Interacting Particle Systems and Quantized Lie Algebras

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We will describe the connection between a vector space of states in two different contexts:

- Quantum Mechanical System
- Markov chain
Quantum Mechanical System:

- vector space of states;
- have linear operators on state space, *observables*;
- eigenvalues of observables are observed quantities, e.g.

\[ \hat{H} \mid \psi \rangle = E \mid \psi \rangle; \]

means that state \( \mid \psi \rangle \) has energy \( E \).
Markov chain (continuous-time):

- $S$ is the state space;
- $\vec{X} = \begin{bmatrix} x_i \\ \vdots \end{bmatrix} \leftarrow x_i =$ probability that the system is in state $i \in S$;
- $\sum_i x_i = 1.$
Time evolution can be given by the differential equation

\[
\frac{d\vec{X}}{dt} = \mathcal{L} \vec{X} \implies \vec{X}(t) = e^{\mathcal{L}t} \vec{X}(0)
\]

where \( \mathcal{L} \) is an \( S \times S \) matrix.

To conserve probability:

1. Rows of \( e^{\mathcal{L}t} \) must sum to 1; or equivalently, rows of \( \mathcal{L} \) must sum to 0.
2. Off-diagonal entries of \( \mathcal{L} \) must be non-negative. Thus the diagonal entries of \( \mathcal{L} \) cannot be positive.
The transition matrix $P_t = \exp(\mathcal{L}t)$ completely determines the continuous time Markov process:

$$\mathcal{L} = \lim_{t \downarrow 0} \frac{P_t - \text{Id}}{t}.$$
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It satisfies:

$$P_{t+s} = P_t \circ P_s.$$
We can describe explicitly $P_t$ in terms of $\mathcal{L}$:

- the amount of time that the system stays in state $x$ has exponential distribution of rate $-\mathcal{L}(x, x)$;
We can describe explicitly $P_t$ in terms of $\mathcal{L}$:

- the amount of time that the system stays in state $x$ has exponential distribution of rate $-\mathcal{L}(x, x)$;
- after leaving state $x$, the system jumps to state $y$ with probability $\frac{-\mathcal{L}(x, y)}{\mathcal{L}(x, x)}$. 

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Interacting Particle Systems
How to make a quantum system of a chain of atoms with spins:

- In a specific representation of $\mathfrak{gl}_n$, this Lie Algebra acts as an algebra of linear operators on a vector space. This vector space is the quantum mechanical state space.
- The Universal Enveloping Algebra is an associative algebra which, in a sense, contains information about every representation of $\mathfrak{gl}_n$.

\[
\begin{array}{ccc}
\mathfrak{gl}_n & \xrightarrow{h} & \mathcal{U}(\mathfrak{gl}_n) \\
 & \downarrow \pi & \\
 & f & \downarrow \\
 & & A
\end{array}
\]
The Asymmetry Parameter $q$

- We can *quantize* our algebra to introduce asymmetry by introducing a parameter $q \in (0, 1]$.
- We deform $\mathcal{U}(g\mathfrak{l}_n)$ into $\mathcal{U}_q(g\mathfrak{l}_n)$, a one-parameter family of algebras formed by deformed relations.
- We have $\mathcal{U}_1(g\mathfrak{l}_n) = \mathcal{U}(g\mathfrak{l}_n)$.
- These are called *Quantum groups*.
The algebra $U_q(\mathfrak{gl}_n)$ has generators

$$\{ E_{i,i+1}, E_{i+1,i} : 1 \leq i \leq n-1 \}, \{ q^{E_{ii}} : 1 \leq i \leq n \}.$$ 

**Gould-Zhang-Bracken '91** In $U_q(\mathfrak{gl}_n)$ there is an element $C$ that commutes with all other elements.

$$C = \sum_{i=1}^{n} q^{2i-2n-1} q^{2E_{ii}} + (q - q^{-1})^2 \sum_{1 \leq i < j \leq n} q^{2j-2n-2} q^{E_{ii} + E_{jj}} E_{ij} E_{ji}.$$
- When we choose the $d = 2j$ representation of $U_q(gl_n)$ (i.e. choose an $A$) we view $U_q(gl_n)$ as a set of operators on particles of spin $j$.

