Bernouilli numbers, Eisenstein Series and Cyclotomic units
Michael Zhao Memorial Student Colloquium

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November 20th, 2019
Fermat Last Theorem

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Let \( x, y, z \in \mathbb{Z} \) such that \( x^p + y^p = z^p \) then \( xyz = 0 \).
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Let $x, y, z \in \mathbb{Z}$ such that $x^p + y^p = z^p$ then $xyz = 0$.

Let $F$ be the cyclotomic field $\mathbb{Q}(\zeta_p)$ where $\zeta_p = e^{2i\pi/p}$. Using the factorization

$$
\prod_{a=0}^{p-1} (x + \zeta_p^a y) = z^p,
$$

one can show that FLT follows for $p$, if we know that $O_F = \mathbb{Z}[\zeta_p]$ is a UFD. It is false in general but Kummer proved that if $p$ does not divide the class number of $\mathbb{Q}(\zeta_p)$, then FLT follows for $p$. Such prime numbers are called regular primes.
Bernouilli numbers

The Bernouilli numbers are the rational numbers $B_n$ for $n \geq 1$ defined by

$$\frac{t}{1 - e^{-t}} = 1 + \sum_{n=1}^{\infty} B_n \cdot \frac{t^n}{n!}$$

Example: $B_1 = \frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_{2n+1} = 0$ for $n \geq 1$, $B_4 = -\frac{1}{30} \ldots$
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Kummer’s Criterion (1850)

If $p \nmid B_2 B_4 \ldots B_{p-3}$, then $p$ is a regular prime.
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Kummer's Criterion (1850)

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More generally, if $\chi$ is a Dirichlet character modulo $N$, we define the generalized Bernoulli numbers $B_{n,\chi}$ as follows:

$$\sum_{a=1}^{N-1} \chi(a) \frac{te^{at}}{e^{Nt} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \cdot \frac{t^n}{n!}$$

Example: $B_{1,\chi} = \frac{1}{N} \sum_{a=1}^{N-1} \chi(a)a$.
A refinement: The theorem of Herbrand-Ribet

Let $G := \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ and let $\omega : G \to (\mathbb{Z}/p\mathbb{Z})^\times$ be the Teichmüller character. Consider the action of $G$ on the $p$-Sylow subgroup $C$ of the class group $\text{Cl}_{\mathbb{Q}(\zeta_p)}$.
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$$C = \bigoplus_{i=0}^{p-1} C(i)$$

where $C(i)$ is the maximal subgroup of $C$ for which $G$ acts via $\omega^i$. We then denotes by $h(i)$ the order of $C(i)$. Then a refinement of Kummer’s criterion is given by the following
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**Theorem (Herbrand(1932)-Ribet(1976))**

Let $i$ be an odd integer between 3 and $p - 2$, then $p$ divides $h(i)$ if and only if $p$ divides $B_{p-i}$.
Further refinement: The theorem of Mazur-Wiles.

Using elementary method, one can see from the definition of Bernoulli numbers that

\[ B_{p-i} \equiv B_{1,\omega-i} \pmod{p} \]
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**Theorem (Mazur-Wiles(1984))**

*Let \( i \) be an odd integer between 3 and \( p - 2 \), then*

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**Theorem (Mazur-Wiles(1984))**

Let $i$ be an odd integer between 3 and $p - 2$, then

$$h(i) \sim B_{1,\omega^{-i}}.$$  

For a Dirichlet character $\chi$, we consider the $L$-series

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad \text{CV for } \Re(s) > 1.$$ 

It has a meromorphic continuation to $\mathbb{C}$ (holomorphic if $\chi$ is non trivial) and the following formula holds

$$L(1 - n, \chi) = - \frac{B_{n, \chi}}{n} \text{ if } n \geq 1 \text{ and } \chi(-1) = (-1)^n.$$
Idea of proof of Mazur-Wiles’s theorem

The idea is an elaborate refinement of Ribet’s idea using Iwasawa theory.

