Notes on Geometry and 3-Manifolds

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Appendices by Paul Norbury

Preface

These are a slightly revised version of the course notes that were distributed during the course on Geometry of 3-Manifolds at the Turán Workshop on Low Dimensional Topology in Budapest, August 1998.

The lectures and tutorials did not discuss everything in these notes. The notes were intended to provide also a quick summary of background material as well as additional material for "bedtime reading." There are "exercises" scattered through the text, which are of very mixed difficulty. Some are questions that can be quickly answered. Some will need more thought and/or computation to complete. Paul Norbury also created problems for the tutorials, which are given in the appendices. There are thus many more problems than could be addressed during the course, and the expectation was that students would use them for self study and could ask about them also after the course was over.

For simplicity in this course we will only consider orientable 3-manifolds. This is not a serious restriction since any non-orientable manifold can be double covered by an orientable one.

In Chapter 1 we attempt to give a quick overview of many of the main concepts and ideas in the study of geometric structures on manifolds and orbifolds in dimension 2 and 3. We shall fill in some "classical background" in Chapter 2. In Chapter 3 we then concentrate on hyperbolic manifolds, particularly arithmetic aspects.

Lecture Plan:

1. Geometric Structures.
2. Proof of JSJ decomposition.
3. Commensurability and Scissors congruence.
4. Arithmetic invariants of hyperbolic 3-manifolds.
5. Scissors congruence revisited: the Bloch group.

CHAPTER 1

Geometric Structures on 2- and 3-Manifolds

1. Introduction

There are many ways of defining what is meant by a "geometric structure" on a manifold in the sense that we mean. We give a precise definition below. Intuitively, it is a structure that allows us to do geometry in our space and which is *locally homogeneous*, *complete*, and of *finite volume*.

"Locally homogeneous" means that the space looks locally the same, where-ever you are in it. I.e., if you can just see a limited distance, you cannot tell one place from another. On a macroscopic level we believe that our own universe is close to locally homogeneous, but on a smaller scale there are certainly features in its geometry that distinguish one place from another. Similarly, the surface of the earth, which on a large scale is a homogeneous surface (a 2-sphere), has on a smaller scale many little wrinkles and bumps, that we call valleys and mountains, that make it non-locally-homogeneous.

"Complete" means that you cannot fall off the edge of the space, as european sailors of the middle ages feared might be possible for the surface of the earth! We assume that our universe is complete, partly because anything else is pretty unthinkable!

The "finite volume" condition refers to the appropriate concept of volume. This is n-dimensional volume for an n-dimensional space, i.e., area when $n = 2$. Most cosmologists and physicists want to believe that our universe has finite volume.

Another way of thinking of a geometric structure on a manifold is as a space that is modeled on some "geometry." That is, it should look locally like the given geometry. A *geometry* is a space in which we can do geometry in the usual sense. That is, we should be able to talk about straight lines, angles, and so on. Most fundamental is that we be able to measure length of "reasonable" (e.g., smooth) curves. Then one can define a "straight line" or *geodesic* as a curve which is the shortest path between any two sufficiently nearby points on the curve, and it is then not hard to define angles and volume and so on. We require a few more conditions of our geometry, the most important being that it is *homogeneous*, that is, it should look the same where-ever you are in it. Formally, this means that the *isometry group* of the geometry—the group of length preserving invertible maps of the geometry to itself—should act transitively on the geometry. A consequence of this is that the geometry is complete—you cannot fall off the edge. Another condition we require of a geometry is that it be *simply-connected*—any closed loop should be continuously deformable to a point of the space; we will come back to this later.

We can describe a geometry by giving its underlying space and the element ds of arc-length. The length of a smooth curve is the path integral of ds along it.

A manifold with geometric structure modeled on a geometry $\mathbb{X}$ is isometric to $\mathbb{X}/\Gamma$ for some discrete subgroup Γ of the isometry group of $\mathbb{X}$. This can be proved by an "analytic continuation" argument (key words: *developing map and holonomy*, cf. e.g., [**46**]). We will not go into this, since for our purposes, we can take it as definition:

DEFINITION 1.1. A *geometric manifold* (or manifold with geometric structure) is a manifold of finite volume of the form $\mathbb{X}/\Gamma$, where $\mathbb{X}$ is a geometry and Γ a discrete subgroup of the isometry group $\mathrm{Isom}(\mathbb{X})$.

We will usually restrict to *orientable* manifolds and orbifolds for simplicity. That is, in the above definition, $\Gamma \subset \mathrm{Isom}^+(\mathbb{X})$, the group of orientation preserving isometries.

2. Geometries in dimension 2.

There are three basic geometries in dimension 2:

$$\begin{aligned}
&\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\},\\
&\qquad ds = \sqrt{dx^2+dy^2+dz^2}, \quad \text{Curvature } K = 1;\\
&\mathbb{E}^2 = \mathbb{R}^2 \text{ with metric },\\
&\qquad ds = \sqrt{dx^2+dy^2}, \quad \text{Curvature } K = 0;\\
&\mathbb{H}^2 = \{z = x+iy \in \mathbb{C} \mid y > 0\},\\
&\qquad ds = \frac{1}{y}\sqrt{dx^2+dy^2}, \quad \text{Curvature } K = -1;
\end{aligned}$$

called *spherical*, *euclidean*, and *hyperbolic* geometry respectively. We will explain "curvature" below.

Historically, euclidean geometry is the "original" geometry. Dissatisfaction with the role of the parallel axiom in euclidean geometry led mathematicians of the 19th century to study geometries in which the parallel axiom was replaced by other versions, and hyperbolic geometry (also called "Lobachevski geometry") and elliptic geometry resulted. Elliptic geometry is what you get if you identify antipodally opposite points in spherical geometry, that is, it is the geometry of real 2-dimensional projective space. It is not a geometry in our sense, because of our requirement of simply connected underlying space, while spherical geometry is not a geometry in the sense of Euclid's axioms (with modified parallel axiom) since those axioms require that distinct lines meet in at most one point, and distinct lines in spherical geometry meet in two antipodally opposite points. But from the point of view of providing a "local model" for geometric structures, spherical and elliptic geometry are equally good. Our requirement of simple connectivity assures that we have a unique geometry with given local structure and serves some other technical purposes, but is not really essential.

2.1. Meaning of Curvature. If Δ is a triangle (with geodesic—i.e., "straight-line"—sides) with angles α, β, γ then $\alpha+\beta+\gamma-\pi = K\mathrm{vol}(\Delta)$ where vol means 2-dimensional volume, i.e., area. Because of our homogeneity assumption, our 2-dimensional geometries have constant curvature K, but general riemannian geometry allows manifolds with geometry of varying curvature, that is, K varies from point to point. In this case $K(p)$ can be defined as the limit of $(\alpha+\beta+\gamma-\pi)/\,\mathrm{vol}(\Delta)$

over smaller and smaller triangles Δ containing the point p, and the above formula must be replaced by $\alpha + \beta + \gamma - \pi = \int_\Delta K\, d(\mathrm{vol})$.

You may already have noticed that we have not listed all possible 2-dimensional geometries above. For example, the 2-sphere of radius 2 with its natural metric is certainly a geometry in our sense, but has curvature $K = 1/4$, which is different from that of $\mathbb{S}^2$. But it differs from $\mathbb{S}^2$ just by scaling. In general, we will not wish to distinguish geometries that differ only by scalings of the metric. Up to scaling, the above three geometries are the only ones in dimension 2.

3. Orbifolds

An n-orbifold is a space that looks locally like $\mathbb{R}^n/G$ where G is a finite subgroup of $GL(n, \mathbb{R})$. Note that G varies from point to point, for example, a neighborhood of $[x] \in \mathbb{R}^n/G$ looks like $\mathbb{R}^n/G_x$ where $G_x = \{g \in G \mid gx = x\}$.

We will restrict, for simplicity, to locally orientable 2-orbifolds (i.e., the above G preserves orientation). Then the only possible local structures are $\mathbb{R}^2/C_p$, $p = 1, 2, 3, \ldots$, where C_p is the cyclic group of order p acting by rotations. the local structure is then a "cone point" with "cone angle $2\pi/p$" (Fig. 1).

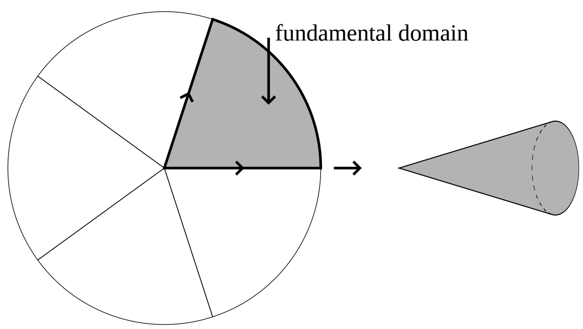

FIGURE 1

4. Some examples of geometric manifolds and orbifolds

EXAMPLE 4.1. Let Λ be a group of translations of $\mathbb{E}^2$ generated by two linearly independent translations e_1 and e_2. Then $T^2 = \mathbb{E}^2/\Lambda =$ gives a Euclidean structure on T^2. We picture a fundamental domain and how the sides of the fundamental domain are identified in Fig. 2.

Note that different choices of $\{e_1, e_2\}$ will give different Euclidean structures on T^2.

EXAMPLE 4.2. A classical example is the subgroup $\mathrm{PSL}(2, \mathbb{Z}) \subset \mathrm{PSL}(2, \mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$, which acts on $\mathbb{H}^2$ with fundamental domain pictured in Fig. 3. The picture also shows how $\mathrm{PSL}(2, \mathbb{Z})$ identifies edges of this fundamental domain. thus the quotient $\mathbb{H}^2/\mathrm{PSL}(2, \mathbb{Z})$ is an *orbifold* with a $2\pi/2$-cone-point at P, a $2\pi/3$-cone-point at Q, and a "cusp" at infinity (Fig. 4). It has finite 2-volume (area) as you can check by integrating the volume form $\frac{1}{y^2}dxdy$ over the fundamental domain: $\mathrm{vol}(\mathbb{H}^2/\mathrm{PSL}(2, \mathbb{Z})) = 2\pi/6$. We give a different proof of this in the next section.

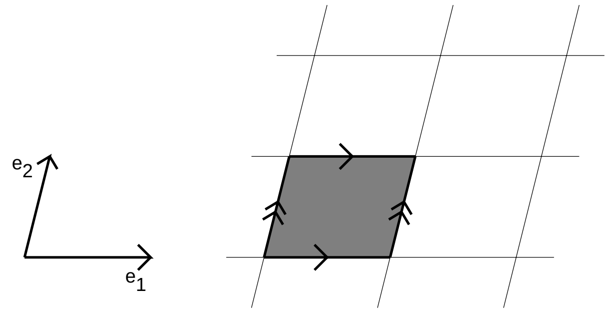

Figure 2

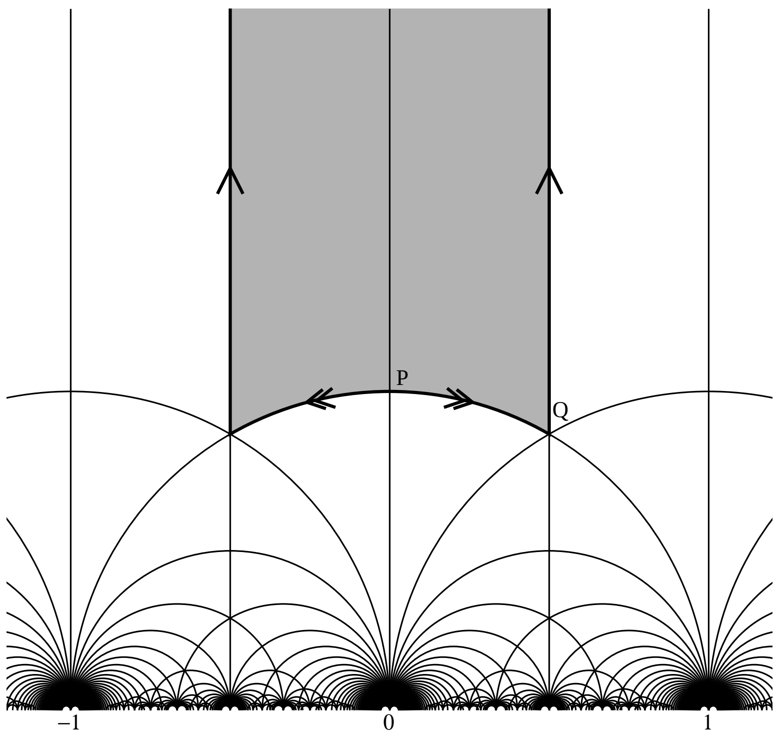

Figure 3

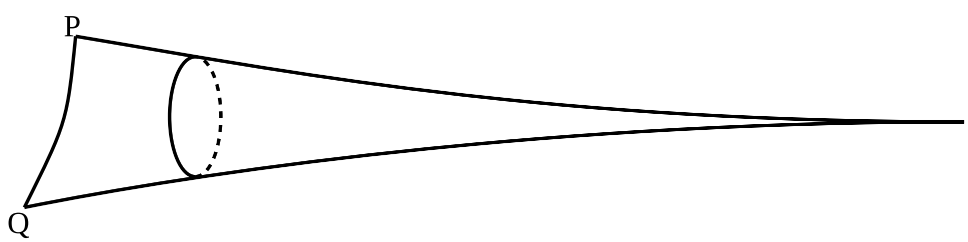

Figure 4

5. Volume and euler characteristic

Denote by F_g the compact orientable surface of genus g (Fig. 5 shows $g = 4$). Suppose it has a geometric structure with (constant) curvature K.

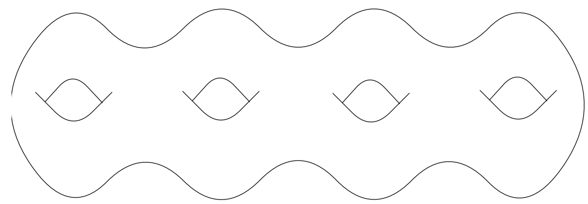

FIGURE 5

Subdivide F_g into many small geodesic triangles

$$\Delta_i\,, \quad i = 1, 2, \ldots, T\,.$$

Let the number of vertices of this triangulation be V, number of edges E. It is known (Euler theorem) that

$$T - E + V = 2 - 2g\,. \tag{1}$$

By 1.1, $K\,\mathrm{vol}(\Delta_i) = \alpha_i + \beta_i + \gamma_i - \pi$, where α_i, β_i, γ_i are the angles of Δ_i, so

$$\begin{aligned} K\,\mathrm{vol}(F_g) &= \sum_{i=1}^{T}(\alpha_i + \beta_i + \gamma_i - \pi) = \sum_{i=1}^{T}(\alpha_i + \beta_i + \gamma_i) - \pi T \\ &= 2\pi V - \pi T \end{aligned} \tag{2}$$

since we are summing *all* angles in the triangulation and around any vertex they sum to 2π. Now $3T = 2E$ (a triangle has 3 edges, each of which is on two triangles), so

$$T = 2E - 2T \tag{3}$$

(2) and (3) imply $K\,\mathrm{vol}(F_g) = 2\pi V - 2\pi E + 2\pi T$. By (1) this gives: $K\,\mathrm{vol}(F_g) = 2\pi(2 - 2g)$.

Now suppose that instead of F_g we have the surface of genus g with s orbifold points with cone angles $2\pi/p_1, \ldots, 2\pi/p_s$. Then in step (2) above we must replace s summands 2π by $2\pi/p_1, \ldots, 2\pi/p_s$, so we get instead:

$$K\,\mathrm{vol}(F) = 2\pi\left(2 - 2g - \sum_{i=1}^{s}\left(1 - \frac{1}{p_i}\right)\right)\,.$$

If we also have h cusps we must correct further by subtracting $2\pi h$. Thus:

THEOREM 5.1. *If the surface F of genus g with h cusps and s orbifold points of cone-angles $2\pi/p_1, \ldots, 2\pi/p_s$ has a geometric structure with constant curvature K then*

$$K\,\mathrm{vol}(F) = 2\pi\chi(F)$$

with

$$\chi(F) = 2 - 2g - h - \sum_{i=1}^{s}\left(1 - \frac{1}{p_i}\right)\,.$$

In particular, the geometry is $\mathbb{S}^2$, $\mathbb{E}^2$, or $\mathbb{H}^2$ according as $\chi(F) > 0$, $\chi(F) = 0$, $\chi(F) < 0$. □

There is a converse (which we will not prove here, but it is not too hard):

THEOREM 5.2 (Geometrization Theorem for Orbifolds). *Let F be the orbifold of Theorem 5.1. Then F has a geometric structure unless F is in the following list:*

1. $g = 0$, $h = 0$, *and either* $s = 1$ *or* $s = 2$ *and* $p_1 \neq p_2$*;*
2. $g = 0$, $h = 1$ *or* 2, $s = 0$*;*
3. $g = 0$, $h = 1$, $s = 1$*;*
4. $g = 0$, $h = 1$, $s = 2$, $p_1 = p_2 = 2$.