- We can put multiple particles with spins together by tensoring together the state spaces (representations).

- The action of algebra elements on second tensor product is defined via an algebra homomorphism called the coproduct, $\Delta : U_q(gl_n) \mapsto U_q(gl_n) \otimes U_q(gl_n)$.

- We take the action of $C$ on this representation to be the Hamiltonian $\tilde{H} = \Delta(C)$. 


The coproduct $\Delta$ tells us how to make a one-site system into a multiple-site system by tensored operators.

For us the effect of $q$ is to introduce an asymmetry in the final generator $\mathcal{L}$.

Particles will be more likely to jump to the right than to the left.

In the limit $q \to 1$, symmetry is regained.
There is a process to turn this QM system into a Markov chain system:

- **Step 1:** Find a definite energy state \( g \), a *ground state*
  \[
  \tilde{H} g = E g.
  \]

  Shift the energy of the Hamiltonian so that it has energy 0:
  \[
  H = \tilde{H} - E \cdot \text{Id}.
  \]

- **Step 2:** Form a diagonal matrix \( G \) out of entries of \( g \):
  \[
  G(i, i) = g(i).
  \]

- **Step 3:** A Markov chain generator is given by
  \[
  \mathcal{L} = G^{-1} H G.
  \]

  Each row of \( \mathcal{L} \) sums to 0 by construction.
Carinci-Giardinà-Redig-Sasamoto ’14 give an explicit form of the generator $\mathcal{L}$ for two-species spin $2j$ ASEP under the action of $\Delta(C^1)$, and prove self-duality of the process. Our research focuses on extending these results to higher powers of $C$ in the multi-species process.
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Asymmetric Simple Exclusion Process

- The ASEP has \( L \) lattice sites, one class (species) of identical particles, and at most one particles at a site.
- Each particle has two clocks, each for left and right jumps, ringing with exponential distributions \( e^{-qt} \) and \( e^{-q^{-1}t} \).
- When the clock rings, the particle jumps to the corresponding adjacent site unless that site is occupied.
What do these Markov Processes look like?

- The ASEP, obtained from \( \mathfrak{gl}_2 \) with the \( d = 1 \ (j = \frac{1}{2}) \) representation (\( \frac{\alpha}{\beta} = q^2 \)): 

![Diagram of ASEP process](image)
The state space that used to be particles with spins at sites can now be interpreted as configuration of particle occupation. Elements of $U_q(gl_2)$ act by transition of configuration.
The new process we obtain from $\mathcal{U}_q(\mathfrak{gl}_3)$ with $n = 3$ introduces second class (“lighter”) particles.

- When first class particles jump to a site occupied by a second class particle, the two particles switch sites.
- If a second class particle attempts to jump to a site occupied by a first class particle, the jump is blocked.

Two-species ASEP corresponding to $\mathfrak{gl}_3$. $\frac{\alpha}{\beta} = q^2$. 
Multi-species spin $2j$ ASEP

- In general, we can have more than two classes of particles at each lattice site of the ASEP.
- The state space for $(n - 1)$-species spin $2j$ ASEP is obtained from a representation of $\mathcal{U}_q(\mathfrak{gl}_n)$.
- The representation corresponds to ways of placing $n - 1$ types of particles with total number $d = 2j$ at a given site.
- An explicit representation of $\mathcal{U}_q(\mathfrak{gl}_n)$ can be given by specifying actions of the generators of $\mathcal{U}_q(\mathfrak{gl}_n)$ on the vector space.
In the physical description of the two-species spin $2j$ ASEP, lighter particles cannot influence heavier particles. Therefore we expect the following:

- The evolution of first-class particles does not depend on second class particles, and gives the usual ASEP.
- Similarly, if both classes of particles are treated identically (i.e. we only look at particle occupation), then our particle system also becomes the ASEP.
It is proved in (Kuan, 2016) that the Markov process obtained from the spin $2j$ representation of $\mathcal{U}_q(gl_n)$ has analogous properties, now with $n - 1$ classes of particles. We say that the process is \textit{lumpable} with respect to a partition of the state space. Thus the abstract Markov process corresponds with the physical description.
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A Markov process with generator $\mathcal{L}$ on a state space $S$ is said to be \textit{self-dual} with respect to a function $D : S \times S \mapsto \mathbb{C}$ if $\mathcal{L}D = D\mathcal{L}^*$. 
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Stochastic Duality

A Markov process with generator $\mathcal{L}$ on a state space $S$ is said to be \textit{self-dual} with respect to a function $D : S \times S \mapsto \mathbb{C}$ if $\mathcal{L}D = DL^*$. 

- There are probabilistic applications in deriving long-time fluctuations (Borodin-Corwin-Sasamoto ’14);
- Most Markov processes do not have self-duality, and coming up with a self-dual process is non-trivial.
The \( n \)-species spin \( 2j \) process has self-duality with respect to the following function:

\[
D(\eta, \xi) = \text{const} \cdot \prod_{x=1}^{L} \frac{[\eta_1^x]!}{q} \prod_{i=1}^{n-1} 1_{\eta_{[1,i]} \geq \xi_{[1,i]}} \prod_{i=1}^{n-1} \left[ \frac{[\eta_{[1,i]} - \xi_{[1,i]}]!}{[\eta_{[1,i]} - \xi_{[1,i]}]!q} \right] \\
\cdot q^{4jx_1^x} q_{x_1}^{\xi_1^x} \left( \sum_{z=x+1}^{L} 2\eta_z^x + \eta_1^x \right)
\]
Generalization to the case $C^m$, $m > 1$
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In such case, if $E$ is an eigenvalue for $\Delta(C)$, then by linearity $E^m$ is an eigenvalue for $\Delta(C^m)$ and therefore its Hamiltonian is

$$H^{(m)} = \Delta(C^m) - E^m. \text{Id} = \Delta(C)^m - E^m. \text{Id}.$$
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This can be rewritten as

$$H^{(m)} = (\Delta(C) - E + E)^m - E^m. \text{Id} = \sum_{i=0}^{m-1} \binom{m}{i} (\Delta(C) - E)^{m-i} E^i. \quad (*)$$
Recall that $\mathcal{L} = G^{-1}HG$ where $G$ is the matrix of the ground state.
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$$G^{-1}H^k G = \underbrace{G^{-1}HG G^{-1}H \ldots G}_{\mathcal{L}} = \mathcal{L}^k.$$
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$$G^{-1}H^{k}G = \underbrace{G^{-1}HG G^{-1}H \ldots G} = \mathcal{L}^{k}.$$ 

Thus we deduce the following

**Proposition**

*The Markov process corresponding to the action of $\Delta(C^m)$ is self-dual with respect to the function $D(\eta, \xi)$, and its generator $\mathcal{L}^{(m)}$ is given by $\mathcal{L}^{(m)} = \sum_{i=0}^{m-1} \binom{m}{i} \mathcal{L}^{m-i} E^i$.***
Knowing now that $\mathcal{L}^{(2)} = \mathcal{L}^2 + 2E\mathcal{L}$, we can ask for what values of the parameter $\alpha$ is $\mathcal{L}^2 - \alpha \mathcal{L}$ a Markov process generator.
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- suffices to ensure non-negativity of the off-diagonal terms;
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- suffices to ensure non-negativity of the off-diagonal terms;
- an upper bound then for \( \alpha \) is

\[
\min_{i \neq j} \left\{ \frac{\mathcal{L}^{(2)}(i,j)}{\mathcal{L}(i,j)} \mid \mathcal{L}(i,j) \neq 0 \right\}.
\]
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Observations

Recall: we have variables:

- $n$ in $\mathfrak{gl}_n$
- $d$, number of particles allowed in each site
- $m$, the power of Casimir element
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- $n$ in $\mathfrak{gl}_n$
- $d$, number of particles allowed in each site
- $m$, the power of Casimir element