Let $L$ be the maximal abelian unramified extension of $\mathbb{Q}(\zeta_p)$. Use Class Field Theory to view $C(i)$ as $C(i) \sim \text{Hom}_G(\text{Gal}(L/\mathbb{Q}(\zeta_p)), \mathbb{Q}_p/\mathbb{Z}_p(\omega - i))$.

Use congruence between Eisenstein series and cusp forms and their associated Galois representations to prove the divisibility $B_1(\omega - i) \sim L(0, \omega - i) | h(i)$ for odd integer $i$ between 3 and $p - 2$.

Use Dirichlet class Number Formulas for $\mathbb{Q}(\zeta_p)$ and $\mathbb{Q}(\zeta_p)$ + that tell us that $p - 2 \prod_{i=3}^{p-2} \text{odd} L(0, \omega - i) \sim h - i = p - 2 \prod_{i=3}^{p-2} \text{odd} h(i)$. 
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$$\prod_{\substack{i=3 \	ext{odd}}}^{p-2} L(0, \omega^{-i}) \sim h^- = \prod_{\substack{i=3 \	ext{odd}}}^{p-2} h(i)$$
Modular forms

A modular forms of weight \( k \) and nebentypus \( \chi \) is an holomorphic function on the Poincaré upper half plane \( \mathbb{H} \) such that

\[
f\left( \frac{az + b}{cz + d} \right) = \chi(d)(cz + d)^k \cdot f(z) \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)
\]

and if \( f(z) \) has a limit at the each of the cusps of \( \Gamma_0(N) \backslash \mathbb{H} \).

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and if $f(z)$ has a limit at the each of the cusps of $\Gamma_0(N) \setminus \mathbb{H}$. It is said cuspidal if the value at the cusps are zero. A modular form $f$ is called an eigenform if it is an eigen vector for all the Hecke operators $T_q$ defined by

$$T_q \cdot f(z) = \sum_{i=0}^{q-1} f\left(\frac{z + i}{q}\right) + \chi(q)f(qz)$$

for $q$ prime to $N$, then

$$f(z) = a_0(f) + \sum_{n=1}^{\infty} a_n(f)e^{2i\pi nz}. \quad (a_1 = 1 \text{ if } f \text{ normalized})$$

where $a_q(f)$ is the eigenvalue of $T_q$ and $a_0(f)$ is called the constant term at the cusp $\infty$. 
Eisenstein series

For each integer $k \geq 3$, one can form the series

$$G_{k,\chi}(z) := \sum_{(c,d)=1 \atop N|c} \chi(d)(cz + d)^{-k}$$

this defines a modular form of weight $k$ and nebentypus $\chi$. After renormalization, one gets

$$E_{k,\chi}(z) = \frac{L(1-k,\chi)}{2} + \sum_{n=1}^{\infty} \left( \sum_{d|n \atop (d,N)=1} \chi(d)d^{k-1} \right) e^{2i\pi nz}$$
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It is an eigenform satisfying

$$T_q \cdot E_{k,\chi} = (1 + \chi(q)q^{k-1}) \cdot E_{k,\chi}$$
Galois representations

If \( f \) is an eigenform of weight \( k \) and nebentypus \( \chi \), it is known thanks to the works of Eichler-Shimura and Deligne, that there exists a continuous Galois representation

\[
\rho_f : G_{\mathbb{Q}} \to GL_2(\mathbb{Z}_p)
\]

unramified away from \( Np \) such that \( tr(\rho_f(Frob_q)) = a_q(f) \) for \( q \nmid Np \) and \( det(\rho_f) = \epsilon_{cyc}^{1-k} \chi \).
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\rho_f|_{I_p} \sim \begin{pmatrix} 1 & \ast \\ 0 & \epsilon^{1-k}_{cyc} \chi \end{pmatrix}
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Ribet proved that \( \rho_f \) is absolutely irreducible whenever \( f \) is cuspidal. On the other hand, if \( f \) is the Eisenstein series \( E_{k,\chi} \), the corresponding Galois representation is reducible.
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Ribet proved that $\rho_f$ is absolutely irreducible whenever $f$ is cuspidal. On the other hand, if $f$ is the Eisenstein series $E_{k,\chi}$, the corresponding Galois representation is reducible. More precisely, it is given by