That is: $F \cong \mathbb{X}/\Gamma$ *where* $\mathbb{X} = \mathbb{S}^2$, $\mathbb{E}^2$, *or* $\mathbb{H}^2$, *and* $\Gamma \subset \mathrm{Isom}^+(\mathbb{X})$ *is a discrete subgroup acting so* $\mathbb{X}/\Gamma$ *has finite volume. (In each of cases 2–4 the orbifold does have an* infinite-*volume Euclidean structure, which is unique up to similarity. The orbifolds of case 1 are "bad orbifolds," that is, orbifolds which have no covering by a manifold at all—see Sect. 14)*

EXERCISE 1. *Here is a hyperbolic structure on* F_2*. Start with a regular octagon in* $\mathbb{H}^2$ *with angles* $2\pi/8$*. Why does it exist? (Hint: work in the Poincaré disk model instead of upper half plane and expand a regular octahedron centered at the origin from very small to very large.) Identify edges as shown in Fig. 6.*

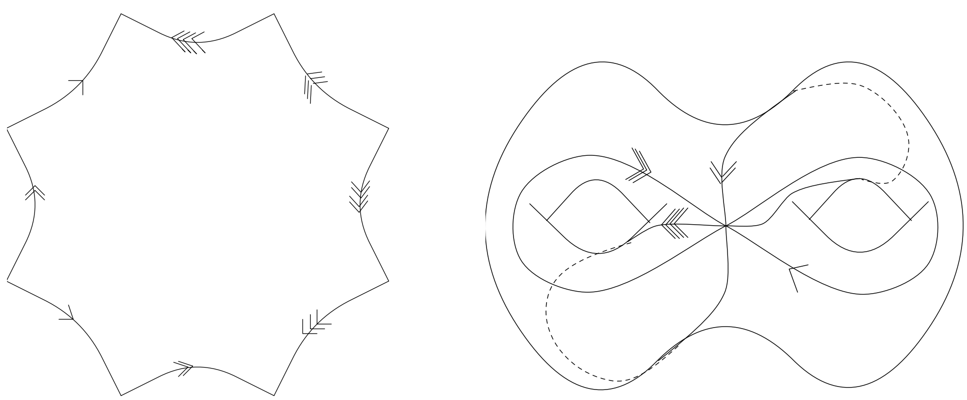

FIGURE 6

Note that we don't really need a regular octahedron for this construction. All we need is:

(i) sum of all eight angles is 2π.
(ii) pairs of edges to be identified have equal length.

It is not hard to see that this gives 8 degrees of freedom for choosing the polygon. This gives 6 degrees of freedom for the geometric structure on F_2 since the choice of the point P in F_2 uses 2 degrees of freedom.

In general,

THEOREM 5.3. *The set of geometric structures on the orbifold of Theorem 5.1 (up to isometry) is a space of dimension* $\max\{0, 6g-6+2s+2h\}$*. It is the so-called* Teichmüller moduli space *of geometric structures on the orbifold. (In the* $\mathbb{E}^2$ *case, we must take structures up to similarity instead of isometry.)*

6. Moduli space of geometric structures on the torus

We consider two Euclidean tori equivalent if they are *similar*, i.e., they are isometric after possibly uniformly scaling the metric on one of them.

See Example 4.1. A Euclidean torus is $T^2 = \mathbb{E}^2/\Lambda$. Choose a basis e_1, e_2 of Λ. By a similarity (scaling) we can assume e_1 has length 1. By a rotation we can make $e_1 = (1, 0)$. If we choose $\{e_1, e_2\}$ as an oriented basis, e_2 is now in the upper half plane. We identify $\mathbb{R}^2 = \mathbb{C}$ and consider e_2 as a complex number τ. This τ is called the *complex parameter* of the torus T^2 *with respect to the chosen basis* e_1, e_2.

If we fix this basis, then τ determines the Euclidean structure on T^2 up to similarities of T^2 *which are isotopic to* id_{T^2}. The space of geometric structures on an orbifold up to equivalences isotopic to the identity is called *Teichmüller space*. Thus Teichmüller space for T^2 is the upper half plane $\mathbb{H}^2$.

If we *change* the basis e_1, e_2 to

$$\begin{aligned} e_2' &= a e_2 + b e_1 \\ e_1' &= c e_2 + d e_1 \end{aligned}$$

with $ad - bc = 1$ (so it is an invertible orientation-preserving change of basis) then τ gets changed to

$$\tau' = a\tau + \frac{b}{c}\tau + d\,.$$

This is the action of $\mathrm{PSL}(2, \mathbb{Z})$ on $\mathbb{H}^2$. Thus:

THEOREM 6.1. *The (Teichmüller) moduli space of geometric structures on* T^2 *is* $\mathbb{H}^2/\operatorname{PSL}(2, \mathbb{Z})$*. (cf. Example 3.2).* □

7. A digression: classifying finite subgroups of SO(3)

Let $G \subset \mathrm{SO}(3) = \mathrm{Isom}^+(\mathbb{S}^2)$ be a finite subgroup. Then $F = \mathbb{S}^2/G$ is an orbifold with spherical geometric structure, so by Theorem 5.1, F satisfies $2 - 2g - h - \sum_{i=1}^{s} \left(1 - \frac{1}{p_i}\right) > 0$. Now $h = 0$ since $\mathbb{S}^2/G$ is compact, and $2 - 2g - \sum \left(1 - \frac{1}{p_i}\right) > 0$ clearly implies $g = 0$, so we get

$$2 - \sum_{i=1}^{s} \left(1 - \frac{1}{p_i}\right) > 0\,.$$

It is easy to enumerate the solutions:

			$\lvert G\rvert$
$s = 0$		$G = \{1\}$	1
$s = 1$	Not allowed ("bad"—see Theorem 5.2		
$s = 2$	$p_1 = p_2 = p$ ($p_1 \neq p_2$ is "bad")	$G = C_p$	p
$s = 3$	$(p_1, p_2, p_3) = (2, 2, p)$	$G = D_{2p}$	$2p$
	$= (2, 3, 3)$	$G = T$	12
	$= (2, 3, 4)$	$G = O$	24
	$= (2, 3, 5)$	$G = I$	60

Here T, O, I are the "tetrahedral group", "octahedral group", "icosahedral group" (group of orientation preserving symmetries of the tetrahedron, octahedron, and icosahedron respectively).

We can compute the size of G from theorem 5.1:

$$|G| = \operatorname{vol}(S^2)/\operatorname{vol}(F) = 4\pi/\left(2\pi\left[2-\sum\left(1-\frac{1}{p_i}\right)\right]\right).$$

Note that $G = \pi_1(F)$. With Theorem 14.3 this gives the standard presentation of the above finite $G \subset \mathrm{SO}(3)$.

8. Dimension 3. The geometrization conjecture

We have seen that in dimension 2 essentially "all" orbifolds have geometric structures. Until 1976, the idea that anything similar might hold for dimension 3 was so absurd as to be unthinkable. In 1976 Thurston formulated his

CONJECTURE 8.1 (Geometrization Conjecture). *Every 3-manifold has a "natural decomposition" into geometric pieces.*

The decomposition in question had already been proved in a topological version by Jaco & Shalen and Johannson, as we describe in Chapter 2. One may assume by earlier results of Knebusch and Milnor (cf. [**27**]) that M^3 is connected-sum-prime, and M^3 then has a natural "JSJ decomposition" (also called "toral decomposition") which cuts M^3 along embedded tori[1] into pieces which are one of

1. Seifert fibered with circle fibers (Seifert fibered means fibered over an orbifold, see Chapter 2);
2. not Seifert fibered with circle fibers but Sefert fibered over a 1-orbifold (i.e., S^1 or I) with T^2-fibers (in the I case there are special fibers—Klein bottles—over the endpoints of I);
3. simple (see below), but not Seifert fibered.

We will discuss the geometrization conjecture for these three cases after defining "simple." We are interested in a manifold M^3 that is the interior of a compact manifold-with-boundary $\overline{M}^3$ such that $\partial\overline{M^3}$ is a (possibly empty) union of tori (briefly "M^3 has toral ends"). This is because M^3 may have resulted via the JSJ decomposition theorem by cutting a compact manifold along tori.

DEFINITION 8.2. M^3 is simple if every essential embedded torus (that is, one that doesn't bound a solid torus in M^3) is isotopic to a boundary component of $\overline{M}$.

The geometrization conjecture is true (and easy) in cases 1 and 2 above. In case 3 it splits into two conjectures:

CONJECTURE 8.3. *A 3-manifold with $|\pi_1(M^3)| < \infty$ is homeomorphic to $\mathbb{S}^3/G$ for some finite subgroup $G \subset \mathrm{Isom}^+(\mathbb{S}^3)$.*

CONJECTURE 8.4. *A simple 3-manifold with $|\pi_1(M^3)| = \infty$ which is not Seifert fibered has a hyperbolic structure.*

Conjecture 8.3 is equivalent to the combination of two old and famous unsolved conjectures:

CONJECTURE 8.5 (Poincaré Conjecture). $\pi_1(M^3) = \{1\} \Rightarrow M^3 \cong S^3$.

CONJECTURE 8.6 (Space-Form-Conjecture). *A free action of a finite group on S^3 is equivalent to a linear action.*

[1]The geometric version of the decomposition uses both tori and Klein bottles, see Sect. 6 of Chapter 2.

9. Geometrization in the "easy" cases

There are 8 geometries relevant to 3-manifolds. They are called $\mathbb{S}^3$, $\mathbb{N}il$, $\mathbb{P}SL$, $\mathbb{S}ol$, $\mathbb{H}^3$, $\mathbb{S}^3 \times \mathbb{E}^1$, $\mathbb{E}^3$, $\mathbb{H}^2 \times \mathbb{E}^1$.

A Seifert fibered manifold with circle fibers is fibered $M^3 \to F^2$ over an *orbifold* F^2 of dimension 2. If M^3 is closed there are two invariants associated with this situation:

- The orbifold euler characteristic $\chi(F^2)$ (see Section 3 of this chapter).
- The euler number of the Seifert fibration $e(M^3 \to F^2)$ (see Section 4 of Chapter 2).

and the relevant geometry is determined by these as:

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e \neq 0$	$\mathbb{S}^3$	$\mathbb{N}il$	$\mathbb{P}SL$
$e = 0$	$\mathbb{S}^2 \times \mathbb{E}^1$	$\mathbb{E}^3$	$\mathbb{H}^2 \times \mathbb{E}^1$

If the Seifert fibered manifold M^3 is not closed then $e(M^3 \to F^2)$ is not well defined (it depends on a choice of "slopes" on the toral ends of M^3) so M^3 can have either a $\mathbb{P}SL$ or a $\mathbb{H}^2 \times \mathbb{E}^1$ structure. There are three exceptional cases: $D^2 \times S^1$, $T^2 \times (0,1)$ and a manifold 2-fold covered by the latter (interval bundle over Klein bottle) each have infinite volume complete $\mathbb{E}^3$ structures but no finite volume geometric structure.

For more details on the above see [**30**] or [**41**].

Case 2 of the "easy cases" is manifolds with $\mathbb{S}ol$-structures. Only closed manifolds occur for this geometry.

This leaves only $\mathbb{H}^3$ to discuss.

10. The hyperbolic geometrization conjecture

There are several models for $\mathbb{H}^3$. One of the often used ones is the *upper half space model*:

$$\{(z,r) \in \mathbb{C} \times \mathbb{R} \mid r > 0\} \quad \text{with metric} \quad ds = \sqrt{dx^2 + dy^2 + dr^2}/r,$$

where $z = x + iy$. The orientation preserving isometry group is $\mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}(2,\mathbb{C})$.

Recall Conjecture 8.4:

CONJECTURE. If M is simple and not Seifert fibered and $|\pi_1(M)| = \infty$ then M has a hyperbolic structure.

This had been proved by Thurston in the following situations

- M^3 is Haken (see Chapter 2 for the definition; this is true if for instance if $|H_1(M)| = \infty$)
- M has a finite symmetry with positive dimensional fixed point set,

but not all details of the second case are published yet (see [**46**], [**29**], [**8**]). This conjecture and the special cases proven so far have had a major effect on 3-manifold theory, including helping toward the solutions of several old conjectures, e.g., the Smith Conjecture ([**29**]) and various conjectures about knots.

11. Examples of a hyperbolic 3-manifold and orbifolds

EXAMPLE 11.1. Take two regular ideal tetrahedra in $\mathbb{H}^3$ (i.e., vertices are "at infinity"). Paste pairs of faces together as directed by Fig. 7, so correspondingly marked edges match up (there is just one way to do this).

FIGURE 7

The result is a hyperbolic structure on the complement in S^3 of the figure-8 knot, pictured in Fig. 8

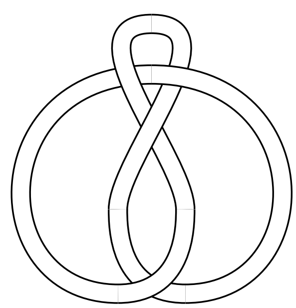

FIGURE 8

EXAMPLE 11.2. Consider the tesselation of $\mathbb{H}^3$ by copies of the regular ideal tetrahedron (I can't draw this, so Fig. 9 shows the analogous picture for dimension $n = 2$). Call this tetrahedral tesselation τ_0.

Let $\Gamma_0 \subset \mathrm{PSL}(2, \mathbb{C})$ be the group of all symmetries (preserving orientation) of this tesselation.

- $Q_0 = \mathbb{H}^3/\Gamma_0$ is the smallest orientable orbifold with a cusp (Meyerhoff [**26**]).
- Q_0 is 24-fold covered by the Figure-8 knot complement.
- $Q_0 = \mathbb{H}^3/PGL(2, \mathcal{O}_3)$, so it is arithmetic (see Chapter 3 for the definition).

EXAMPLE 11.3. τ_1 = tesselation of $\mathbb{H}^3$ by regular ideal octahedra. Γ_1 = group of orientation preserving symmetries of τ_1. $Q_1 = \mathbb{H}^3/\Gamma_1$.

- Q_1 is the second smallest orientable cusped orbifold (Adams [**2**]).
- $Q_1 = \mathrm{PGL}(2, \mathcal{O}_1)$.
- Q_1 has a double cover which is the orbifold whose volume is the smallest "limit volume" (see Sect. 16) (Adams [**1**]).
- Q_1 is 24-fold covered by the Whitehead link complement.

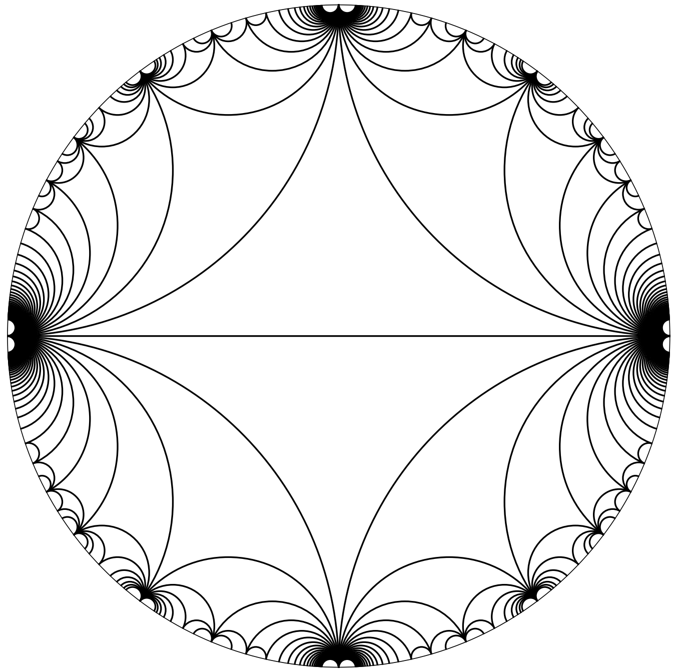

FIGURE 9

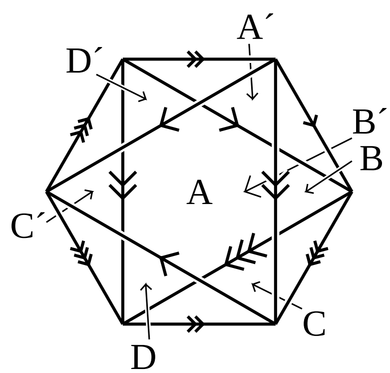

FIGURE 10

EXAMPLE 11.4. Consider a regular ideal octahedron with faces identified as in Fig. 10 (match A to A', B to B', etc.)

Result: Complement of the Whitehead link (Fig. 11).

12. Rigidity in dimension 2: triangle orbifolds

A *triangle orbifold* is a 2-dimensional orbifold with $g = 0$, $s + h = 3$. A triangle orbifold is obtained by gluing together two copies of a triangle of the appropriate geometry (this triangle has ideal vertices if $h > 0$). We speak of the (p_1, p_2, p_3)-triangle if the angles of the triangle are π/p_1, π/p_2, π/p_3 with $p_i \in \{2, 3, 4, \ldots, \infty\}$. For example $\mathbb{H}^2/\operatorname{PSL}(2, \mathbb{Z})$ is the $(2, 3, \infty)$-triangle orbifold.

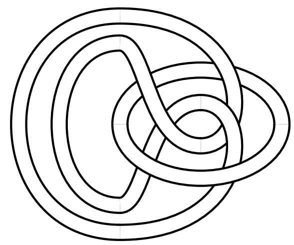

FIGURE 11

A triangle orbifold has unique geometric structure, as does the orbifold $\mathbb{S}^2/C_p$ with $g = 0$, $h = 0$, $s = 2$. In *all* other cases the dimension $6g - 6 + 2s + 2h$ of Teichmüller space is positive, so there are infinitely many geometric structures.

