Problem 1.

(a) \[ \left( \bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c. \]

(\subseteq, 1 pt) \( x \in (\bigcup A_i)^c \implies x \notin \cup A_i \implies x \notin A_i \forall i \in I \implies x \in A_i^c \forall i \in I \implies x \in \cap A_i^c. \)

(\supseteq, 1 pt) \( x \in \cap A_i^c \implies x \in A_i^c \forall i \in I \implies x \notin A_i \forall i \in I \implies x \notin \cup A_i \implies x \in (\bigcup A_i)^c. \)

(b) \[ \left( \bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} A_i^c. \]

(\subseteq, 1 pt) \( x \in (\bigcap A_i)^c \implies x \notin \cap A_i \implies x \notin A_j \text{ for some } j \in I \implies x \in A_j^c \implies x \in \cup A_j^c. \)

(\supseteq, 1 pt) \( x \in \cup A_j^c \implies x \in A_j^c \text{ for some } j \in I \implies x \notin A_j \implies x \notin \cap A_i \implies x \in (\bigcap A_i)^c. \)

Problem 2. Answer (1 pt): The intersection may be empty. Example (1 pt): \( A_n = (0, \frac{1}{n}). \)

Problem 3.

\( \bullet \) \( \sup A = \cos \frac{1}{2}. \)

(1 pt) Clearly \( \sup A \geq \frac{|\cos 1|}{1+1} = \frac{\cos 1}{2}. \)

(1 pt) Suppose \( \frac{|\cos n|}{n+1} > \frac{\cos 1}{2} \) for some \( n > 1. \) Then

\[ \frac{(n+1)\cos 1}{2} < |\cos n| \leq 1, \]

which given that \( \cos 1 \sim 0.54 \) is only possible if \( n = 2. \) But \( \frac{|\cos 2|}{3} \sim 0.14 < 0.27 \sim \frac{\cos 1}{2} \implies \frac{|\cos n|}{n+1} \leq \frac{\cos 1}{2} \forall n \in \mathbb{N}, \) so indeed \( \sup A = \cos \frac{1}{2}. \)

\( \bullet \) \( \inf A = 0. \)

(1 pt) \( |\cos(n)| \geq 0 \) and \( n+1 > 0 \forall n \in \mathbb{N}, \) so clearly 0 is a lower bound of \( A. \)

(1 pt) Suppose \( \inf A > 0, \) then by setting \( n = \lfloor \frac{1}{\inf A} \rfloor \) we get

\[ \frac{|\cos n|}{n+1} < \frac{1}{\inf A} = \inf A, \]

contradiction with \( \inf A \) being a lower bound of \( A. \)

Problem 4.

\[ x_0 = \sup A \iff \forall \varepsilon > 0 \exists x \in A \text{ such that } x_0 - x < \varepsilon. \]

(\( \implies \), 1 pt) If for some \( \varepsilon > 0 \) none such \( x \) exists, then \( x_0 - \varepsilon \) is an upper bound of \( A, \) contradicting minimality of \( x_0 = \sup A. \)

(\( \impliedby \), 1 pt) If \( x_0 \neq \sup A, \) then \( x_0 > \sup A \) and by setting \( \varepsilon = x_0 - \sup A > 0 \) we obtain \( x \in A \) such that \( x_0 - x < \varepsilon \implies x > \sup A, \) contradicting \( \sup A \) being an upper bound of \( A. \)

Date: June 4, 2019.
Problem 5.

- (1 pt) Suppose \( x_0 \notin \mathbb{Z} \), then by applying Problem 4 with \( \varepsilon = 1 \) we get \( m \in A \) such that \( x_0 - 1 < m < x_0 \), the second inequality coming from the fact that \( x_0 \notin \mathbb{Z} \). Another application with \( \varepsilon = x_0 - m \) gives \( n \in A \) such that \( x_0 - 1 < m < n < x_0 \), but then \( 1 = x_0 - (x_0 - 1) > n - m = 1 \), contradiction.
- (1 pt) Suppose \( x_0 \notin A \), then \( x \leq x_0 - 1 \) \( \forall x \in A \) since \( A \) consists of integers, so \( x_0 - 1 \) is an upper bound of \( A \) less than \( x_0 = \sup A \), contradiction.

Problem 6.

(a) \( \sup(A \cup B) = \sup\{\sup A, \sup B\} \).

\( \leq, 1 \text{ pt} \)

\[ x \in A \cup B \implies x \in A \text{ or } x \in B \implies x \leq \sup A \text{ or } x \leq \sup B \implies x \leq \sup\{\sup A, \sup B\} \]

\( \geq, 1 \text{ pt} \)

\[ A \subseteq A \cup B \implies \sup A \leq \sup A \cup B \]
\[ B \subseteq A \cup B \implies \sup B \leq \sup A \cup B \implies \sup\{\sup A, \sup B\} \leq \sup(A \cup B) \]

(b) Answer (1 pt): The statement is false. Example (1 pt): \( A = \{0, 1\}, B = \{0, 2\} \).

Problem 7.

(a) \( |t| \leq a \iff -a \leq t \leq a \).

\( \implies, 1 \text{ pt} \)

\[ |t| \leq a \implies \begin{cases} t \leq a & t \geq 0 \\ -t \leq a & t < 0. \end{cases} \]

The first inequality gives \( 0 \leq t \leq a \), and the second one \( -a \leq t < 0 \), so combining them we get \( -a \leq t \leq a \).

\( \iff, 1 \text{ pt} \)

\[ -a \leq t \leq a \implies \begin{cases} |t| = t \leq a \text{ (second inequality)} & t \geq 0 \\ |t| = -t \leq a \text{ (first inequality)} & t < 0. \end{cases} \]

In any case, \( |t| \leq a \).

(b) By the triangle inequality

\[ \|x\| = \|(x - y) + y\| \leq \|x - y\| + \|y\| \implies \|x\| - \|y\| \leq \|x - y\| \text{ (1 pt)} \]
\[ \|y\| = \|(y - x) + x\| \leq \|y - x\| + \|x\| \implies \|x\| - \|y\| \geq -\|x - y\| \text{ (1 pt)} \]

Therefore by (a) we have \( \|x\| - \|y\| \leq \|x - y\| \).