$$\rho_{E_{k,\chi}} = \begin{pmatrix} \epsilon_1^{1-k} \chi & 0 \\ 0 & 1 \end{pmatrix}$$
Ordinary Eisenstein congruences

If $p^m | L(1 - k, \chi)$, then $E_{k,\chi}$ looks cuspidal modulo $p^m$. In fact, one can show that there exists $g$ cuspidal of the same weight and level as $E_{k,\chi}$ such that

$$g \equiv E_{k,\chi} \pmod{p^m}$$
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If $g$ is an eigenform, we deduce that $\rho_g$ becomes reducible modulo $p^m$. Since $\rho_g$ is itself irreducible, one can show (Ribet's lemma) that there exists a stable lattice in the space of the representation $\rho_g$ such that modulo $p^m$, we have

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$$\rho_g \equiv \begin{pmatrix} \chi \epsilon_{\text{cyc}}^{1-k} & \ast \\ 0 & 1 \end{pmatrix} \pmod{p^m}$$

where the upper right shoulder $\ast$ defines a non-trivial extension and therefore an element in $H^1(G_{\mathbb{Q}}, \mathbb{Z}/p^m\mathbb{Z}(-1))$ which is unramified at $p$. By refining this argument and making it precise, Mazur and Wiles prove that $B_{1,\omega_{-i}}$ divides $h(i)$. 
Cyclotomic units and the Kummer map

For any number field $F \subset \bar{Q}$ and integer $n$, recall that we have the Kummer map

$$k_F: F^\times \to H^1(F, \mathbb{Z}_p(1))$$
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$$c_N = k_{\mathbb{Q}(\zeta_N)}(1 - \zeta_N)$$
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It satisfies the Norm Relations of an Euler system:

$$\text{Cores}_{Q(\zeta_N)}^{Q(Q_N)}(c_{N\ell}) = \begin{cases} (1 - Fr_{\ell}^{-1}) \cdot c_N & \text{if } \ell \nmid N \\ c_N & \text{if } \ell | N \end{cases}$$
Another proof of MW theorem using this Euler system:

Using these classes, following ideas of Thaine and Kolyvagin, K. Rubin (1990) gave another proof of the Mazur-Wiles theorem. Grosso modo, his proof uses the classes $c_{N}$ to construct torsion classes $\kappa_{N} \in H^{1}(\mathbb{Q}, \mu_{p^{n}})$ satisfying precise ramification conditions. He uses Tate and Poitou-Tate duality theorems, the reciprocity law of Class Field Theory to exhibit many relations among elements in $C_{i}$. This allows him to bound the size of the class groups $C_{i}(\omega - i)$ for odd integer $i$ between 3 and $p - 2$. He uses the Dirichlet class Number Formula as in the original proof of Mazur-Wiles:

$$p - 2 \prod_{i=3 \text{ odd}} \omega - i \sim h_{-i} = p - 2 \prod_{i=3 \text{ odd}} h_{i}$$
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$$\prod_{\substack{i=3 \\ \text{odd}}}^{p-2} L(0, \omega^{-i}) \sim h^- = \prod_{\substack{i=3 \\ \text{odd}}}^{p-2} h(i)$$
Ordinary Eisenstein congruences and Euler systems combined

Using Eisenstein congruences and the Euler system of cyclotomic units one obtains a proof of MW theorem without invoking Dirichlet class number formula. Here is another example where these two technics are combined.