Dimension 3 is in sharp contrast to this:

13. Mostow-Prasad rigidity

THEOREM 13.1 (Mostow-Prasad). *The hyperbolic structure on a hyperbolic 3-orbifold is unique. In fact (stronger formulation): If* Γ_1 *and* Γ_2 *are discrete subgroups of* $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$ *such that*

(i) $\mathbb{H}^3/\Gamma_1$ *is a finite volume orbifold, and*
(ii) $\Gamma_1 \cong \Gamma_2$,

then any isomorphism $\Gamma_1 \to \Gamma_2$ *is induced by an inner automorphism (conjugation) in* $\mathrm{PSL}(2,\mathbb{C})$*. In particular,* $\mathbb{H}^3/\Gamma_1 \cong \mathbb{H}^3/\Gamma_2$ *(isometry).*

This is a remarkable result. The geometrization conjecture says that "almost every" 3-manifold has a hyperbolic structure, and rigidity says this structure is unique! Thus any information we extract from the geometry is actually a *topological* invariant of the manifold. Usually it is hard to find topological descriptions of the resulting invariants, but there is an elegant topological invariant of a 3-manifold called "Gromov norm" (after its inventor) which equals a constant multiple of the volume for a hyperbolic 3-manifold.

In later lectures we will describe arithmetic invariants of the hyperbolic structure. Again, by rigidity, these invariants are topological invariants.

14. Fundamental Group and Covering Spaces

We have referred to fundamental groups and covering spaces of orbifolds above. For our purposes we can define the fundamental group of an orbifold similarly to the standard definition for spaces (our definition depends on the fact that we only consider orientable orbifolds—the codimension 1 sets of orbifold points that occur in non-orientable ones need extra technicalities):

Let F be an orientable orbifold, $* \in F$ a base-point. $\pi_1(F) =$ set of "homotopy classes" of closed paths $\gamma\colon [0,1] \to F$, $\gamma\{0,1\} = \{*\}$ which do *not* pass through any orbifold points. "Homotopy" now means deformation of paths in which the deformation *may* pass through orbifold points, but when it does, the deformation in the "local picture" U/G must be the image of a deformation in U, cf. Fig. 12 (a $\frac{2\pi}{2}$-cone-point).

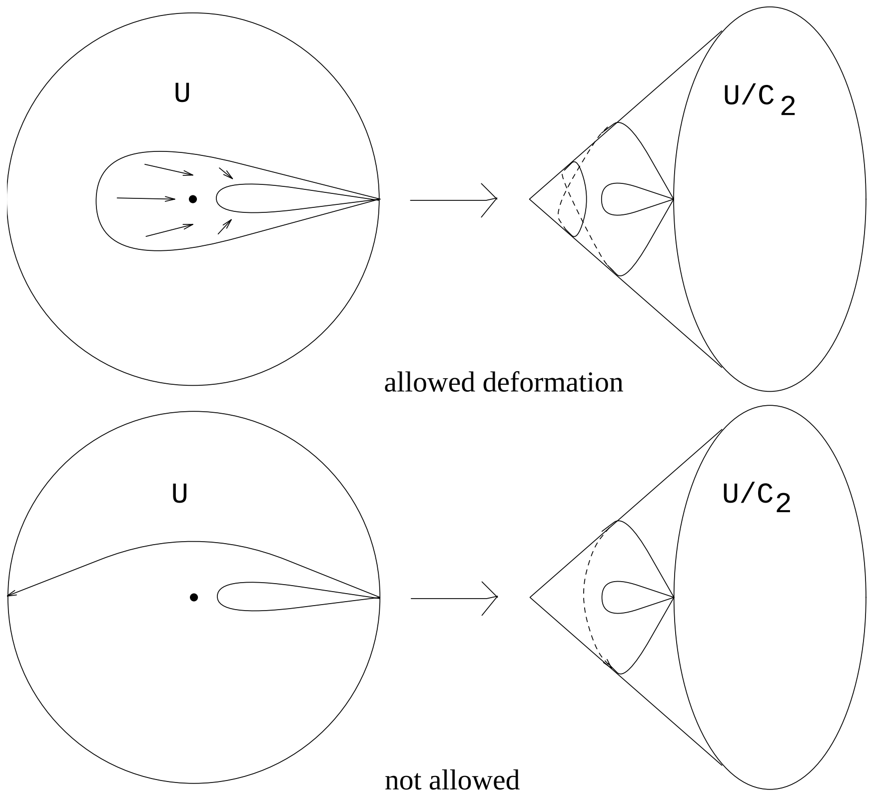

FIGURE 12

14.1. Covering Spaces. Recall that, for us, orbifolds arose as orbit spaces X/Γ of properly discontinuous group actions on manifolds (or other orbifolds).

From this point of view it is easy to define coverings.

DEFINITION 14.1. $f\colon M \to N$ is a covering map of orbifolds if one can express M and N, as orbifolds, as X/Λ and X/Γ, for some X, so that $\Lambda \subset \Gamma$ and $f\colon M \to N$ is the natural map $X/\Lambda \to X/\Gamma$.

THEOREM 14.2. (Connection between $\pi_1(F)$ and coverings) *Let F be a connected oriented orbifold, $\Gamma = \pi_1(F)$. then there exists an orbifold $\widetilde{F}$ with a properly discontinuous action of Γ such that $F = \widetilde{F}/\Gamma$. Any covering $M \to F$ with M connected has the form $\widetilde{F}/\Lambda \to F = \widetilde{F}/\Gamma$ for some group $\Lambda \subset \Gamma$.*

$\widetilde{F} \to F$ is called the *universal covering* of F. If $\widetilde{F}$ is a manifold then F is called a *good* orbifold.

Theorem 14.2 is proved for spaces in many text-books on topology. It is a good exercise to take such a proof and re-write it for orbifolds. It is also a good exercise to take the computation of the fundamental group of a surface, that can be found in many textbooks, and generalize it to show:

THEOREM 14.3. *The orbifold $F(g;h;p_1,\ldots,p_s)$ of genus g with h punctures and s orbifold points of the types $p_1,\ldots,p_s$ has*

$$\pi_1(F(g;h;p_1,\ldots,p_s)) = \langle a_1,\ldots,a_g,b_1,\ldots,b_g,q_1,\ldots q_{s+h} \mid q_1^{p_1}=1,\ldots,q_s^{p_s}=1, q_1\ldots q_{s+h}[a_1,b_1]\ldots[a_g,b_g]=1\rangle$$

.

15. Cusps of hyperbolic 3-orbifolds

Where a hyperbolic 3-manifold is non-compact, the local structures is that of a *cusp*

$$C = \{(x+iy,r) \mid r > \mathbf{K}\}/\Lambda$$

where Λ is a lattice of horizontal translations. Horizontal cross-sections ($r =$ constant) of C are Euclidean tori which are shrinking in size as $r \to \infty$.

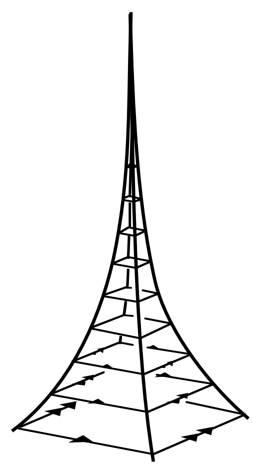

FIGURE 13

The ($r =$ constant) cross-sections of C are called *horosphere sections.*

In a hyperbolic 3-orbifold the picture is the same except that the horosphere sections of a cusp can be any Euclidean orbifold. The Euclidean orbifolds are easily classified.

EXERCISE 2. *Do this analogously to Section 7—solve the equation $\chi = 0$ to show they are*

- *the Euclidean tori $\mathbb{E}^2/\Lambda$;*
- *the triangle orbifolds (cf. Section 12) with $(p_1,p_2,p_3) = (2,4,4)$, $(2,3,6)$, or $(3,3,3)$;*
- *orbifolds with $g = 0$, $h = 0$, $s = 4$, and $(p_1,p_2,p_3,p_4) = (2,2,2,2)$. We call these "pillow orbifolds."*

EXERCISE 3. *Fig. 14 shows a pillow orbifold with an embedding of it as a the surface of a tetrahedron in $\mathbb{E}^3$. Show that if one allows degenerate (flat) tetrahedra, every pillow orbifold has a unique embedding as the surface of a euclidean tetrahedron up to isometry.*

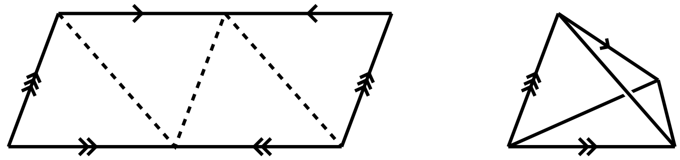

FIGURE 14

The Euclidean triangle orbifolds are rigid (see Section 12) but the Euclidean tori and pillow orbifolds have 2-dimensional spaces of deformations of the Euclidean structure. In fact, every Euclidean torus double covers a unique Euclidean pillow orbifold and vice versa (the torus $\mathbb{E}^2/\Lambda$ double covers $\mathbb{E}^2/\Gamma$, where $\Gamma \subset \mathrm{Isom}^+(\mathbb{E}^2)$ is generated by Λ and the map $\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right) : \mathbb{E}^2 \to \mathbb{E}^2$), so the Teichmüller moduli space of pillow orbifolds is the same as for tori, namely $\mathbb{H}^2/\,\mathrm{PSL}(2,\mathbb{Z})$—see Theorem 6.1

DEFINITION 15.1. If a cusp of a hyperbolic 3-orbifold has horosphere sections which are tori or pillow orbifolds the cusp is called *non-rigid*, if the horosphere sections are triangle orbifolds the cusp is *rigid*.

The non-rigid cusps are important for "hyperbolic Dehn surgery," as we shall describe later. One effect of this is that they affect the volume of the orbifold in a way that we can already describe.

16. Volumes of hyperbolic orbifolds

THEOREM 16.1. (Thurston, Jørgenson) *The set* $\mathrm{Vol} = \{v \in \mathbb{R} \mid v$ is the volume of some hyperbolic 3-orbifold$\}$ *is a well ordered closed subset of* $\mathbb{R}$ *of order type* ω^ω. *To each* $v \in \mathrm{Vol}$ *there are just finitely many orbifolds of this volume.*

Otherwise expressed: the elements of Vol are ordered

$$v_0 < v_1 < v_2 < \cdots < v_\omega < v_{\omega+1} < \cdots < v_{2\omega} < \cdots < v_{3\omega} < \cdots$$
$$\cdots < v_{\omega^2} < \cdots < v_\kappa < \cdots .$$

The general index κ is an infinite ordinal number

$$\kappa = a_n\omega^n + a_{n-1}\omega^{n-1} + \cdots + a_0 \quad \text{and } a_i \in \{0,1,2,\dots\}$$

If κ is divisible by ω then v_κ is the limit of the v_λ, $\lambda < \kappa$; we say v_κ is a *limit volume*. If κ is divisible by ω^2 then v_κ is a limit of limit volumes—it is a 2-*fold limit volume*. If κ is divisible by ω^n then v_κ is an n-fold limit volume (limit of $(n-1)$-fold limit volumes).

THEOREM 16.2. *If M is a hyperbolic orbifold with n non-rigid cusps, then* $\mathrm{vol}(M)$ *is an n-fold limit volume.*

A few of the v_κ are known: Colin Adams has found v_ω, $v_{2\omega}$, and $v_{3\omega}$. He has also found the six *non-compact* orbifolds of least volume (with rigid cusps; the smallest of these has been earlier found by Meyerhoff). See [**1**] and [**2**].

No-one knows v_0 (although there is a guess, namely $0.03905\ldots$, known to be the smallest volume in the arithmetic case, [**6**]). The smallest hyperbolic *manifold* is even harder to determine, but again there is a guess, with volume about $.942707\ldots$, again known to be smallest among arithmetic hyperbolic 3-manifolds [**7**].

The minimal examples found so far are all arithmetic (see Chapter 3 or [**33**] for a definition). This is striking, because arithmetic examples are very sparse overall—Borel showed in [**5**] that there are only finitely many with volume below any given bound.

17. Hyperbolic Dehn surgery

Let M be a hyperbolic orbifold with a cusp C whose horosphere section (cf. Section 15) is a torus. If we cut off the cusp C, we obtain a manifold M_0 with boundary: $\partial M_0 = T^2$.

If we write $T^2 = \mathbb{E}^2/\Lambda$ and choose a basis e_1, e_2 for Λ, then for any coprime pair of integers (p, q), the element $pe_1 + qe_2$ determines a simple curve on T^2.

We can glue a solid torus $D^2 \times S^1$ to M_0 along the boundary T^2 in such a way that the curve $pe_1 + qe_2$ in ∂M_0 matches up with the "meridian curve" $\partial D^2 \times \{1\} \subset \partial(D^2 \times S^1)$. The result of this pasting will be called $M(p, q)$:

$$M(p,q) = M_0 \bigcup_{T^2} D^2 \times S^1 \qquad \text{such that} \quad pe_1 + qe_2 = \partial D^2 \times \{1\}\,.$$

If p and q are *not* coprime we define $M(p, q)$ as an orbifold as follows. Let $d = gcd(p, q)$, $p' = p/d$, $q' = q/d$. The underlying space of $M(p, q)$ is $M(p', q')$ but we put a cone-angle of $2\pi/d$ transverse to the core circle of $D^2 \times S^1$. That is, instead of $D^2 \times S^1$ we use the orbifold $(D^2/C_d) \times S^1$.

Terminology $M(p, q)$ is a (p, q)*-Dehn-filling of* M.

Theorem 17.1. (Thurstons' Dehn Surgery Theorem) *For all but at most finitely many* $(p, q) \in \mathbb{Z}\times\mathbb{Z}-\{(0, 0)\}$*, the orbifold* $M(p, q)$ *has a hyperbolic structure. Moreover*

$$\mathrm{vol}(M(p,q)) < \mathrm{vol}(M) \qquad \textit{and} \qquad \lim_{(p,q)\to\infty} \mathrm{vol}(M(p,q)) = \mathrm{vol}(M)\,.$$

If M has a pillow cusp (see Section 15), then one can still define (p, q)-Dehn filling. Remember that the pillow orbifold is T^2/C_2 where C_2 acts on $T^2 = \mathbb{E}^2/\Lambda$ by the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. This action of C_2 extends to the solid torus $D^2 \times S^1$, so instead of pasting in $D^2 \times S^1$, one pastes in the orbifold $(D^2 \times S^1)/C_2$. This orbifold is a 3-disk with two curves through it with transverse cone angle $2\pi/2$, cf. Fig. 15a. If $gcd(p, q) = d$, there are additional orbifold points; cf. Fig. 15b.

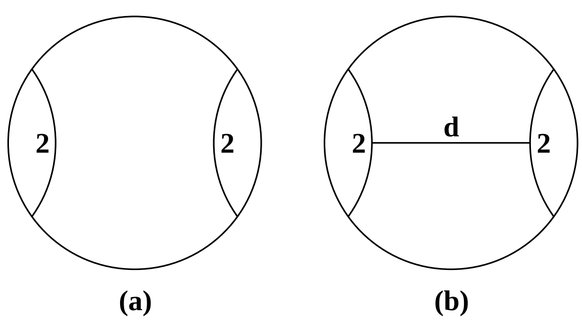

Figure 15

The Dehn surgery theorem explains how a non-rigid cusp of M makes $\mathrm{vol}(M)$ a limit volume (so n non-rigid cusps makes $\mathrm{vol}(M)$ an n-fold limits volume).

Theorem 16.1 follows from the Dehn surgery theorem and the following

Proposition 17.2. *For any bound V there is a finite collection of hyperbolic orbifolds such that every hyperbolic orbifold with volume $\leq V$ results from Dehn surgery on a member of the collection.*

18. Postscript

If one believes the geometrization conjecture, and most topologists do, then understanding hyperbolic 3-manifolds is by far the biggest part of understanding all 3-manifolds. Many general topological questions remain unsolved for hyperbolic manifolds, for instance, does every hyperbolic manifold have a finite covering space with infinite homology, or which even fibers over S^1 (Thurston's question)? This is not even known for the much more restricted class of arithmetic hyperbolic 3-manifolds.

Of course, one of the most basic questions in most fields is the classification question. Can one find a reasonable classification of hyperbolic 3-manifolds? Although we are far from such a thing at present, it may not be an entirely hopeless project in the long run—Proposition 17.2 could be a starting point. An excellent tool for the computational study of hyperbolic 3-manifolds exists in Jeff Weeks' computer program "snappea," available for various computers, and the program "snap" based on it. These have already been used to help classify non-compact hyperbolic manifolds that can be triangulated with few ideal simplices, for example.

19. Non-orientable 2-orbifolds

If a 3-manifold is Seifert fibred over a 2-orbifold the 2-orbifold may be non-orientable even if the 3-manifold is orientable. The 2-orbifold will, however, be locally orientable. We will use $g < 0$ for genus of non-orientable surfaces (so $g = -1, -2, \ldots$ is projective plane, Klein bottle, ...). The euler characteristic of the closed surface of genus $g < 0$ is $2 + g$, so in theorem 5.1 we must replace $2 - 2g$ by $2 + g$ if $g < 0$. Theorem 5.2 remains true with the additional exception:

$2'$. $g = 0$, $h = 1$, $s = 0$.

If one drops the condition of local orientability then one also has the local structures given by the dihedral groups of order $2n, n = 1, 2, \ldots$.

