the middle equality follows from the fact that Δ is the free group generated by $\mathbf{P}^1(\mathbf{Q})$. Therefore $H^1_c(\mathcal{H}, V) = \mathrm{Hom}(\Delta_0, V)$, and so a spectral sequence computation, combined with the fact that $H^0_c(\mathcal{H}\Gamma, \widetilde{V}) = 0$, implies that

$$\mathrm{Symb}_\Gamma(V) = \mathrm{Hom}(\Delta_0, V)^\Gamma = H^1_c(\mathcal{H}, \widetilde{V})^\Gamma = H^1_c(\mathcal{H}/\Gamma, \widetilde{V}).$$

□

But we care about the whole H^i, so we need to see to what extent the modular symbols contribute to it.

Definition 3.2. The *interior cohomology* $H^i_!(\Gamma, V)$ is by definition the image of the natural map

$$H^i_c(\Gamma, V) \to H^i(\Gamma, V)$$

In fact, Poincaré duality induces a perfect pairing

$$H^1_!(\Gamma, V) \times H^1_!(\Gamma, V^\vee) \to \mathbf{C}$$

(here V is a $\mathbf{C}$-vector space and Γ acts $\mathbf{C}$-linearly).

4. Cusp Forms

In practice, we let

$$V = V_k(\mathbf{C}) = \mathrm{Sym}^{k-2}(\mathbf{C}^2),$$

which is the exact analog of specifying the weight k in the definition of a modular form.

So how do you associate a modular symbol to a cusp form? We need to construct a map that takes a "path" $[s]-[r]$ and spits out an element of $V_k(\mathbf{C}) = \mathrm{Sym}^{k-2}(\mathbf{C}^2)$. First we give a more concrete description of V. Let $P_k(\mathbf{C})$ denote the space of homogeneous polynomials of degree $k-2$ in two variables, with the action

$$(P \cdot \gamma)(X, Y) = P(aX + bY, cX + dY).$$

Then $V_k(\mathbf{C}) \cong P_k(\mathbf{C})^\vee$. Note also that there is an isomorphism

$$\Theta_k : V_k(\mathbf{C}) \xrightarrow{\sim} P_k(\mathbf{C})$$

which follows from the classification of highest weight representations of $\mathrm{SL}_2(\mathbf{C})$. Combined with Poincaré duality, we get a perfect pairing

$$H^1_!(\Gamma, V) \times H^1_!(\Gamma, V) \xrightarrow{1\times\Theta_k} H^1_!(\Gamma, V) \times H^1_!(\Gamma, V^\vee) \to \mathbf{C}$$

We also have a pairing on $S_k(\Gamma)$:

Definition 4.1. The Petersson inner product is given by

$$(f, g)_\Gamma = \int_{\mathcal{H}/\Gamma} f(z)\overline{g(z)}y^{k-2}dxdy$$

It is a perfect sesquilinear pairing.

In a moment we will relate the two pairings. First we associate a modular symbol to each cusp form.

Definition 4.2. Given a cusp form $f \in S_k(\Gamma)$, we may define

$$I_f([s]-[r])(P) = \int_r^s f(z)P(z,1)dz$$

and extend $\phi_k(f)$ to all of Δ^0 (exercise: check that this is independent of the path chosen between r and s on the modular curve, and that this "extension to Δ_0" is well-defined). Convergence of this integral relies on the fact that f decays rapidly as it approaches the cusps, eventually hitting 0.

Lemma 4.3. *The map I_f is Γ-equivariant.*

Proof. If $D \in \Delta_0$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we just compute:

$$\begin{aligned}
I_f(\gamma \cdot ([s]-[r]))(P) &= I_f(\gamma \cdot [s] - \gamma \cdot [r])(P) \\
&= \int_{\gamma \cdot r}^{\gamma \cdot s} f(z)P(z,1)dz \\
&= \int_r^s f(\gamma \cdot z)P(\gamma \cdot z, 1)d(\gamma \cdot z) \\
&= \int_r^s (cz+d)^k f(z) P\left(\frac{az+b}{cz+d}, 1\right)(cz+d)^{-2}dz \\
&= \int_r^s (cz+d)^k f(z)(cz+d)^{-(k-2)}(P \cdot \gamma)(z,1)(cz+d)^{-2}dz \\
&= \int_r^s f(z)(P \cdot \gamma)(z,1)dz \\
&= I_f([s]-[r])(P \cdot \gamma) \\
&= [\gamma \cdot (I_f([s]-[r])](P)
\end{aligned}$$

Here we used the automorphy condition on f, the homogeneity of P, and the chain rule for dz. □

So we get a map

$$\phi_k : S_k(\Gamma) \xrightarrow{f \mapsto I_f} \mathrm{Symb}_\Gamma(V_k(\mathbf{C})) \xrightarrow{\sim} H^1_c(\Gamma, V_k(\mathbf{C})) \to H^1(\Gamma, V_k(\mathbf{C}))$$