Let $E$ be an elliptic curve over the rational having good ordinary reduction at $p$ such that $L(E, 1) \neq 0$ and having no $p$-torsion point over an abelian extension of $\mathbb{Q}$. Then the $p$-part of the BSD formula holds

$$L(E, 1) \Omega_E \sim \#X_p(E) \prod_{\ell \mid N_E} c_{\ell}(E)$$

The Euler system construction using Siegel units and the upper bound is obtained by the work of K. Kato. The Eisenstein congruences argument uses Klingen-Eisenstein series for $GU(2, 2)$ to obtain the right lower bound by the work of Skinner-U.
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Non-ordinary Eisenstein congruences

Consider the Eisenstein series

\[ E_{2, \chi}^{\text{crit}}(z) := E_{2, \chi}(z) - E_{2, \chi}(pz) \]

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One can patch these congruences and use Ribet’s lemma to construct a class $c_\chi \in H^1(Q, Q_p(1)(\chi)) = H^1(Q(\zeta_N), Q_p(1))^\chi$. 
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Consider the Eisenstein series

\[ E_{2, \chi}^{\text{crit}}(z) := E_{2, \chi}(z) - E_{2, \chi}(pz) \]

It is \( p \)-adically cuspidal. Using the theory of overconvergent forms, for each integer \( n \), one can find a cusp form \( g_n \) of weight \( k_n \) such that

\[ g_n \equiv E_{k, \chi}^{\text{crit}} \pmod{p^n} \text{ and } k_n \to 2 \text{ \( p \)-adically} \]

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(i) The classes are integrals
(ii) They satisfy some norm relations
Non-ordinary Eisenstein congruences II

This raises the following questions:

Question 1: Can we normalize the classes $c^\chi$'s in a canonical way so that

$$c^N := \#(\mathbb{Z}/N\mathbb{Z}) \times \sum_{\chi \in \hat{(\mathbb{Z}/N\mathbb{Z})}} c^\chi \in H^1(\mathbb{Q}(\zeta_N), \mathbb{Z}_p(1))$$

Question 2: If the answer to Q1 is positive, can we relate the classes $c^N$'s to $L$-values or Bernoulli numbers? Or equivalently, are the $c^N$'s related to the cyclotomic units $(1 - \zeta_N)$ via the Kummer map?

Question 3: Can we prove Mazur-Wiles theorem using only Eisenstein congruences? (i.e. without invoking the class number formula)
Non-ordinary Eisenstein congruences II

This raises the following questions:

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Can we normalize the classes $c_\chi$’s in a canonical way so that

$$c_N := \frac{1}{\#(\mathbb{Z}/N\mathbb{Z})^\times} \sum_{\chi \in (\mathbb{Z}/N\mathbb{Z})^\times} c_\chi \in H^1(Q(\zeta_N), \mathbb{Z}_p(1))?$$

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Euler system via Eisenstein congruences

Theorem (U.)

There is a positive answer to three questions above. In other words, let $a$ be an odd integer distinct from 1 modulo $p - 1$. For each integer $N$, there exists a class $C_{N,a} \in H^1_{Iw}(\mathbb{Q}(\zeta_N), \mathbb{Z}_p(1))^\omega_a$ constructed using Eisenstein congruences and satisfying the norm relation of an Euler system. Moreover, the image of $C_{1,a}$ in $H^1_{Iw}(\mathbb{Q}_p, \mathbb{Z}_p(1))^\omega_a$ is related to the $\omega_a$-branch of the Kubota-Leopold $p$-adic $L$-function.

Remarks:

(i) The non-triviality of the classes come from the non-triviality of the $c\chi$'s constructed above. The norm relations follow from the very definition of the construction where a normalization factor comes from the ordinary Eisenstein congruence number.