Exercise 4. *Generalise Theorems 5.1 and 5.2 to allow non-locally-orientable 2-orbifolds and then use the method of Section 7 to classify* all *compact spherical and euclidean orbifolds. This gives the classifications of all finite subgroups of $O(3)$ and of the so called "seventeen wallpaper groups" (a misnomer, since most of the seventeen have positive dimensional Teichmüller spaces and are thus infinite families of groups).*

CHAPTER 2

Classical Theory and JSJ Decomposition

1. Dehn's Lemma, Loop and Sphere Theorems

THEOREM 1.1 (Dehn's Lemma). *If M^3 is a 3-manifold and $f\colon D^2 \to M^3$ a map of a disk such that for some neighbourhood N of ∂D^2 the map $f|N$ is an embedding and $f^{-1}(f(N)) = N$. Then $f|\partial D^2$ extends to an embedding $g\colon D^2 \to M^3$.*

Dehn's proof of 1910 [**10**] had a serious gap which was pointed out in 1927 by Kneser. Dehn's Lemma was finally proved by Papakyriakopoulos in 1956, along with two other results, the loop and sphere theorems, which have been core tools ever since. These theorems have been refined by various authors since then. The following version of the loop theorem contains Dehn's lemma. It is due to Stallings [**43**].

THEOREM 1.2 (Loop Theorem). *Let F^2 be a connected submanifold of ∂M^3, N a normal subgroup of $\pi_1(F^2)$ which does not contain* $\ker(\pi_1(F^2) \to \pi_1(M^3))$. *Then there is a proper embedding $g\colon (D^2, \partial D^2) \to (M^3, F^2)$ such that $[g|\partial D^2] \notin N$.*

THEOREM 1.3 (Sphere Theorem). *If N is a $\pi_1(M^3)$-invariant proper subgroup of $\pi_2(M^3)$ then there is an embedding $S^2 \to M^3$ which represents an element of $\pi_2(M^3) - N$.*

(These theorems also hold if M^3 is non-orientable except that in the Sphere Theorem we must allow that the map $S^2 \to M^3$ may be a degree 2 covering map onto an embedded projective plane.)

The proofs of the results in this section and the next can be found in several books on 3-manifolds, for example [**25**].

2. Some Basics

DEFINITION 2.1. An embedded 2-sphere $S^2 \subset M^3$ is *essential* or *incompressible* if it does not bound an embedded ball in M^3. M^3 is *irreducible* if it contains no essential 2-sphere.

Note that if M^3 has an essential 2-sphere that separates M^3 (i.e., M^3 falls into two pieces if you cut along S^2), then there is a resulting expression of M as a *connected sum* $M = M_1 \# M_2$ (to form connected sum of two manifolds, remove the interior of a ball from each and then glue along the resulting boundary components S^2). If M^3 has no essential separating S^2 we say M^3 is *prime*

EXERCISE 5. *M^3 prime $\Leftrightarrow$ Either M^3 is irreducible or $M^3 \simeq S^1 \times S^2$.* Hint[1].

[1] Don't read this footnote unless you want a hint. If M^3 is prime but not irreducible then there is an essential non-separating S^2. Consider a simple path γ that departs this S^2 from one side in M^3 and returns on the other. Let N be a closed regular neighbourhood of $S^2 \cup \gamma$. What is ∂N? What is $M^3 - N$?

Theorem 2.2 (Kneser and Milnor). *Any 3-manifold has a unique connected sum decomposition into prime 3-manifolds (the uniqueness is that the list of summands is unique up to order).*

We next discuss embedded surfaces other than S^2. Although we will mostly consider closed 3-manifolds (i.e., compact without boundary), it is sometimes necessary to consider manifolds with boundary. If M^3 has boundary, then there are two kinds of embeddings of surfaces that are of interest: embedding F^2 into ∂M^3 or embedding F^2 so that $\partial F^2 \subset \partial M^3$ and $(F^2 - \partial F^2) \subset (M^3 - \partial M^3)$. The latter is usually called a "proper embedding." Note that ∂F^2 may be empty. In the following we assume without saying that embeddings of surfaces are of one of these types.

Definition 2.3. If M^3 has boundary, then a properly embedded disk $D^2 \subset M^3$ is *essential* or *incompressible* if it is not "boundary-parallel" (i.e., it cannot be isotoped to lie completely in ∂M^3, or equivalently, there is no ball in M^3 bounded by this disk and part of ∂M^3). M^3 is *boundary irreducible* if it contains no essential disk.

If F^2 is a connected surface $\neq S^2, D^2$, an embedding $F^2 \subset M^3$ is *incompressible* if $\pi_1(F^2) \to \pi_1(M^3)$ is injective. An embedding of a disconnected surface is incompressible if each component is incompressibly embedded.

It is easy to see that if you slit open a 3-manifold M^3 along an incompressible surface, then the resulting pieces of boundary are incompressible in the resulting 3-manifold. The loop theorem then implies:

Proposition 2.4. *If $F^2 \neq S^2, D^2$, then a two-sided embedding $F^2 \subset M^3$ is compressible (i.e., not incompressible) if and only if there is an embedding $D^2 \to M^3$ such that the interior of D^2 embeds in $M^3 - F^2$ and the boundary of D^2 maps to an essential simple closed curve on F^2.*

(For a one-sided embedding $F^2 \subset M^3$ one has a similar conclusion except that one must allow the map of D^2 to fail to be an embedding on its boundary: ∂D^2 may map 2-1 to an essential simple closed curve on F^2. Note that the boundary of a regular neighbourhood of F^2 in M^3 is a two-sided incompressible surface in this case.)

Exercise 6. *Show that if M^3 is irreducible then a torus $T^2 \subset M^3$ is compressible if and only if either*[2]

- *it bounds an embedded solid torus in M^3, or*
- *it lies completely inside a ball of M^3 (and bounds a knot complement in this ball).*

A 3-manifold is called *sufficiently large* if it contains an incompressible surface, and is called *Haken* if it is irreducible, boundary-irreducible, and sufficiently large. Fundamental work of Haken and Waldhausen analysed Haken 3-manifolds by repeatedly cutting along incompressible surfaces until a collection of balls was reached (it is a theorem of Haken that this always happens). A main result is

Theorem 2.5 (Waldhausen). *If M^3 and N^3 are Haken 3-manifolds and we have an isomorphism $\pi_1(N^3) \to \pi_1(M^3)$ that "respects peripheral structure" (that*

[2]In the version of these notes distributed at the course the second case was omitted. I am grateful to Patrick Popescu for pointing out the error.

is, it takes each subgroup represented by a boundary component of N^3 *to a a conjugate of a subgroup represented by a boundary component of* M^3*, and similarly for the inverse homomorphism). Then this isomorphism is induced by a homeomorphism* $N^3 \to M^3$ *which is unique up to isotopy.*

The analogous theorem for surfaces is a classical result of Nielsen.

We mention one more "classical" result that is a key tool in Haken's approach.

DEFINITION 2.6. Two disjoint surfaces $F_1^2, F_2^2 \subset M^3$ are *parallel* if they bound a subset isomorphic to $F_1 \times [0,1]$ between them in M^3.

THEOREM 2.7 (Kneser-Haken finiteness theorem). *For given* M^3 *there exists a bound on the number of disjoint pairwise non-parallel incompressible surfaces that can be embedded in* M^3*.*

3. JSJ Decomposition

We shall give a recent quick proof of the main "JSJ decomposition theorem" which describes a canonical decomposition of any irreducible boundary-irreducible 3-manifold along tori and annuli.

We shall just describe it in the case that the boundary of M^3 is empty or consists of tori. Then only tori occur in the JSJ decomposition (see section 5). An analogous proof works in the general torus-annulus case (see [**34**]), but the general case can also be deduced from the case we prove here.

The theory of such decompositions for Haken manifolds with toral boundaries was first outlined by Waldhausen in [**49**]; see also [**50**] for his later account of the topic. The details were first fully worked out by Jaco and Shalen [**20**] and independently Johannson [**24**].

DEFINITION 3.1. M is *simple* if every incompressible torus in M is boundary-parallel.

If M is simple we have nothing to do, so suppose M is not simple and let $S \subset M$ be an essential (incompressible and not boundary-parallel) torus.

DEFINITION 3.2. S will be called *canonical* if any other properly embedded essential torus T can be isotoped to be disjoint from S.

Take a disjoint collection $\{S_1, \dots, S_s\}$ of canonical tori in M such that

- no two of the S_i are parallel;
- the collection is maximal among disjoint collections of canonical tori with no two parallel.

A maximal system exists because of the Kneser-Haken finiteness theorem. The result of splitting M along such a system will be called a *JSJ decomposition* of M. The maximal system of pairwise non-parallel canonical tori will be called a *JSJ-system.*

The following lemma shows that the JSJ-system $\{S_1, \dots, S_s\}$ is unique up to isotopy.

LEMMA 3.3. *Let* $S_1, \dots, S_k$ *be pairwise disjoint and non-parallel canonical tori in* M*. Then any incompressible torus* T *in* M *can be isotoped to be disjoint from* $S_1 \cup \dots \cup S_k$*. Moreover, if* T *is not parallel to any* S_i *then the final position of* T *in* $M - (S_1 \cup \dots \cup S_k)$ *is determined up to isotopy.*

By assumption we can isotop T off each S_i individually. Writing $T = S_0$, the lemma is thus a special case of the stronger:

LEMMA 3.4. *Suppose* $\{S_0, S_1, \ldots, S_k\}$ *are incompressible surfaces in an irreducible manifold* M *such that each pair can be isotoped to be disjoint. Then they can be isotoped to be pairwise disjoint and the resulting embedded surface* $S_0 \cup \ldots \cup S_k$ *in* M *is determined up to isotopy.*

PROOF. We just sketch the proof. We start with the uniqueness statement. Assume we have $S_1, \ldots, S_k$ disjointly embedded and then have two different embeddings of $S = S_0$ disjoint from $T = S_1 \cup \ldots \cup S_k$. Let $f\colon S \times I \to M$ be a homotopy between these two embeddings and make it transverse to T. The inverse image of T is either empty or a system of closed surfaces in the interior of $S \times I$. Now use Dehn's Lemma and Loop Theorem to make these incompressible and, of course, at the same time modify the homotopy (this procedure is described in Lemma 1.1 of [**48**] for example). We eliminate 2-spheres in the inverse image of T similarly. If we end up with nothing in the inverse image of T we are done. Otherwise each component T' in the inverse image is a parallel copy of S in $S \times I$ whose fundamental group maps injectively into that of some component S_i of T. This implies that S can be homotoped into S_i and its fundamental group $\pi_1(S)$ is conjugate into some $\pi_1(S_i)$. It is a standard fact (see, e.g., [**45**]) in this situation of two incompressible surfaces having comparable fundamental groups that, up to conjugation, either $\pi_1(S) = \pi_1(S_j)$ or S_j is one-sided and $\pi_1(S)$ is the fundamental group of the boundary of a regular neighbourhood of T and thus of index 2 in $\pi_1(S_j)$. We thus see that either S is parallel to S_j and is being isotoped across S_j or it is a neighbourhood boundary of a one-sided S_j and is being isotoped across S_j. The uniqueness statement thus follows.

A similar approach to proves the existence of the isotopy using Waldhausen's classification [**47**] of proper incompressible surfaces in $S \times I$ to show that S_0 can be isotoped off all of $S_1, \ldots, S_k$ if it can be isotoped off each of them. □

The thing that makes decomposition along incompressible annuli and tori special is the fact that they have particularly simple intersection with other incompressible surfaces.

LEMMA 3.5. *If a properly embedded incompressible torus* T *in an irreducible manifold* M *has been isotoped to intersect another properly embedded incompressible surface* F *with as few components in the intersection as possible, then the intersection consists of a family of parallel essential simple closed curves on* T*.*

PROOF. Suppose the intersection is non-empty. If we cut T along the intersection curves then the conclusion to be proved is that T is cut into annuli. Since the euler characteristics of the pieces of T must add to the euler characteristic of T, which is zero, if not all the pieces are annuli then there must be at least one disk. The boundary curve of this disk bounds a disk in F by incompressibility of F, and these two disks bound a ball in M by irreducibility of M. We can isotop over this ball to reduce the number of intersection components, contradicting minimality. □

Let $M_1, \ldots, M_m$ be the result of performing the JSJ-decomposition of M along the JSJ-system $\{S_1 \cup \cdots \cup S_s\}$.

THEOREM 3.6. *Each* M_i *is either simple or Seifert fibered by circles (or maybe both).*

Proof. Suppose N is one of the M_i which is non-simple. We must show it is Seifert fibered by circles.

Since N is non-simple it contains essential tori. Consider a maximal disjoint collection of pairwise non-parallel essential tori $\{T_1, \dots, T_r\}$ in N. Split N along this collection into pieces $N_1, \dots, N_n$. We shall analyze these pieces and show that they are of one of nine basic types, each of which is evidently Seifert fibered. Moreover, we will see that the fibered structures match together along the T_i when we glue the pieces N_i together again to form N.

Consider N_1, say. It has at least one boundary component that is a T_j. Since T_j is not canonical, there exists an essential torus T' in N which essentially intersects T_1. We make the intersection of T' with the union $T = T_1 \cup \dots \cup T_r$ minimal, and then by Lemma 3.5 the intersection consists of parallel essential curves on T'.

Let s be one of the curves of $T_j \cap T'$. Let P be the part of $T' \cap N_1$ that has s in its boundary. P is an annulus. Let s' be the other boundary component of P. It may lie on a T_k with $k \neq j$ or it may lie on T_j again. We first consider the case

Case 1: s' lies on a different T_k.

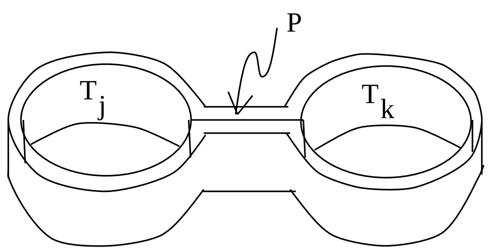

Figure 1

In Fig. 1 we have drawn the boundary of a regular neighbourhood of the union $T_j \cup T_k \cup P$ in N_1. The top and the bottom of the picture should be identified, so that the whole picture is fibered by circles parallel to s and s'. The boundary torus T of the regular neighbourhood is a new torus disjoint from the T_i's, so it must be parallel to a T_i or non-essential. If T is parallel to a T_i then N_1 is isomorphic to $X \times S^1$, where X is a the sphere with three disks removed. Moreover all three boundary tori are T_i's. If T is non-essential, then it is either parallel to a boundary component of N or it is compressible in N. In the former case N_1 is again isomorphic to $X \times S^1$, but with one of the three boundary tori belonging to ∂N. If T is compressible then it must bound a solid torus in N_1 and the fibration by circles extends over this solid torus with a singular fiber in the middle (there must be a singular fiber there, since otherwise the two tori T_j and T_k are parallel).

We draw these three possible types for N_1 in items 1,2, and 3 of Fig. 2, suppressing the circle fibers, but noting by a dot the position of a possible singular fiber. Solid lines represent part of ∂N while dashed lines represent T_i's. We next consider

Case 2. s' also lies on T_j, so both boundary components s and s' of P lie on T_j.

Now P may meet T_j along s and s' from the same side or from opposite sides, so we split Case 2 into the two subcases:

Case 2a. P meets T_j along s and s' both times from the same side;

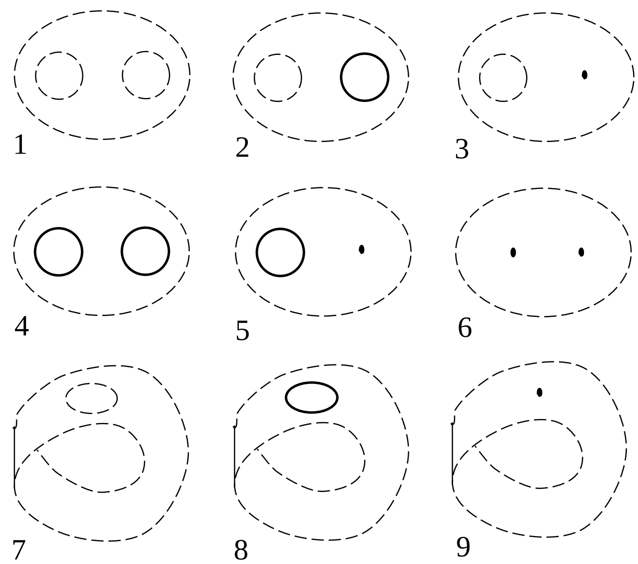

FIGURE 2

Case 2b. P meets T_j along s and s' from opposite sides.

It is not hard to see that after splitting along T_j, Case 2b behaves just like Case 1 and leads to the same possibilities. Thus we just consider Case 2a. This case has two subcases 2a1 and 2a2 according to whether s and s' have the same or opposite orientations as parallel curves of T_j (we orient s and s' parallel to each other in P). We have pictured these two cases in Fig. 3 with the boundary of a regular neighbourhood of $T_j \cup P$ also pictured.

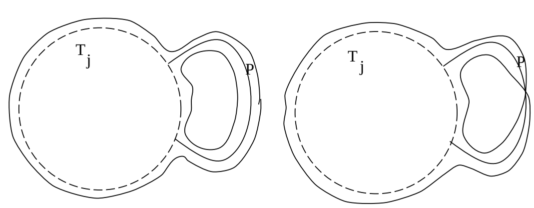

FIGURE 3

In Case 2a1 the regular neighbourhood is isomorphic to $X \times S^1$ and there are two tori in its boundary, each of which may be parallel to a T_i, parallel to a boundary component of N, or bound a solid torus. This leads to items 1 through 6 of Fig. 2.