We experiment with different values of $n, m, d$, and made the interesting observation that for a chosen triple $(n, m, d)$, the candidates for $\alpha$ all have the same value at 1, the symmetric case.
Observations

For example, for $n = 3$, $d = 2$, $m = 2$, we arrange the bases such that all non-zero $\mathcal{L}(x, y)$ are in blocks,
Observations

For example, for $n = 3$, $d = 2$, $m = 2$, we arrange the bases such that all non-zero $\mathcal{L}(x, y)$ are in blocks, and the $2 \times 2$ block would include:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & \rightarrow & 1 & 0
\end{bmatrix}
\]
Observations

The corresponding minimum in each block is:

<table>
<thead>
<tr>
<th>Block Size</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 × 2 blocks</td>
<td>$q^7 + q^5 + q^{-1} + q^{-3} + 2q^{-5}$</td>
</tr>
<tr>
<td>3 × 3 blocks</td>
<td>$q^7 + q^5 - q^3 + q + 2q^{-1} + 2q^{-5}$</td>
</tr>
<tr>
<td></td>
<td>$2q^5 + q^3 - q + q^{-1} + q^{-3} + 2q^{-5}$</td>
</tr>
<tr>
<td>4 × 4 blocks</td>
<td>$q^7 + 2q^5 - q^3 - q + 2q^{-1} + q^{-3} + 2q^{-5}$</td>
</tr>
</tbody>
</table>

Here, $q^7 + q^5 - q^3 + q + 2q^{-1} + 2q^{-5}$ is the minimal α, occurring at the site

$$
\begin{array}{ccc}
2 & 2 & 1 & 2 \\
1 & 1 & \rightarrow & 1 & 2
\end{array}
$$
Choosing $n$ to be 2, 3, 4, 5, $m$ between 1 and 10, $d$ between 1 and 5 what we found was that:

- $\alpha$, the quotient of two polynomials, is always a Laurent polynomial
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- the value at 1, the symmetric case, or the sum of coefficients ($\Sigma$) of $\alpha$ does not depend on $d$
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- $\alpha$, the quotient of two polynomials, is always a Laurent polynomial
- the value at 1, the symmetric case, or the sum of coefficients ($\Sigma$) of $\alpha$ does not depend on $d$
- $\Sigma = mn^{m-1}$ for all $n, m, d$, tested.
In all cases we consider, the minimal value of $\alpha$ always occurs at the site

$$(d - 1, 0, 0, \ldots, 1), (1, 0, 0, \ldots, d - 1) \rightarrow (d, 0, 0, \ldots, 0), (0, 0, 0, \ldots, d)$$

where 1 is the first class particle, $k$ is the lowest class particle.
Proposition

\[ \Sigma = mn^{m-1} \text{ for all } n, m, d. \]
Proposition

\[ \sum = mn^{m-1} \text{ for all } n, m, d. \]

\[ \mathcal{L}^{(m)} = \mathcal{L}^m + \ldots + \binom{m}{1} E^{m-1} \mathcal{L} \]

and,

\[ \alpha = \frac{\mathcal{L}^{(m)}}{\mathcal{L}}. \]

In addition, the sum of coefficients of \( E \) is \( n \); then, the sum of coefficients of \( \alpha \) is

\[ \sum = \binom{m}{1} n^{m-1} = mn^{m-1}. \]
A sample for our observations:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>$d$</th>
<th>value at 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>2, 3</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2, 3</td>
<td>6</td>
</tr>
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<td>405</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>2, 3</td>
<td>1458</td>
</tr>
</tbody>
</table>
Conjectures

- $\alpha$ is always a Laurent polynomial
Conjectures

- $\alpha$ is always a Laurent polynomial
- The minimal value of $\alpha$ always occurs at the site $(d - 1, 0, 0, \ldots, 1) \otimes (1, 0, 0, \ldots, d - 1) \rightarrow (d, 0, 0, \ldots, 0) \otimes (0, 0, 0, \ldots, d)$
Thank you!