(ii) On the other hand, proving that the classes are integral is difficult. There are two proofs. One uses the Stickelberger theorem and is therefore not generalizable. The second proof (which is generalizable) uses the local-global compatibility property for the $p$-adic Langlands correspondence for $GL_2(\mathbb{Q}_p)$ using V. Pa˘sk¯unas formalism.
Euler system via Eisenstein congruences

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Sketch of the construction I

We fix $N > 1$ prime to $p$ and denote by $t_N$ the Hecke algebra generated by $T_\ell$'s for $\ell$ prime to $Np$. There is a pseudorepresentation

$$t_N: G_\mathbb{Q} \to T_N$$

unramified away from $Np$ and such that $t_N(Frob_\ell) = T_\ell$. We consider $T_N^{\text{red}}$ the maximal quotient of $T_N$ such that the push forward of $t_N$ to $T_N^{\text{red}}$ becomes reducible after restriction to $G_{\mathbb{Q}_p}$. We have a natural maps

$$T_N \to T_N^{\text{red}} \to T_N^{\text{ord}}$$

and we set $Q_N := \text{Ker}(T_N \to T_N^{\text{ord}})$ and we define $Q_N^{\text{red}}$ similarly.
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Let

$$\lambda_N : T_N \to \Lambda_N := \mathbb{Z}_p[[1 + p\mathbb{Z}_p]][\mathbb{Z}/N\mathbb{Z})^\times]$$

interpolating the Hecke eigenvalues of Eisenstein series.
Sketch of the construction II

There exists a canonical exact sequence

$$0 \to \lim_{\to} \to \mathbb{N} \to \Lambda \to \lim_{\to} \to \mathbb{N} \to \mathbb{Q} \otimes \Lambda \to \lim_{\to} \to \mathbb{N} \to \mathbb{Q} \otimes \Lambda \to 0$$

Using the pseudo-representation $t$, it is possible to construct a canonical element $(y) \in H_1(Q, \lim_{\to} \to \mathbb{Q} \otimes \Lambda(1))$

The difficulty left is to show that $\mathbb{Q} \otimes \Lambda$ is annihilated by an element $L \in \Lambda$ defined such that for all even character $\chi$ of level $Np^\infty$ with $\chi \mid (\mathbb{Z}/p\mathbb{Z}) \not= 1$, we have $\chi(L) = L_{p}(1, \chi) \prod_{\ell \mid N} (1 - \chi(\ell) - 1)$. 

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Sketch of the construction II

- There exists a canonical exact sequence

\[ 0 \to \lim_{\leftarrow N} \Lambda_N \to \lim_{\leftarrow N} (Q_N \otimes_{\Lambda_N} \Lambda_N) \to \lim_{\leftarrow N} (Q_{N}^{\text{red}} \otimes_{\Lambda_N} \Lambda_N) \to 0 \]

Using the pseudo-representation \( t_N \), it is possible to construct a canonical element \( (y_N) \in H^1(Q, \lim_{\leftarrow N} Q_N \otimes_{\Lambda_N} (1)) \)

The difficulty left is to show that \( Q_{N}^{\text{red}} \otimes_{\Lambda_N} \Lambda_N \) is annihilated by an element \( L_N \in \Lambda_N \) defined such that for all even character \( \chi \) of level \( Np \infty \) with \( \chi \mid (\mathbb{Z}/p\mathbb{Z}) \times \neq 1 \), we have

\[ \chi(L_N) = L_{Np}(1, \chi) \prod_{\ell \mid N} (1 - \chi(\ell) - 1) \]

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Sketch of the construction II

- There exists a canonical exact sequence

\[ 0 \to \lim_{\leftarrow N} \Lambda_N \to \lim_{\leftarrow N} (Q_N \otimes_{\Lambda_N} \Lambda_N) \to \lim_{\leftarrow N} (Q_{red}^N \otimes_{\Lambda_N} \Lambda_N) \to 0 \]