In Case 2b the regular neighbourhood is a circle bundle over a möbius band with one puncture (the unique such circle bundle with orientable total space). The torus in its boundary may be parallel to a T_i, parallel to a component of ∂N, or bound a solid torus. This leads to cases 7, 8, and 9 of Fig. 2. In all cases but case

9 a dot signifies a singular fiber, but in case 9 it signifies a fiber which may or may not be singular.

We now know that N_1 is of one of the types of Fig. 2 and thus has a Seifert fibration by circles, and therefore similarly for each piece N_i. Moreover, on the boundary component T_j that we are considering, the fibers of N_1 are parallel to the intersection curves of T_j and T' and therefore match up with fibers of the Seifert fibration on the piece on the other side of T_j. We must rule out the possibility that, if we do the same argument using a different boundary component T_k of N_1, it would be a different Seifert fibration which we match across that boundary component. In fact, it is not hard to see that if N_1 is as in Fig. 2 with more than one boundary component, then its Seifert fibration is unique. To see this up to homotopy, which is all we really need, one can use the fact that the fiber generates a normal cyclic subgroup of $\pi_1(N_1)$, and verify by direct calculation that $\pi_1(N_1)$ has a unique such subgroup in the cases in question.

(In fact, the only manifold of a type listed in Fig. 2 that does not have a unique Seifert fibration is case 6 when the two singular fibers are both degree 2 singular fibers and case 9 when the possible singular fiber is in fact not singular. These are in fact two Seifert fibrations of the same manifold T^1Mb, the unit tangent bundle of the Möbius band Mb. This manifold can also be fibered by lifting the fibration of the Möbius band by circles to a fibration of the total space of the tangent bundle of Mb by circles.) □

An alternative characterisation of the JSJ decomposition is as a minimal decomposition of M along incompressible tori into Seifert fibered and simple pieces. In particular, if some torus of the JSJ-system has Seifert fibered pieces on both sides of it, the fibrations do not match up along the torus.

Exercise 7. *Verify the last statement.*

4. Seifert fibered manifolds

In this section we describe all three-manifolds that can be Seifert fibered with circle or torus fibers.

Seifert's original concept of what is now called "Seifert fibration" referred to 3-manifolds fibered with circle fibers, allowing certain types of "singular fibers." For orientable 3-manifolds this gives exactly fibrations over 2-orbifolds, so it is reasonable to use the term "Seifert fibration" more generally to mean "fibration of a manifold over an orbifold" as we did in section 8 of Chapter 1.

That is, a map $M \to N$ is a Seifert fibration if it is locally isomorphic to maps of the form $(U \times F)/G \to U/G$, with U/G an orbifold chart in N (so U is isomorphic to an open subset of $\mathbb{R}^n$ with an action of the finite group G) and F a manifold with G-action such that the diagonal action of G on $U \times F$ is a free action. The freeness of the action is to make M a manifold rather than just an orbifold.

4.1. Seifert circle fibrations.

We start with "classical" Seifert fibrations, that is, fibrations with circle fibers, but with some possibly "singular fibers." We first describe what the local structure of the singular fibers is. This has already been suggested by the proof of JSJ above.

We have a manifold M^3 with a map $\pi\colon M^3 \to F^2$ to a surface such that all fibers of the map are circles. Pick one fiber f_0 and consider a regular neighbourhood N of it. We can choose N to be a solid torus fibered by fibers of π. To have a reference, we will choose a longitudingal curve l and a meridian curve m on the boundary torus $T = \partial N$. The typical fiber f on T is a simple closed curve, so it is homologous to $pl + rm$ for some coprime pair of integers p, r. We can visualise the solid torus N like an onion, made up of toral layers parallel to T (boundaries of thinner and thinner regular neighbourhoods) plus the central curve f_0. Each toral shell is fibered just like the boundary T, so the typical fibers converge on pf_0 as one moves to the center of N.

EXERCISE 8. *Let s be a closed curve on T that is a section to the boundary there. Then (with curves appropriately oriented) one has the homology relation $m = ps + qf$ with $qr \equiv 1$ (mod p).*

The pair $(p.q)$ is called the *Seifert pair* for the fiber f_0. It is important to note that the section s is only well defined up to multiples of f, so by changing the section s we can alter q by multiples of p. If we have chosen things so $0 \leq q < p$ we call the Seifert pair *normalized*.

By changing orientation of f_0 if necessary, we may assume $p \geq 0$. In fact:

EXERCISE 9. *If M^3 contains a fiber with $p = 0$ then M^3 is a connected sum of lens spaces. (A lens space is a 3-manifold obtained by gluing two solid tori along their boundaries; it is classified by a pair of coprime integers (p, q) with $0 \leq q < p$ or $(p, q) = (0, 1)$. One usually writes it as $L(p, q)$. Special cases are $L(0, 1) = S^2 \times S^1$, $L(1, 0) = S^3$. For $p \geq 0$ $L(p, q)$ can also be described as the quotient of $S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$ by the action of $\mathbb{Z}/p$ generated by $(z, w) \mapsto (e^{2\pi i/p}z, e^{2\pi iq/p}w)$.)*

We therefore rule out $p = 0$ and assume from now on that every fiber has $p > 0$. Note that $p = 1$ means that the fiber f_0 is a non-singular fiber, i.e., the whole neighbourhood N of f_0 is fibered as the product $D^2 \times S^1$. If $p > 1$ then f_0 is a singular fiber, but the rest of N consists only of non-singular fibers. In particular, singular fibers are isolated, so there are only finitely many of them in M^3.

Now let $f_0, \dots, f_r$ be a collection of fibers which includes all singular fibers. For each one we choose a fibered neighbourhood N_i and a section s_i on ∂N_i as above, giving a Seifert pair (p_i, q_i) with $p_i \geq 1$ for each fiber. Now on $M_0 := M^3 - \bigcup \int(N_i)$ we have a genuine fibration by circles over a surface with boundary. Such a fibration always has a section, so we can assume that our sections s_i on ∂M_0 have come from a global section on M_0. This section on M_0 is not unique. If we change it, then each s_i is replaced by $s_i + n_i f$ for some integers n_i, and a homological calculation shows that $\sum n_i$ must equal 0. The effect on the Seifert pairs (p_i, q_i) is to replace each by $(p_i, q_i - n_i p_i)$. In summary, we see that changing the choice of global section on M_0 changes the Seifert pairs (p_i, q_i) by changing each q_i, keeping fixed:

- the congruence class q_i (mod p_i)
- $e := \sum \frac{q_i}{p_i}$

The above number e is called the *euler number* of the Seifert fibration. We have not been careful about describing our orientation conventions here. With a standard choice of orientation conventions that is often used in the literature, e is more usually defined as $e := -\sum \frac{q_i}{p_i}$.

Note that we can also change the collection of Seifert pairs by adding or deleting pairs of the form $(1, 0)$, since they correspond to non-singular fibers with choice of local section that extends across this fiber. Up to these changes the topology of the base surface F and the collection of Seifert pairs is a complete invariant of M^3. A convenient normalization is to take f_0 to be a non-singular fiber and $f_1, \dots, f_s$ to be all the singular fibers and normalize so that $0 < q_i < p_i$ for $i \geq 1$. This gives a complete invariant:

$$(g; (1, q_0), (p_1, q_1), \dots, (p_r, q_r)) \quad \text{with } g = \text{genus}(F)$$

which is unique up to permuting the indices $i = 1, \dots, r$. A common convention is to use negative g for the genus of non-orientable surfaces (even though we are assuming M^3 is oriented, the base surface F need not be orientable).

EXERCISE 10. *Explain why the base surface F most naturally has the structure of an orbifold of type $(g; p_1, \dots, p_r)$.*

As discussed in the first chapter, the orbifold euler characteristic of this base orbifold and the euler number e of the Seifert fibration together determine the type of natural geometric structure that can be put on M^3.

There exist a few manifolds M^3 that have more than one Seifert fibration. For example, the lens space $L(p, q)$ has infinitely many, all of them with base surface S^2 and at most two singular fibers (but if one requires the base to be a good orbifold, then $L(p, q)$ has only one Seifert fibration up to isomorphism).

4.2. Seifert fibrations with torus fiber. There are two basic ways a 3-manifold M^3 can fiber with torus fibers. The base must be 1-dimensional so it is either the circle, or the 1-orbifold that one obtains by factoring the circle by the involution $z \mapsto \overline{z}$. The latter is the unit interval $[0, 1]$ considered as an orbifold.

In the case M^3 fibers over the circle, we can obtain it by taking $T^2 \times [0, 1]$ and then pasting $T^2 \times \{0\}$ to $T^2 \times \{1\}$ by an automorphism of the torus. Thinking of the torus as $\mathbb{R}^2/\mathbb{Z}^2$, it is clear that an automorphism is given by a 2×2 integer matrix of determinant 1 (it is orientation preserving since we want M^3 orientable), that is, by an element $A \in \text{SL}(2, \mathbb{Z})$.

EXERCISE 11. *Show the resulting M^3 is Seifert fibered by circles if $|\operatorname{tr}(A)| \leq 2$. Work out the Seifert invariants.*

If $|\operatorname{tr}(A)| > 2$ then the natural geometry for a geometric structure on M is the $\mathbb{S}ol$ geometry.

In case M^3 fibers over the orbifold $[0, 1]$ we can construct it as follows. There is a unique interval bundle over the Klein bottle with oriented total space X (it can be obtained as $(T^2 \times [-1, 1])/\mathbb{Z}/2$ where $\mathbb{Z}/2$ acts diagonally, its action on T^2 being the free action with quotient the Klein bottle). X is fibered by tori that are the boundaries of thinner versions of X obtained by shrinking the interval I, with the Klein bottle zero-section as special fiber. Gluing two copies of X by some identification of their torus boundaries gives M^3. This M^3 has a double cover that fibers over the circle, and it is Seifert fibered by circles if and only if this double cover is Seifert fibered by circles, otherwise it again belongs to the $\mathbb{S}ol$ geometry.

5. Simple Seifert fibered manifolds

We said earlier that if M^3 is irreducible and all its boundary components are tori then only tori occur in the JSJ decomposition. This is essentially because of the following:

EXERCISE 12. *Let M^3 be an orientable manifold, all of whose boundary components are tori, which is simple (no essential tori) and suppose M^3 contains an essential embedded annulus (i.e., incompressible and not boundary parallel). Then*

M^3 is Seifert fibered over D^2 with two singular fibers, or over the annulus or the Möbius band with at most one singular fiber.

For manifolds with boundary, "simple" is often defined by the absence of essential annuli and tori, rather than just tori. The difference between these definitions is just the manifolds of the above exercise. $D^2 \times S^1$ is simple by either definition. The only other simple Seifert fibered manifolds are those that are Seifert fibered over S^2 with at most three singular fibers or over $\mathbb{P}^2$ with at most one singular fiber and which moreover satisfy $e(M^3 \to F) \neq 0$.

6. Geometric versus JSJ decomposition

The JSJ decomposition does not give exactly the desired decomposition of M^3 into pieces with geometric structure. This is because of the fact that the I-bundle over the Klein bottle Kl may occur as a Seifert fibered piece in the decomposition, but, as mentioned in Sect. 9 of Chapter 1, it does not admit a geometric structure.

Thus, whenever the I-bundle over Kl occurs as a piece in the JSJ decomposition, instead of including the boundary of this piece as one of the surfaces to split M^3 along, we include its core Klein bottle. The effect of this is simply to eliminate all such pieces without affecting the topology of any other piece. The modified version of JSJ-decomposition that one gets this way is called *geometric decomposition.*

CHAPTER 3

Arithmetic Invariants

1. Introduction

A *Kleinian* group is a subgroup $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ such that $\mathbb{H}^3/\Gamma$ is a finite-volume 3-orbifold. A *Fuchsian* group is a subgroup $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ such that $\mathbb{H}^2/\Gamma$ is a finite-volume 2-orbifold. (We are dropping some extra adjectives for convenience, since we don't want to consider more general kinds of Kleinian or Fuchsian groups.)

A Kleinian group is torsion free if and only if the quotient $\mathbb{H}^3/\Gamma$ is a manifold. Thus classification of hyperbolic manifolds is the same thing as classification of torsion free Kleinian groups.

A full classification of hyperbolic 3-manifolds up to isometry is probably an impossible task. It therefore makes sense to consider weaker equivalence relations. We will discuss two in some detail, "commensurability" and "scissors congruence."

Two hyperbolic manifolds or orbifolds are *commensurable* if they have finite covers which are isometric to each other. We discuss this in more detail in Sect. 4. They are *scissors congruent* if you can cut the one into pieces that can be reassembled to form the other, see Sect. 10.

We can also study special classes of hyperbolic 3-manifolds and orbifolds. A particularly important one is the class of "arithmetic manifolds and orbifolds" which are quotients of $\mathbb{H}^3$ by arithmetic Kleinian groups. Arithmetic Kleinian groups, although surprisingly common among small volume examples, are exceedingly rare among all Kleinian groups. Nevertheless, their special properties make them attractive for study. We shall also describe how some of their arithmetic aspects can be carried over to arbitrary Kleinian groups.

One application of this is to compute "commensurators" of hyperbolic orbifolds. The commensurator is a generalization of the isometry group that includes all isometries of finite covers too, see Sect. 5.

Most of the time we will talk about Kleinian groups, but the discussion is also applicable to Fuchsian groups.

2. Quaternion algebras and number fields

A *number field* is a finite extension field of $\mathbb{Q}$. Abstractly, any number field is isomorphic to a field of the form $K = \mathbb{Q}(x)/(f(x))$, where $f(x) \in \mathbb{Q}[x]$ is an irreducible poynomial.

If τ is a root of the polynomial $f(x)$ then K embeds in $\mathbb{C}$ by the map induced by $x \mapsto \tau$. Conversely, any embedding $K \to \mathbb{C}$ is obtained this way, so the *complex embeddings* of K are in 1-1 correspondence with the roots of $f(x)$. Such an embedding is called a *real embedding* if its image lies in $\mathbb{R}$, i.e., if the corresponding root of $f(x)$ is real.

If we denote by r_1 the number of real embeddings of K (real roots of $f(x)$) and r_2 the number of conjugate pairs of complex embeddings (conjugate pairs of complex roots of $f(x)$) then we see that $r_1 + 2r_2 = d$, the degree of the extension $K/\mathbb{Q}$ (this is also the degree of the polynomial $f(x)$.

An element of $\mathbb{C}$ is an *algebraic integer* if it satisfies a monic polynomial equation with integral coefficients. The algebraic integers in a number field K form a subring $\mathcal{O}_K$ of K called the *ring of integers* of K. As a $\mathbb{Z}$-module, $\mathcal{O}_K$ has rank $d = [K : \mathbb{Q}]$ and it is a lattice in K considered as a $\mathbb{Q}$-vector space.

A *quaternion algebra* over a field K of characteristic 0 is a central simple algebra of dimension 4 over K. If a, b are nonzero elements of K then there is a unique quaternion algebra A over K generated by elements i, j satisfying $i^2 = a$, $j^2 = b$, $ij = -ji$. The elements $1, i, j, ij$ will form a K-basis of A as a vector space. The pair (a, b) is called the *Hilbert symbol* for A. Every quaternion algebra arises this way and thus has a Hilbert symbol, but the Hilbert symbol is far from unique.

A is a division algebra (no zero divisors) if and only if the equation $aX^2 + bY^2 - Z^2 = 0$ has no nontrivial solutions in K. Otherwise A is isomorphic to the algebra $M(2, K)$ of 2×2 matrices over K. The classification of quaternion algebras over some important fields is:

- $\mathbb{C}$: Only $M(2, \mathbb{C})$.
- $\mathbb{R}$: The usual quaternions with Hilbert symbol $(-1, -1)$, $M(2, \mathbb{R})$.

More generally, over any complete field with absolute value, e.g., a p-adic completion of a number field, there are exactly two quaternion algebras: an analog of the Hamiltonian quaternions and $M(2, K)$. The former is called "ramified" and the latter "unramified."

There are infinitely many quaternion algebras over any number field K but they have an elegant classification. Such a quaternion algebra A is determined by the set of real or p-adic completions K' of K for which the algebra $A \otimes_K K'$ is ramified. Real completions are just real embeddings, while p-adic completions are determined by the primes of K (prime ideals in the ring of integers of K). So the isomorphism type of A is determined by a collection of real embeddings and primes of K. This collection is always finite with an even number of elements, and any finite such collection of even size occurs as the ramification set of some quaternion algebra over K.

Given a quaternion algebra A over K with generators i, j as above, any element $q \in A$ can be written $q = a + bi + cj + dij$ with $a, b, c, d \in K$. Denote $\overline{q} = a - bi - cj - dij$. The *norm* and *trace* of q are

$$N(q) = q\overline{q} \in K$$
$$\operatorname{tr}(q) = q + \overline{q} \in K$$

An element $q \in A$ is *integral* if $N(q)$ and $\operatorname{tr}(q)$ are both in $\mathcal{O}_K$. Unfortunately, sum of integral quaternions need not be integral in general, so integral quaternions may not form a ring. A subring $\mathcal{O} \subset A$ is an *order* if it satisfies

- $\mathcal{O}$ is a ring consisting of integral quaternions $q \in A$
- $\mathcal{O}_K \subset \mathcal{O}$
- $\mathcal{O} \otimes_{\mathcal{O}_K} K = A$.

Every quaternion algebra contains orders, and every order is contained in a maximal order.