Note the existence of the $\mathbf{R}$-vector subspace $H^1_{(c)}(\Gamma, V_k(\mathbf{R}))$ which satisfies

$$H^1_{(c)}(\Gamma, V_k(\mathbf{R})) \otimes_{\mathbf{R}} \mathbf{C} \cong H^1_{(c)}(\Gamma, V_k(\mathbf{C}))$$

so we define a map

$$\mathrm{ES}_k : S_k(\Gamma) \oplus \overline{S_k(\Gamma)} \cong S_k(\Gamma) \otimes_{\mathbf{R}} \mathbf{C} \xrightarrow{2\Re(\phi_k)\otimes 1} H^1_c(\Gamma, V_k(\mathbf{R})) \otimes_{\mathbf{R}} \mathbf{C} = H^1_c(\Gamma, V_k(\mathbf{C})) \to H^1_!(\Gamma, V_k(\mathbf{C}))$$

Theorem 4.4. *The map* ES_k *is an isomorphism.*

Proof. We first show injectivity of $\Re(\phi_k) : S_k(\Gamma) \to H^1_c(V_k(\mathbf{R}))$. To check injectivity, we use the following computation, due to Shimura:

$$(\Re(\phi_k)(f), \Re(\phi_k)(g)) = c_k((f,g)_\Gamma + (-1)^{k+1}\overline{(f,g)_\Gamma})$$

for some nonzero constant $c_k \in \mathbf{C}$ (to do this computation, you need to trace through the cohomological formalism and figure out what the pairing is: basically it's given by a cup product, and by using a comparison with de Rham cohomology, you can get a handle on computing this cup product). Then if we assume $\Re(\phi_k)(f) = 0$, then applying the above formula to g and ig we see that $\Re(f,g)_\Gamma = 0$ and $\Im(f,g)_\Gamma = 0$. Since $(\cdot,\cdot)_\Gamma$ is perfect and $(f,g)_\Gamma = 0$ for all $g \in S_k(\Gamma)$, we must have $f = 0$.

As for surjectivity, there is a dimension formula in Section 6 of [Hid93]. Following Bellaiche's book, we will sketch a proof when $k = 0$. In this case, $S_2(\Gamma)$ can actually be identified with the space $\Omega^1(X(\Gamma))$, the space of holomorphic differential 1-forms on X_Γ. But by definition $\dim_{\mathbf{C}} \Omega^1(X(\Gamma))$ is the genus of X_Γ, so $\dim_{\mathbf{R}} S_2(\Gamma) = 2g$. On the other hand we have a factorization (note $V_0(\mathbf{R}) = \mathbf{R}$)

$$\begin{array}{ccc}
H^1_c(Y_\Gamma, \mathbf{R}) & \longrightarrow & H^1(Y_\Gamma, \mathbf{R}) \\
\downarrow & \nearrow & \\
H^1(X_\Gamma, \mathbf{R}) & &
\end{array}$$

but $\dim_{\mathbf{R}} H^1(X_\Gamma, \mathbf{R}) = 2g$. □

5. Hecke Operators

As mentioned in the beginning, we should show that this map has "nice properties". In this case, this means that the map respects Hecke operators.

First we define Hecke operators for modular forms. Take the monoid $S = \mathrm{GL}_2^+(\mathbf{Q}) \cap M_2(\mathbf{Z})$, and form the double coset $\Gamma s \Gamma$. If we pick a decomposition $\Gamma s \Gamma = \bigsqcup_i \Gamma s_i$, for $s_i \in S$, then we can define a right action on:

- $S_k(\Gamma)$:
$$f \mapsto f|_{\Gamma s \Gamma} = \sum_i f[s_i]_k$$
where we define the slash operator
$$f[s_i]_k(z) = (\det s_i)^{k-1} j(s_i, z)^{-k} f(s_i \cdot z)$$
- $\mathrm{Symb}_\Gamma(V_k(\mathbf{R}))$: note $V_k(\mathbf{R})$ is actually an S-module because S acts on $\mathbf{R}^2$ in the same way as Γ.
$$\phi \mapsto \phi|_{\Gamma s \Gamma} : D \mapsto \sum_i \phi(s_i \cdot D) \cdot s_i$$

Remark 5.1. When $\Gamma = \Gamma_1(N)$, the matrices $1, 0, 0, p$ give you the T_p operators, which have the decomposition

Proposition 5.2. *The map $\Re(\phi_k) : S_k(\Gamma) \to H^1(\Gamma, V_k(\mathbf{R}))$ respects the Hecke operators.*

Proof. First we show that $S_k(\Gamma) \to \mathrm{Symb}(V_k(\mathbf{R}))$ respects the Hecke operators. Then the proposition follows from the fact that
$$\mathrm{Symb}_\Gamma(V(\mathbf{R})) \to H^1(\Gamma, V_k(\mathbf{R}))$$
is Hecke-equivariant, which follows from the fact that $0 \to V \to \mathrm{Hom}(\Delta, V) \to \mathrm{Hom}(\Delta_0, V) \to 0$ is actually an exact sequence of S-modules, and the above map is the connecting homomorphism in the long exact sequence associated to taking Γ-invariants of this sequence.

But the proof that $\mathrm{Symb}_\Gamma(V_k(\mathbf{R})) \to H^1(\Gamma, V_k(\mathbf{R}))$ is Hecke-equivariant is an only slightly more general version of Lemma 4.3. □

References

[Hid93] Haruzo Hida. *Elementary theory of L-functions and Eisenstein series*, volume 26 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1993.

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