- Using the pseudo-representation \( t_N \), it is possible to construct a canonical element

\[ (y_N)_N \in H^1(Q, \lim_{\leftarrow N} Q_N \otimes \Lambda_N(1)) \]
There exists a canonical exact sequence

$$0 \rightarrow \left\lim_{\mathbb{N}} \Lambda_{\mathbb{N}} \rightarrow \left\lim_{\mathbb{N}} (\mathbb{Q}_{\mathbb{N}} \otimes_{\Lambda_{\mathbb{N}}} \Lambda_{\mathbb{N}}) \rightarrow \left\lim_{\mathbb{N}} (\mathbb{Q}_{\mathbb{N}}^{red} \otimes_{\Lambda_{\mathbb{N}}} \Lambda_{\mathbb{N}}) \rightarrow 0$$

Using the pseudo-representation $t_{\mathbb{N}}$, it is possible to construct a canonical element

$$(y_{\mathbb{N}})_{\mathbb{N}} \in H^1(\mathbb{Q}, \left\lim_{\mathbb{N}} \mathbb{Q}_{\mathbb{N}} \otimes \Lambda_{\mathbb{N}}(1))$$

The difficulty left is to show that $\mathbb{Q}_{\mathbb{N}}^{red} \otimes_{\Lambda_{\mathbb{N}}} \Lambda_{\mathbb{N}}$ is annihilated by an element $\mathcal{L}_{\mathbb{N}} \in \Lambda_{\mathbb{N}}$ defined such that for all even character $\chi$ of level $Np^{\infty}$ with $\chi|_{(\mathbb{Z}/p\mathbb{Z})^\times} \neq 1$, we have

$$\chi(\mathcal{L}_{\mathbb{N}}) = L^{Np}(1, \chi) \prod_{\ell|\mathbb{N}} (1 - \chi(\ell)^{-1})$$
Sketch of the construction III

Using some local-global compatibility with $p$-adic Landlands correspondence, we can replace $Q_N \otimes \Lambda_N$ by a suitable quotient such that the exact sequence still holds and the corresponding co-kernel is killed by $\mathcal{L}_N$. 

Let us call $y'_N$ the projection of $y_N$ the Galois cohomology of this quotient. Then, we have

$$C_N, a := e_{\omega a} \cdot \mathcal{L}_N \cdot y'_N \in H_1(Q, \Lambda_N(1))_{\omega a} = H_1(Iw(Q(\zeta_N), Z_p(1)))_{\omega a}$$

The classes are integral and the norm relation will follow from the ones satisfied by $\pi_N, \ell(\mathcal{L}_N \ell) = (1 - <\ell> - 1_N) \cdot \mathcal{L}_N$ where $\pi_N, \ell$ is the projection $\Lambda_N \ell \to \Lambda_N$. 

Sketch of the construction III

Using some local-global compatibility with \( p \)-adic Landlands correspondence, we can replace \( Q_N \otimes \Lambda_N \) by a suitable quotient such that the exact sequence still holds and the corresponding co-kernel is killed by \( \mathcal{L}_N \).

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Using some local-global compatibility with $p$-adic Landlands correspondence, we can replace $Q_N \otimes \Lambda_N$ by a suitable quotient such that the exact sequence still holds and the corresponding co-kernel is killed by $\mathcal{L}_N$.

Let us call $y_N'$ the projection of $y_N$ the Galois cohomology of this quotient. Then, we have

$$C_{N,a} := e_{\omega^a} \cdot \mathcal{L}_N \cdot y_N' \in H^1(Q, \Lambda_N(1))^{\omega^a} = H^1_{Iw}(Q(\zeta_N), \mathbb{Z}_p(1))^{\omega^a}$$

The classes are integral and the norm relation will follow from the one satisfied by $\mathcal{L}_N$:

$$\pi_{N,\ell}(\mathcal{L}_{N\ell}) = (1 - <\ell>_{N}^{-1}) \cdot \mathcal{L}_N$$

where $\pi_{N,\ell}$ is the projection $\Lambda_{N\ell} \rightarrow \Lambda_N$. 
THANK YOU!