3. Arithmetic Kleinian groups

The general definition of an arithmetic group is in terms of the set of $\mathbb{Z}$-points of an algebraic group that is defined over $\mathbb{Q}$. It is a result of Borel that all arithmetic subgroups of $\mathrm{PSL}(2,\mathbb{C})$ can be obtained as follows. Take:

- A number field k having exactly one non-real complex embedding $k \subset \mathbb{C}$ up to complex conjugation;
- A quaternion algebra A over k which is ramified at all real embeddings of k (i.e., for any real embedding $\sigma\colon k \mapsto \mathbb{R}$ one has $A \otimes_{\sigma(k)} \mathbb{R} = H$, the Hamiltonian quaternions);
- An order $\mathcal{O} \subset A$

Under the complex embedding of k the quaternion algebra becomes $M(2,\mathbb{C})$ so the group of units $\mathcal{O}^*$ of $\mathcal{O}$ becomes a subgroup of $\mathrm{GL}(2,\mathbb{C})$. Let Γ be the image of $\mathcal{O}^*$ in the quotient $\mathrm{PSL}(2,\mathbb{C})$ of $\mathrm{GL}(2,\mathbb{C})$. This Γ is an arithmetic subgroup of $\mathrm{PSL}(2,\mathbb{C})$ and any subgroup commensurable to it is also arithmetic. Any arithmetic subgroup arises this way.

In fact, up to commensurability, the arithmetic group Γ is determined by *and* determines k and A alone—changing the order $\mathcal{O}$ gives commensurable groups. There are infinitely many quaternion algebras A for any given k, hence infinitely many commensurability classes of Γ for given k. We call k and A the *defining field* and *defining quaternion algebra* for Γ. The quotient $\mathbb{H}^3/\Gamma$ is usually compact—it is noncompact if and only if $k = \mathbb{Q}(\sqrt{-d})$ and $A = M_2(\mathbb{Q}(\sqrt{-d}))$ (the algebra of 2×2 matrices) for some $d \in \mathbb{N}$.

4. Commensurability

Two Kleinian groups Γ_1 and Γ_2 in $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$ are *strictly commensurable* if $\Gamma_1 \cap \Gamma_2$ has finite index in each of Γ_1 and Γ_2. They are *commensurable* if Γ_1 is strictly commensurable with $\Gamma_2^g \colon = g^{-1}\Gamma_2 g$ for some $g \in \mathrm{Isom}(\mathbb{H}^3)$.

For many purposes one is only interested in a group up to commensurability, since commensurable groups differ in a controlled way.

We also say the orbifolds $\mathbb{H}^3/\Gamma_1$ and $\mathbb{H}^3/\Gamma_2$ are *commensurable* if their fundamental groups Γ_1 and Γ_2 are. Geometrically, this means that an orbifold exists which is a finite covering of both $\mathbb{H}^3/\Gamma_1$ and $\mathbb{H}^3/\Gamma_2$.

Note that it is an easy consequence of Mostow-Prasad rigidity that commensurability of Γ_1 and Γ_2 is a *group-theoretic* property and is independent of their embeddings in $\mathrm{PSL}(2,\mathbb{C})$. Namely, Γ_1 and Γ_2 are commensurable if and only if they have finite index subgroups $\overline{\Gamma}_1 \subset \Gamma_1$ and $\overline{\Gamma}_2 \subset \Gamma_2$ with $\overline{\Gamma}_1 \cong \overline{\Gamma}_2$.

We can speak of commensurability of subgroups in other groups G rather than $\mathrm{PSL}(2,\mathbb{C})$. For instance, $G = \mathrm{PSL}(2,\mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$ interests us also. In this case commensurability is *not* the same as group-theoretic commensurability, since Mostow-Prasad rigidity is false for $\mathrm{PSL}(2,\mathbb{R})$.

5. The commensurator of a group

Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ be a Kleinian group. The *commensurator of* Γ is defined as

$$\mathrm{Comm}(\Gamma) := \{g \in \mathrm{Isom}(\mathbb{H}^3) \mid \Gamma \cap \Gamma^g \text{ has finite index in } \Gamma \text{ and } \Gamma^g\}\,.$$

Its orientation preserving subgroup, $\mathrm{Comm}(\Gamma) \cap \mathrm{PSL}(2,\mathbb{C})$, will be denoted $\mathrm{Comm}^+(\Gamma)$.

$\mathrm{Comm}(\Gamma)$ can be defined purely group-theoretically. Call a triple $(\varphi, \Gamma_1, \Gamma_2)$, consisting of two finite index subgroups Γ_i of Γ and an isomorphism $\varphi\colon \Gamma_1 \to \Gamma_2$, a *germ of an automorphism of* Γ. Call two germs $(\varphi, \Gamma_1, \Gamma_2)$ and $(\varphi', \Gamma_1', \Gamma_2')$ *equivalent* if there exists a finite index subgroup $\Gamma_0 \subset \Gamma_1 \cap \Gamma_1'$ with $\varphi \mid \Gamma_0 = \varphi' \mid \Gamma_0$. Equivalence classes of germs of automorphisms form a group under composition, which we call the *abstract commensurator* of Γ, denoted $\mathrm{Comm}^{\mathrm{abs}}(\Gamma)$.

EXERCISE 13.

i) *Use Mostow-Prasad rigidity to show* $\mathrm{Comm}(\Gamma) = \mathrm{Comm}^{\mathrm{abs}}(\Gamma)$ *for a Kleinian group.*
ii) $\mathrm{Comm}^{\mathrm{abs}}(\mathbb{Z}^n) = \mathrm{GL}(n, \mathbb{Q})$
iii) $\mathrm{Comm}(\Gamma)$ *only depends on the commensurability class of* $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$.
iv) $\mathrm{Comm}^{\mathrm{abs}}(\Gamma)$ *only depends on the group-theoretic (or "abstract") commensurability class of* Γ.

REMARK 5.1. By part (iv) of this exercise, the abstract commensurator $CS = \mathrm{Comm}^{\mathrm{abs}}(\pi_1(F_g))$ of the fundamental group of a surface of genus $g \geq 2$ does not depend on g. One can show that CS embeds naturally in the homeomorphism group $\mathrm{Homeo}(S^1)$ of a circle. It is a very interesting group about which very little is known. It deserves more study.

Similar remarks apply to CF, the abstract commensurator of a nonabelian free group of finite rank, which embeds in the homeomorphism group of a Cantor set.

The commensurator of a Kleinian group Γ contains the normalizer of Γ:

$$N(\Gamma) = \{g \in \mathrm{Isom}(\mathbb{H}^3) \mid \Gamma^g = \Gamma\}\,.$$

Note that

$$N(\Gamma) = \mathrm{Isom}(\mathbb{H}^3/\Gamma)\,.$$

There has been quite some effort expended in the past to compute the isometry groups of such spaces as hyperbolic knot complements (for all knots in the standard tables). It turns out that computing the commensurator is often not much harder. But it tells one much more—not just the isometries of $\mathbb{H}^3/\Gamma$, but also all isometries of finite covers of $\mathbb{H}^3/\Gamma$.

6. The commensurator and arithmeticity

The commensurator gives a geometric content to arithmeticity. Let Γ be a Kleinian group.

THEOREM 6.1.
1. Γ *is non-arithmetic if and only if* Γ *has finite index in* $\mathrm{Comm}^+(\Gamma)$. *In this case* $\mathrm{Comm}^+(\Gamma)$ *is the unique maximal element in the commensurability class of* Γ.
2. Γ *is arithmetic if and only if* $\mathrm{Comm}^+(\Gamma)$ *is dense in* $\mathrm{PSL}(2, \mathbb{C})$. *In this case, there are infinitely many maximal elements in the commensurability class of* Γ.

This theorem is due to Margulis (cf. [**52**], Ch. 6), except for the last sentence, which is due to Borel [**5**]. It holds in much more general situations.

Here is a very geometric way of thinking of the above theorem. A Kleinian (or Fuchsian) group is the symmetry group of some "pattern" in $\mathbb{H}^3$ (respectively $\mathbb{H}^2$). This pattern might just be a tesselation—for instance, a tesselation by fundamental domains—or it might be an Escher-style drawing. If one superposes two copies of this pattern, displaced with respect to each other, one will usually get a pattern which no longer has a Kleinian (or Fuchsian) symmetry group in our sense—the symmetry group has become too small to have finite volume quotient. But in the arithmetic case—and *only* in the arithmetic case—one can always change the displacement very slightly to make the superposed pattern have a symmetry group that is of finite index in the original group. (One might formulate this: any Moiré pattern made from the original pattern is close to a very symmetric Moiré pattern.)

EXERCISE 14. *Consider a lattice Λ of Euclidean translations of $\mathbb{E}^2$, generated by translations e_1 and e_2. Recall that the* complex parameter *of Λ is the ratio e_2/e_1 after identifying the set of translations with $\mathbb{C}$. Show that the commensurator of Λ in $\mathrm{Isom}^+(\mathbb{E}^2)$ is dense in $\mathrm{Isom}^+(\mathbb{E}^2)$ if and only if the complex parameter of Λ is a quadratic irrationality (i.e., an element of some $\mathbb{Q}(\sqrt{-d})$ with $d \in \mathbb{N}$).*

EXERCISE 15. *The tesselations of $\mathbb{H}^3$ by regular ideal tetrahedra, cubes, or octahedra are "arithmetic tesselations"—the symmetry groups are commensurable with $\mathrm{PSL}(2, \mathcal{O}_3)$, $\mathrm{PSL}(2, \mathcal{O}_3)$ and $\mathrm{PSL}(2, \mathcal{O}_1)$ respectively. The tesselation of $\mathbb{H}^3$ by regular ideal dodecahedra is* not *arithmetic (and regular ideal icosahedra do not tesselate $\mathbb{H}^3$, since the dihedral angle is $108°$ which is not an integral submultiple of $360°$).*

Here is a way of "seeing" the arithmeticity in the tetrahedral (and cubical) case. First notice that an ideal cube can be obtained by gluing a regular ideal tetrahedron onto each of the four faces of a fifth regular ideal tetrahedron (each edge of the central tetrahedron becomes a diagonal of a face of the cube). The cube can be constructed from five tetrahedra this way in two different ways. Start with the tetrahedral tesselation and pick one tetrahedron. Using it as the central tetrahedron for one cube, group the tetrahedra, five at a time, to get the cubical tesselation. Dis-assemble the cubes the other way to get a new tetrahedral tesselation. Repeat using a new "central tetrahedron" chosen at random. Watch what happens—you will not easily return to the starting position, but all the tetrahedral tesselations you generate have symmetry group commensurable with the original one.

7. The invariant field and quaternion algebra of a Kleinian group

Let Γ be any Kleinian group (arithmetic or not). Let $\overline{\Gamma} \subset \mathrm{SL}(2, \mathbb{C})$ be the inverse image of Γ under the projection $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{PSL}(2, \mathbb{C})$.

DEFINITION 7.1. The *invariant trace field* of Γ is the field $k(\Gamma)$ generated by all traces of elements of the group $\overline{\Gamma}^{(2)}$ generated by squares of elements of $\overline{\Gamma}$.

THEOREM 7.2 (Alan Reid). (i) $k(\Gamma)$ *is a commensurability invariant of* Γ.
(ii) $k(\Gamma) = \mathbb{Q}(\{(\mathrm{tr}(\gamma))^2 \mid \gamma \in \overline{\Gamma}\})$.

DEFINITION 7.3. The *invariant quaternion algebra* of Γ is the $k(\Gamma)$-subalgebra of $M_2(\mathbb{C})$ (2×2 matrices over $\mathbb{C}$) generated over $k(\Gamma)$ by the elements of $\overline{\Gamma}^{(2)}$. It is denoted $A(\Gamma)$. $A(\Gamma)$ is also a commensurability invariant of Γ. If Γ is arithmetic, then $k(\Gamma)$ and $A(\Gamma)$ equal the defining field and defining quaternion algebra of Γ (and are hence a complete commensurability invariant in this case—see Section 1—but they are not in the non-arithmetic case).

It follows that a necessary condition for arithmeticity is that $k(\Gamma)$ have exactly one non-real complex embedding (it always has at least one). Alan Reid has shown that necessary and sufficient is that, in addition, for each $\gamma \in \overline{\Gamma}$ $\mathrm{trace}(\gamma^2)$ should be an algebraic integer whose absolute value at all real embeddings of k is bounded by

2. Alternatively, all traces should be algebraic integers and $A(\Gamma)$ should be ramified at all real places of k.

Despite their algebraic definition $k(\Gamma)$ and $A(\Gamma)$ turn out to have a lot of geometric and group theoretic content. For example, $A(\Gamma)$ tells one a lot about abelianizations of subgroups of Γ (remark: a fundamental open problem, even in the arithmetic case, is to show that a Kleinian group Γ always has a finite index subgroup with infinite abelianization). Two other examples:

- For an ideally triangulated hyperbolic manifold $\mathbb{H}^3/\Gamma$, $k(\Gamma)$ is generated by the tetrahedral parameters of the ideal tetrahedra (the shape of an ideal tetrahedron is determined by a complex "tetrahedral parameter"—the cross ratio of its four vertices.)
- $k(\Gamma)$ is invariant under cutting $\mathbb{H}^3/\Gamma$ along an embedded surface and gluing by an isometry of this surface.

8. Computing commensurators

As an application of these arithmetic aspects, the commensurators of links in the class of "chain links" are computed in [**33**]. I will not repeat the result here, but I will describe some ingredients. The main plan is:

(1) Check if the example is arithmetic. If so, you are done (for instance, $\mathrm{Comm}^+(\mathrm{PSL}(2, \mathcal{O}_d)) = \mathrm{PGL}(2, \mathbb{Q}(\sqrt{-d})))$.

(2) If not, find the maximal element in the commensurability class.

For example, to step (1), Alan Reid showed that the figure 8 knot complement is the *only arithmetic knot complement*, so for knot complements that step is done (we conjectured that for all other knot complements the commensurator equals the normalizer, but this turns out to fail for Aitcheson and Rubinstein's "dodecahedral knots"([**3**]), but there is reasonable reason to believe they may be the only such examples).

Non-arithmeticity can be detected by a result that arithmetic orbifolds cannot have very short geodesics. Actually, in the compact case, this result is still conjectural, and is essentially equivalent to a classical conjecture of number theory—the Lehmer conjecture. It is, in fact, a quite promising approach to try to prove the Lehmer conjecture. Even without the Lehmer conjecture, geodesic length considerations plus a little extra work can eliminate arithmeticity in large classes of compact examples.

Once one knows an example is non-arithmetic, lower bounds for volume such as those of [**1**], [**2**], [**27**], [**17**] limit the amount of computation needed to actually find the commensurator.

9. Scissors Congruence and Hilbert's 3rd Problem

It was known to Euclid that two plane polygons of the same area are related by scissors congruence: one can always cut one of them up into polygonal pieces that can be re-assembled to give the other. In the 19th century the analogous result was proved with euclidean geometry replaced by 2-dimensional hyperbolic geometry or 2-dimensional spherical geometry.

The 3rd problem in Hilbert's famous 1900 Congress address [**19**] posed the analogous question for 3-dimensional euclidean geometry: are two euclidean polytopes of the same volume "scissors congruent," that is, can one be cut into subpolytopes that can be re-assembled to give the other. Hilbert made clear that he expected a negative answer.

One reason for the nineteenth century interest in this question was the interest in a sound foundation for the concepts of area and volume. By "equal area" Euclid *meant* scissors congruent, and the attempt in Euclid's Book XII to provide the same approach for 3-dimensional euclidean volume involved what was called an "exhaustion argument" — essentially a continuity assumption — that mathematicians of the nineteenth century were uncomfortable with (by Hilbert's time mostly for aesthetic reasons).

The negative answer that Hilbert expected to his problem was provided the same year by Max Dehn [**10**]. His answer applies equally well to scissors congruence of polytopes in hyperbolic or spherical geometry, so let $\mathbb{X}$ denote any one of $\mathbb{E}^3$, $\mathbb{H}^3$, or $\mathbb{S}^3$.

Definition 9.1. Consider the free $\mathbb{Z}$-module generated by the set of congruence classes of 3-dimensional polytopes in $\mathbb{X}$. The *scissors congruence group* $\mathcal{P}(\mathbb{X})$ is the quotient of this module by the relations of scissors congruence. That is, if polytopes $P_1, \dots, P_n$ can be glued along faces to form a polytope P then we set

$$[P] = [P_1] + \cdots + [P_n] \quad \text{in } \mathcal{P}(\mathbb{X}).$$

(A *polytope* is a compact domain in $\mathbb{X}$ that is bounded by finitely many planar polygonal "faces.")

Volume defines a map

$$\mathrm{vol}\colon\, \mathcal{P}(\mathbb{X}) \to \mathbb{R}$$

and Hilbert's problem asks[1] about injectivity of this map for $\mathcal{P}(\mathbb{E}^3)$.

Dehn defined a new invariant of scissors congrence, now called the *Dehn invariant*, which can be formulated as a map $\delta\colon\, \mathcal{P}(\mathbb{X}) \to \mathbb{R}\otimes\mathbb{R}/\pi\mathbb{Q}$, where the tensor product is a tensor product of $\mathbb{Z}$-modules (in this case the same as tensor product as $\mathbb{Q}$-vector spaces).

Definition 9.2. If E is an edge of a polytope P we will denote by $\ell(E)$ and $\theta(E)$ the length of E and dihedral angle (in radians) at E. For a polytope P we define the *Dehn invariant* $\delta(P)$ as

$$\delta(P) := \sum_E \ell(E)\otimes\theta(E) \quad \in \quad \mathbb{R}\otimes(\mathbb{R}/\pi\mathbb{Q}), \quad \text{sum over all edges } E \text{ of } P.$$

We then extend this linearly to a homomorphism on $\mathcal{P}(\mathbb{X})$.

Exercise 16. *Show:*

- *δ is well-defined on $\mathcal{P}(\mathbb{X})$, that is, it is compatible with scissors congruence;*
- *δ and* vol *are independent on $\mathcal{P}(\mathbb{E}^3)$ in the sense that their kernels generate $\mathcal{P}(\mathbb{E}^3)$ (whence $\mathrm{Im}(\delta|\,\mathrm{Ker}(\mathrm{vol})) = \mathrm{Im}(\delta)$ and $\mathrm{Im}(\mathrm{vol}\,|\,\mathrm{Ker}(\delta)) = \mathbb{R}$);*
- *the image of δ on $\mathcal{P}(\mathbb{E}^3)$is uncountable.*

In particular, for euclidean geometry $\mathbb{E}^3$, ker(vol) is not just non-trivial, but even uncountable, giving a strong answer to Hilbert's question. To give an explicit example, the regular simplex and cube of equal volume are not scissors congruent:

[1]Strictly speaking this is not quite the same question since two polytopes P_1 and P_2 represent the same element of $\mathcal{P}(\mathbb{E}^3)$ if and only if they are *stably scissors congruent* rather than scissors congruent, that is, there exists a polytope Q such that P_1+Q (disjoint union) is scissors congruent to P_2+Q. But, in fact, Zylev showed that stable scissors congruence implies scissors congruence, see [**39**] for an exposition).

a regular simplex has non-zero Dehn invariant, and the Dehn invariant of a cube is zero.

Of course, this answer to Hilbert's problem is really just a start. It immediately raises other questions:

- Are volume and Dehn invariant sufficient to classify polytopes up to scissors congruence?
- What about other dimensions?
- What is the full structure of the scissors congruence groups?

The answer to the first question is "yes" for $\mathbb{E}^3$. The question captured the interest of an Austrian librarian called Sydler, who finally proved in 1965 that

$$(\mathrm{vol}, \delta)\colon\ \mathcal{P}(\mathbb{E}^3) \to \mathbb{R} \oplus (\mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Q})$$

is injective. The answer to the third question is also known in this case, in the sense that the image of this map is known. Sydler's argument was difficult, and was simplified somewhat by Jessen [**22, 23**], who also proved an analogous result for $\mathcal{P}(\mathbb{E}^4)$ and the argument has been further simplified in [**13**]. Except for these results and the classical results for dimensions ≤ 2 no complete answers are known. In particular, the analogous questions are not resolved for $\mathcal{P}(\mathbb{H}^3)$ and $\mathcal{P}(\mathbb{S}^3)$.

The Dehn invariant is a more "elementary" invariant than volume since it is defined in terms of 1-dimensional measure. For this reason (and other reasons that will become clear later) we are particularly interested in the kernel of Dehn invariant, so we will abbreviate it

$$\mathcal{D}(\mathbb{X}) := \mathrm{Ker}(\delta\colon\ \mathcal{P}(\mathbb{X}) \to \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Q})$$

In terms of this notation Sydler's theorem that volume and Dehn invariant classify scissors congruence for $\mathbb{E}^3$ can be reformulated:

$$\mathrm{vol}\colon\ \mathcal{D}(\mathbb{E}^3) \to \mathbb{R} \quad \text{is injective.}$$

It is believed that volume and Dehn invariant classify scissors congruence also for hyperbolic and spherical geometry:

CONJECTURE 9.3 (Dehn Invariant Sufficiency). $\mathrm{vol}\colon\ \mathcal{D}(\mathbb{H}^3) \to \mathbb{R}$ is injective and $\mathrm{vol}\colon\ \mathcal{D}(\mathbb{S}^3) \to \mathbb{R}$ is injective.

On the other hand in contrast to the fact that $\mathrm{vol}\colon\ \mathcal{D}(\mathbb{E}^3) \to \mathbb{R}$ is also surjective, which results from the existence of similarity transformations in euclidean space, Dupont [**11**] proved:

THEOREM 9.4. $\mathrm{vol}\colon\ \mathcal{D}(\mathbb{H}^3) \to \mathbb{R}$ *and* $\mathrm{vol}\colon\ \mathcal{D}(\mathbb{S}^3) \to \mathbb{R}$ *have countable image.*

Thus the Dehn invariant sufficiency conjecture would imply:

CONJECTURE 9.5 (Scissors Congruence Rigidity). $\mathcal{D}(\mathbb{H}^3)$ and $\mathcal{D}(\mathbb{S}^3)$ are countable.

The following surprising result collects results of Bökstedt, Brun, Dupont, Parry, Sah and Suslin ([**4**], [**12**], [**40**], [**44**]).

THEOREM 9.6. *$\mathcal{P}(\mathbb{H}^3)$ and $\mathcal{P}(\mathbb{S}^3)$ and their subspaces $\mathcal{D}(\mathbb{H}^3)$ and $\mathcal{D}(\mathbb{S}^3)$ are uniquely divisible groups, so they have the structure of $\mathbb{Q}$-vector spaces. As $\mathbb{Q}$-vector spaces they have infinite rank. The rigidity conjecture thus says $\mathcal{D}(\mathbb{H}^3)$ and $\mathcal{D}(\mathbb{S}^3)$ are $\mathbb{Q}$-vector spaces of countably infinite rank.*

COROLLARY 9.7. *The subgroups* $\mathrm{vol}(\mathcal{D}(\mathbb{H}^3))$ *and* $\mathrm{vol}(\mathcal{D}(\mathbb{S}^3))$ *of $\mathbb{R}$ are $\mathbb{Q}$-vector subspaces of countable dimension.*

9.1. Further comments. Many generalizations of Hilbert's problem have been considered, see, e.g., [**39**] for an overview. There are generalizations of Dehn invariant to all dimensions and the analog of the Dehn invariant sufficiency conjectures have often been made in greater generality, see e.g., [**39**], [**12**], [**18**]. The particular Dehn invariant that we are discussing here is a codimension 2 Dehn invariant.

Conjecture 9.3 appears in various other guises in the literature. For example, as we shall see, the $\mathbb{H}^3$ case is equivalent to a conjecture about rational relations among special values of the dilogarithm function which includes as a very special case a conjecture of Milnor [**28**] about rational linear relations among values of the dilogarithm at roots of unity. Conventional wisdom is that even this very special case is a very difficult conjecture which is unlikely to be resolved in the forseeable future. In fact, Dehn invariant sufficiency would imply the ranks of the vector spaces of volumes in Corollary 9.7 are infinite, but at present these ranks are not even proved to be greater than 1. Even worse: although it is believed that the volumes in question are always irrational, it is not known if a single one of them is!

Work of Bloch, Dupont, Parry, Sah, Wagoner, and Suslin connects the Dehn invariant kernels with algebraic K-theory of $\mathbb{C}$, and the above conjectures are then equivalent to standard conjectures in algebraic K-theory. In particular, the scissors congruence rigidity conjectures for H^3 and S^3 are together equivalent to the rigidity conjecture for $K_3(\mathbb{C})$, which can be formulated that $K_3^{ind}(\mathbb{C})$ (indecomposable part of Quillen's K_3) is countable. This conjecture is probably much easier than the Dehn invariant sufficiency conjecture.

The conjecture about rational relations among special values of the dilogarithm has been broadly generalized to polylogarithms of all degrees by Zagier (sect. 10 of [**51**]). The connections between scissors congruence and algebraic K-theory have been generalised to higher dimensions, in part conjecturally, by Goncharov [**18**].

We will return to some of these issues later. We also refer the reader to the very attractive exposition in [**14**] of these connections in dimension 3.

10. Scissors congruence for hyperbolic 3-manifolds

Suppose that $M = \mathbb{H}^3/\Gamma$ is a hyperbolic three-manifold. For the moment we will also assume M is compact, though we will be able to relax this assumption later. We can subdivide M into small geodesic tetrahedra, and then the sum of these tetrahedra represents a class $\beta_0(M) \in \mathcal{P}(\mathbb{H}^3)$ which is an invariant of M. We call this the *scissors congruence class of M*.

Note that when we apply the Dehn invariant to $\beta_0(M)$ the contributions coming from each edge E of the triangulation sum to $\ell(E) \otimes 2\pi$ which is zero in $\mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Q}$. Thus

PROPOSITION 10.1. *The scissors congruence class $\beta_0(M)$ lies in $\mathcal{D}(\mathbb{H}^3)$.* □

EXERCISE 17. *An analogous result holds for geometric manifolds belonging to the geometry $\mathbb{E}^3$ or $\mathbb{S}^3$. Assuming what you have been told in the previous section, show that if $\mathbb{X}$ is either of these geometries, the volume of an $\mathbb{X}$-manifold determines the element $\beta_0(M^3) \in \mathcal{D}(\mathbb{X})$.*

Of course the Dehn invariant sufficiency conjecture would say that the result of this exercise is also true in the hyperbolic case. In particular, two hyperbolic 3-manifolds would be "scissors congruence commensurable" (a union of several copies of one of them scissors congruent to a union of several copies of the other) if and only if their volumes were rationally commensurable. Although we will see that it is easy to compute whether manifolds are scissors congruence equal or scissors congruence commensurable, to show their volumes are rationally incommensurable has not been done in a single case!

The Dehn invariant sufficiency conjecture would also imply that scissors congruence cannot see orientation of a manifold, since reversing orientation does not change volume. This is true, as Gerling pointed out in a letter to Gauss on 15 April 1844: any polytope is scissors congruent to its mirror image[2]. This raises the question:

QUESTION 10.2. Is there some way to repair the orientation insensitivity of scissors congruence?

The answer to this question is "yes" and lies in the so called "Bloch group" (invented by Bloch, Wigner and Suslin). To explain this we start with a result of Dupont and Sah [**12**] about ideal polytopes — hyperbolic polytopes whose vertices are at infinity (such polytopes exist in hyperbolic geometry, and still have finite volume).

PROPOSITION 10.3. *Ideal hyperbolic tetrahedra represent elements in* $\mathcal{P}(\mathbb{H}^3)$ *and, moreover,* $\mathcal{P}(\mathbb{H}^3)$ *is generated by ideal tetrahedra.*

To help understand this proposition observe that if $ABCD$ is a non-ideal tetrahedron and E is the ideal point at which the extension of edge AD meets infinity then $ABCD$ can be represented as the difference of the two tetrahedra $ABCE$ and $DBCE$, each of which have one ideal vertex. One can iterate this argument for the second and third vertices, but showing that one can rewrite $ABCD$ in terms of simplices with all four vertices at infinity is harder.

We must also define the Dehn invariant of an ideal polytope. To do so we first cut off each ideal vertex by a horoball based at that vertex. We then have a polytope with some horospherical faces but with all edges finite. We now compute the Dehn invariant using the geodesic edges of this truncated polytope (that is, only the edges that come from the original polytope and not those that bound horospherical faces). This is well defined in that it does not depend on the sizes of the horoballs we used to truncate our polytope. (To see this, note that dihedral angles of the edges incident on an ideal vertex sum to a multiple of π, since they are the angles of the horospherical face created by truncation, which is an euclidean polygon. Changing the size of the horoball used to truncate these edges thus changes the Dehn invariant by a multiple of something of the form $l \otimes \pi$, which is zero in $\mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Q}$.)

One consequence of the above proposition is a gain in convenience: a non-ideal tetrahedron needs six real parameters satisfying complicated inequalities to characterise it up to congruence while an ideal tetrahedron can be neatly characterised by a single complex parameter in the upper half plane.

We shall denote the standard compactification of $\mathbb{H}^3$ by $\overline{\mathbb{H}}^3 = \mathbb{H}^3 \cup \mathbb{CP}^1$. An ideal simplex Δ with vertices $z_1, z_2, z_3, z_4 \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ is determined up to

[2]Gauss, Werke, Vol. 10, p. 242; the argument for a tetrahedron is to barycentrically subdivide by dropping perpendiculars from the circumcenter to each of the faces; the resulting 24 tetrahedra occur in 12 mirror image pairs.

congruence by the cross ratio

$$z = [z_1 : z_2 : z_3 : z_4] = \frac{(z_3 - z_2)(z_4 - z_1)}{(z_3 - z_1)(z_4 - z_2)}.$$

Permuting the vertices by an even (i.e., orientation preserving) permutation replaces z by one of

$$z, \quad z' = \frac{1}{1-z}, \quad \text{or} \quad z'' = 1 - \frac{1}{z}.$$

The parameter z lies in the upper half plane of $\mathbb{C}$ if the orientation induced by the given ordering of the vertices agrees with the orientation of $\mathbb{H}^3$.

There is another way of describing the cross-ratio parameter $z = [z_1 : z_2 : z_3 : z_4]$ of a simplex. The group of orientation preserving isometries of $\mathbb{H}^3$ fixing the points z_1 and z_2 is isomorphic to the multiplicative group $\mathbb{C}^*$ of nonzero complex numbers. The element of this $\mathbb{C}^*$ that takes z_4 to z_3 is z. Thus the cross-ratio parameter z is associated with the edge z_1z_2 of the simplex. The parameter associated in this way with the other two edges z_1z_4 and z_1z_3 out of z_1 are z' and z'' respectively, while the edges z_3z_4, z_2z_3, and z_2z_4 have the same parameters z, z', and z'' as their opposite edges. See fig. 1. This description makes clear that the dihedral angles at the edges of the simplex are $\arg(z)$, $\arg(z')$, $\arg(z'')$ respectively, with opposite edges having the same angle.

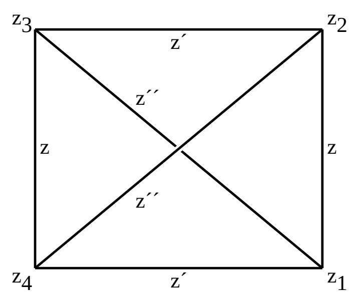

FIGURE 1

Now suppose we have five points $z_0, z_1, z_2, z_3, z_4 \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Any four-tuple of these five points spans an ideal simplex, and the convex hull of these five points decomposes in two ways into such simplices, once into two of them and once into three of them. We thus get a scissors congruence relation equating the two simplices with the three simplices. It is often called the "five term relation." To express it in terms of the cross-ratio parameters we need an orientation convention.

We allow simplices whose vertex ordering does not agree with the orientation of $\mathbb{H}^3$ (so the cross-ratio parameter is in the lower complex half-plane) but make the convention that this represents the negative element in scissors congruence. An odd permutation of the vertices of a simplex replaces the cross-ratio parameter z by

$$\frac{1}{z}, \quad \frac{z}{z-1}, \quad \text{or} \quad 1 - z,$$

so if we denote by $[z]$ the element in $\mathcal{P}(\mathbb{H}^3)$ represented by an ideal simplex with parameter z, then our orientation rules say:

$$(4) \qquad [z] = [1 - \frac{1}{z}] = [\frac{1}{1-z}] = -[\frac{1}{z}] = -[\frac{z-1}{z}] = -[1-z].$$

Exercise 18. *These orientation rules make the five-term scissors congruence relation described above particularly easy to state:*

$$\sum_{i=0}^{4}(-1)^i[z_0 : \cdots : \hat{z}_i : \cdots : z_4] = 0.$$

The cross ratio parameters occuring in this formula can be expressed in terms of the first two as

$$[z_1 : z_2 : z_3 : z_4] =: x \qquad [z_0 : z_2 : z_3 : z_4] =: y$$

$$[z_0 : z_1 : z_3 : z_4] = \frac{y}{x} \quad [z_0 : z_1 : z_2 : z_4] = \frac{1-x^{-1}}{1-y^{-1}} \quad [z_0 : z_1 : z_2 : z_3] = \frac{1-x}{1-y}$$

so the five-term relation can also be written:

$$[x] - [y] + [\frac{y}{x}] - [\frac{1-x^{-1}}{1-y^{-1}}] + [\frac{1-x}{1-y}] = 0. \tag{5}$$

We lose nothing if we also allow degenerate ideal simplices whose vertices lie in one plane so the parameter z is real (we always require that the vertices are distinct, so the parameter is in $\mathbb{R} - \{0, 1\}$), since the five-term relation can be used to express such a "flat" simplex in terms of non-flat ones, and one readily checks no additional relations result. Thus we may take the parameter z of an ideal simplex to lie in $\mathbb{C} - \{0, 1\}$ and every such z corresponds to an ideal simplex.

One can show that relations (4) follow from the five-term relation (5), so we consider the quotient

$$\mathcal{P}(\mathbb{C}) := \mathbb{Z}\langle\mathbb{C} - \{0, 1\}\rangle/(\text{five-term relations (5)})$$

of the free $\mathbb{Z}$-module on $\mathbb{C} - \{0, 1\}$. Proposition 10.3 can be restated that there is a natural surjection $\mathcal{P}(\mathbb{C}) \to \mathcal{P}(\mathbb{H}^3)$. In fact Dupont and Sah (loc. cit.) prove:

Theorem 10.4. *The scissors congruence group $\mathcal{P}(\mathbb{H}^3)$ is the quotient of $\mathcal{P}(\mathbb{C})$ by the relations $[z] = -[\overline{z}]$ which identify each ideal simplex with its mirror image.*

Thus $\mathcal{P}(\mathbb{C})$ is a candidate for the orientation sensitive scissors congruence group that we were seeking.

The analog of the Dehn invariant has a particularly elegant expression in these terms. $\mathcal{P}(\mathbb{C})$ is known to be a $\mathbb{Q}$-vector space and is therefore the sum of its ±1 eigenspaces under the map induced by complex conjugation. The above theorem expresses $\mathcal{P}(\mathbb{H}^3)$ as the "imaginary part" $\mathcal{P}(\mathbb{C})^-$ (negative eigenspace for complex conjugation) of $\mathcal{P}(\mathbb{C})$.

Proposition/Definition 10.5. *The Dehn invariant $\delta\colon \mathcal{P}(\mathbb{H}^3) \to \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Q}$ is twice the "imaginary part" of the map*

$$\delta_{\mathbb{C}}\colon \mathcal{P}(\mathbb{C}) \to \mathbb{C}^* \wedge \mathbb{C}^*, \quad [z] \mapsto (1-z) \wedge z$$

so we shall call this map the "complex Dehn invariant." We denote the kernel of complex Dehn invariant

$$\mathcal{B}(\mathbb{C}) := \operatorname{Ker}(\delta_{\mathbb{C}}),$$

and call it the "Bloch group of $\mathbb{C}$."

Exercise 19. *Compute the real and imaginary parts of the above $\delta_{\mathbb{C}}$ to confirm the above proposition.*

A hyperbolic 3-manifold M has an "orientation sensitive scissors congruence class" which lies in this Bloch group. The way one defines this is to subdivide M^3 into ideal tetrahedra. This cannot be done in the usual sense if M^3 is compact, but it can be done in a weak sense which suffices for our purposes. We just require a "degree one ideal triangulation" of M^3, by which we mean that we should have a collection of ideal simplices glued together by pairing faces plus a map of the resulting "ideal complex" to M, locally isometric on each simplex, and of degree one almost everywhere in M. It is not hard to show that such degree one ideal triangulations exist (e.g., start from an ordinary "finite" triangulation by geodesic simplices and then move vertices continuously off to infinity, taking care not to let simplices degenerate).

THEOREM 10.6 ([**37**]). *Given a degree one ideal triangulation of a hyperbolic manifold M^3 using simplices with cross-ratio parameters $z_1, \dots, z_n$ say, the element $\beta(M) := [z_1] + \cdots + [z_n] \in \mathcal{P}(\mathbb{C})$ is independent of the choice of triangulation and lies in $\mathcal{B}(\mathbb{C})$.*

Note that the definition of $\mathcal{B}(\mathbb{C})$ is purely algebraic, and so the same definition could be made with $\mathbb{C}$ replaced by any subfield k.

Here are some facts about these groups which collate results of several people (see [**31**] for detailed references).

THEOREM 10.7. *$\mathcal{B}(\mathbb{C})$ is a uniquely divisible group, so it has the natural structure of a $\mathbb{Q}$-vector space. If $k \subset \mathbb{C}$ is a number field then $\mathcal{B}(k) \to \mathcal{B}(\mathbb{C})$ is injective modulo torsion and has image isomorphic to $\mathbb{Z}^{r_2}$, where r_2 is the number of complex embeddings of k. In particular, $\mathcal{B}(k)_{\mathbb{Q}} := \mathcal{B}(k) \otimes \mathbb{Q}$ is an r_2-dimensional $\mathbb{Q}$-vector-subspace of $\mathcal{B}(\mathbb{C})$.*

THEOREM 10.8 ([**37**]). *If M is a hyperbolic 3-manifold and k is the invariant trace field of M, then $\beta(M) \in \mathcal{B}(k)_{\mathbb{Q}} \subset \mathcal{B}(\mathbb{C})$.*

11. Computing $\beta(M)$

Let D_2 be the so called "Bloch-Wigner dilogarithm function" given by:

$$D_2(z) = \operatorname{Im} \ln_2(z) + \log|z| \arg(1-z), \quad z \in \mathbb{C} - \{0,1\},$$

where $\ln_2(z)$ is the classical dilogarithm function. This function has geometric meaning: $D_2(z)$ is the signed volume of the ideal simplex with cross-ratio parameter z. It follows the map D_2 is compatible with the five-term relation and therefore induces a map which we call vol from $\mathcal{P}(\mathbb{C})$ to $\mathbb{R}$

If k is a number field with r_2 complex embeddings up to conjugation, let us list these embeddings as $\sigma_1, \dots, \sigma_{r_2}$ and denote

$$\mathrm{vol}_i := \mathrm{vol} \circ \sigma_i \colon \mathcal{P}(k) \to \mathbb{R}$$

DEFINITION 11.1. The *Borel regulator* is the map

$$(\mathrm{vol}_1, \dots, \mathrm{vol}_{r_2}) \colon \mathcal{B}(k) \to \mathbb{R}^{r_2}.$$

THEOREM 11.2 (Borel, Suslin). *The Borel regulator maps $\mathcal{B}(k)/\mathit{Torsion}$ injectively onto a full sublattice of $\mathbb{R}^{r_2}$.*

This is the theorem which makes scissors congruence of hyperbolic 3-manifolds computable. In fact, there is a computer program "snap" written by O. Goodman using ingredients from Jeff Weeks' program "snappea," which does these computations (and much more). It is described in detail, along with several tables of computations, in [**9**]. Both this paper and the actual program are available from http://www.ms.unimelb.edu/˜snap.

We close these notes with one final exercise. We first remind that if k is an imaginary quadratic number field, then 3-manifolds with invariant trace field k do exist. In fact infinitely many non-commensurable ones do, since one can take arithmetic manifolds with different quaternion algebras.

Exercise 20. *Show that two hyperbolic 3-manifolds with imaginary quadratic invariant trace fields are scissors congruence commensurable if and only if they have the same invariant trace fields.*

Contrast this exercise with the Dehn invariant sufficiency conjecture: that conjecture would imply that the volumes of two manifolds with different imaginary quadratic invariant trace fields cannot be rational multiples of each other, but it is not even proved that any volume of any hyperbolic 3-manifold is actually irrational!

12. Solution to Exercise 19

Consider the ideal tetrahedron $\Delta(z)$ with parameter z. We may position its vertices at $0, 1, \infty, z$. There is a Klein 4-group of symmetries of this tetrahedron and it is easily verified that it takes the following horoballs to each other:

- At ∞ the horoball $\{(w,t) \in \mathbb{C} \times \mathbb{R}^+ | t \geq a\}$;
- at 0 the horoball of euclidean diameter $|z|/a$;
- at 1 the horoball of euclidean diameter $|1-z|/a$;
- at z the horoball of euclidean diameter $|z(z-1)|/a$.

After truncation, the vertical edges thus have lengths $2\log a - \log|z|$, $2\log a - \log|1-z|$, and $2\log a - \log|z(z-1)|$ respectively, and we have earlier said that their angles are $\arg(z), \arg(1/(1-z)), \arg((z-1)/z)$ respectively. Thus, adding contributions, we find that these three edges contribute $\log|1-z| \otimes \arg(z) - \log|z| \otimes \arg(1-z)$ to the Dehn invariant. By symmetry the other three edges contribute the same, so the Dehn invariant is:

$$\delta(\Delta(z)) = 2\big(\log|1-z| \otimes \arg(z) - \log|z| \otimes \arg(1-z)\big) \in \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Q}.$$

On the other hand, to understand the "imaginary part" of $(1-z) \wedge z \in \mathbb{C}^* \wedge \mathbb{C}^*$ we use the isomorphism

$$\mathbb{C}^* \to \mathbb{R} \oplus \mathbb{R}/2\pi\mathbb{Z}, \quad z \mapsto \log|z| \oplus \arg z,$$

to represent

$$\begin{aligned}\mathbb{C}^* \wedge \mathbb{C}^* &= (\mathbb{R} \oplus \mathbb{R}/2\pi\mathbb{Z}) \wedge (\mathbb{R} \oplus \mathbb{R}/2\pi\mathbb{Z}) \\ &= (\mathbb{R} \wedge \mathbb{R}) \oplus (\mathbb{R}/2\pi\mathbb{Z} \wedge \mathbb{R}/2\pi\mathbb{Z}) \quad \oplus \quad (\mathbb{R} \otimes \mathbb{R}/2\pi\mathbb{Z}) \\ &= (\mathbb{R} \wedge \mathbb{R}) \oplus (\mathbb{R}/\pi\mathbb{Q} \wedge \mathbb{R}/\pi\mathbb{Q}) \quad \oplus \quad (\mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Q}),\end{aligned}$$

(the equality on the third line is because tensoring over $\mathbb{Z}$ with a divisible group is effectively the same as tensoring over $\mathbb{Q}$). Under this isomorphism we have

$$\begin{aligned}(1-z) \wedge z = \big(&\log|1-z| \wedge \log|z| \oplus \arg(1-z) \wedge \arg z\big) \\ &\oplus \quad \big(\log|1-z| \otimes \arg z - \log|z| \otimes \arg(1-z)\big),\end{aligned}$$

confirming the Proposition 10.5.

APPENDIX A

Examples

In this appendix we will work through some explicit examples related to the preceding notes. These were presented during tutorials at the Turán Workshop.

1. Trefoil complement

We will put two geometric structures on the trefoil complement. Since the trefoil is a torus knot its complement is Seifert fibred with an orbifold quotient F given by the disk with two orbifold points of orders 2 and 3. Thus $\chi(F) = 1-(1-1/2)-(1-1/3) = -1/6$ so F admits a hyperbolic structure. We can realise this explicitly with the orbifold given in example 4.2 of Chapter 1 of the notes, $F \cong PSL(2,\mathbb{Z})\backslash\mathbb{H}^2$. Now pull back the metric using the map $S^3 - K \to F$. This construction is not so explicit and in fact we can pull back the metric in different ways to get $\mathbb{H}^2 \times \mathbb{R}$ structures or a $\mathbb{P}SL$ structure.

A better way to see things is as follows. We use the fact that the trefoil is a fibred knot. This can be seen from the fundamental group

$$\begin{aligned} \pi_1(S^3 - K) &= \{a_1, a_2, a_3 | a_1a_2 = a_2a_3 = a_3a_1\} \\ &\cong \{a, x, y | a^{-1}xa = y, a^{-1}ya = x^{-1}y\} \end{aligned}$$

or as the link of the singularity $Y^2 + X^3 = 0$. We can perturb this to $Y^2 = -X^3 + \epsilon$ which describes the Milnor fibre, or the Seifert surface S, of the trefoil as a branched cover of the disk with branch points $X_0, \zeta X_0, \zeta^2 X_0$ where $\zeta^3 = 1$. This makes the holonomy of the fibration $S^3 - K \to S^1$ clear—it is given by rotation of the disk by $2\pi/3$.

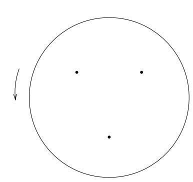

FIGURE 1

Thus if we have a hyperbolic metric on the orbifold given by the disk with degree 2 orbifold points at the branch points that is invariant under the holonomy this gives an explicit $\mathbb{H}^2 \times \mathbb{R}$ structure on $S^3 - K$. Notice that if we quotient S by the $\mathbb{Z}/3$ action we get the orbifold F described above. The hyperbolic metric on F lifts to a hyperbolic metric on S invariant under rotation by $2\pi/3$ so we see the $\mathbb{H}^2 \times \mathbb{R}$ structure explicitly.

We can also get the $\mathbb{P}SL$ structure as follows. If we put a left invariant metric on $PSL(2,\mathbb{R})$ then it pushes down to the quotient $PSL(2,\mathbb{Z})\backslash PSL(2,\mathbb{R})$. This quotient is the complement of the trefoil. We can see this by studying the action of $SO(2)$ on the right and seeing that the quotient $PSL(2,\mathbb{Z})\backslash PSL(2,\mathbb{R})/SO(2)$ is a disk with two orbifold points of degrees 2 and 3. By the classification of Seifert fibred spaces this must be the trefoil complement. Notice that the Seifert surface is totally geodesic in the $\mathbb{H}^2 \times \mathbb{R}$ case, but not so in the $\mathbb{P}SL$ case.

This example is one of the few cases when two types of geometries can be put on the manifold. We cannot put a hyperbolic structure on the trefoil complement due to the following general fact about Seifert fibred spaces. A fibre of the Seifert fibring gives a normal $\mathbb{Z}$ in the fundamental group of the trefoil complement. Thus, the square of any element of the fundamental group commutes with the element represented by a fibre. This gives two parabolic elements and hence a cusp. But there are more commuting parabolic elements and they also define the same cusp which contradicts the fact that a lattice in the boundary $\mathbb{C}$ cannot have more than two generators.

2. JSJ decomposition

We will give the JSJ decomposition of the link of the singularity

$$f = (x^2 + y^3)^2 + xy^5 + z^5.$$

The plumbing graph of this link is given in the following appendix. It is equivalent to the splice diagram in figure 2. See [**15**]. For this example it is sufficient to

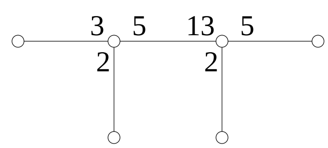

FIGURE 2

understand the following fact about splice diagrams. The simplest type of splice diagram is of the form in figure 3 which represents a Seifert fibred space with three

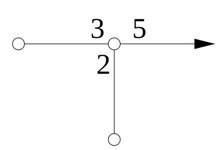

FIGURE 3

singular fibres of multiplicities $2, 3$ and 5, respectively. The arrow indicates that we wish to specify the singular fibre of multiplicity 5 as a link component in the Seifert fibred space. (In this case the link is a knot.) The splice diagram in figure 2 is obtained by "splicing" together the Seifert fibred spaces represented in figures 3 and 4 along the two specified knots. We splice by removing a neighbourhood of each knot and identifying the respective boundary tori where the longitude and meridian of one torus is mapped to the meridian and longitude of the other torus.

This is exactly the canonical decompositon of the three-manifold into two Seifert fibred spaces. They can each be given either the geometry $\mathbb{H} \times \mathbb{R}$ or $\mathbb{P}SL$.

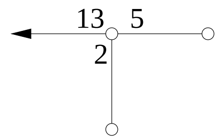

FIGURE 4

3. Ideal tetrahedra and the invariant trace field

We will show that the invariant trace field of an ideally triangulated hyperbolic manifold with cusps is generated by its tetrahedra parameters. See [**32**] for further details.

Given M^3 non-compact and ideally triangulated notate its set of cusps by $C \subset \partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$. Define k_Δ to be the field generated by the simplex parameters of the ideal tetrahedra. Take one of the ideal tetrahedra of M^3 and put three of the vertices of the tetrahedron at $0, 1$ and ∞.

(i) First we will show that $C \subset k_\Delta \cup \{\infty\}$ and C generates $k_\Delta \cup \{\infty\}$.

This is proved by induction using the fact that if three of $z_1, ...z_4$ are in $k_\Delta \cup \{\infty\}$ then the last one is in $k_\Delta \cup \{\infty\}$ if an only if the cross ratio is in $k_\Delta \cup \{\infty\}$. Thus, we begin with the tetrahedron with three vertices at $0, 1$ and ∞ and use the preceding fact to see that the fourth vertex lies in $k_\Delta \cup \{\infty\}$. Any ideal tetrahedron that shares a face with this tetrahedron now has three of its vertices in $k_\Delta \cup \{\infty\}$ so the induction continues to this tetrahedron. Thus $C \subset k_\Delta \cup \{\infty\}$ and C generates $k_\Delta \cup \{\infty\}$ since each simplex parameters is the cross ratio of four cusp points.

Thus it is sufficent to show that $k_\Delta = k(\Gamma)$.

(ii) $k_\Delta \subset k(\Gamma)$.

Since $\Gamma \subset k(\Gamma) \cdot \bar{\Gamma} = Q(\Gamma)$, the quaternion algebra associated to Γ, then the existence of parabolic elements implies that $Q(\Gamma) \subset M(2, \mathbb{C})$ has zero divisors—if P is parabolic then $(P - I)^2 = 0$—so $Q(\Gamma) \cong M(2, k(\Gamma))$. Therefore we can conjugate Γ to get a subgroup of $PGL(2, k(\Gamma))$. Notice that if $g \in PGL(2, k(\Gamma))$ then $g^2/\det(g) \in PSL(2, k(\Gamma))$ and thus we can conjugate $\Gamma^{(2)}$ to a subset of $PSL(2, k(\Gamma))$. Now, a cusp is a solution of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

where the matrix lies in $PGL(2, k(\Gamma))$ and in fact by using the square of an element of Γ we may assume that the matrix lies in $PSL(2, k(\Gamma))$. Therefore $x/y \in k(\Gamma)$.

(iii) If Γ is a group fixing a subset of $k_\Delta \cup \{\infty\}$ then $\Gamma \subset PGL(2, k_\Delta)$.

We can see this simply by observing that the following equation:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \left.\begin{matrix} \gamma(0) = \alpha \\ \gamma(1) = \beta \\ \gamma(\infty) = \gamma \end{matrix}\right\} \in k_\Delta \cup \{\infty\}.$$

has a solution in k_Δ:

$$b = \alpha d, \;\; a + b = \beta c + \beta d, \;\; a = c\gamma.$$

Thus, $\Gamma \subset PGL(2, k_\Delta)$ and $\Gamma^{(2)} \subset PSL(2, k_\Delta)$ so it follows that $k(\Gamma) \subset k_\Delta$.

4. Scissors congruence

Here we will prove that two Euclidean polygons are scissors congruent if and only if they have the same area.

Given two polygons of equal area we can reduce the problem of scissors congruence to two equal area triangles as follows. Assume we can solve the equal area triangle problem. Cut the two polygons into triangles and take a triangle from each polygon. If they have equal area then we use the solution of the equal area triangle problem to cut the two triangles into equal pieces. If they have unequal area then cut the bigger one into two triangles one of which has the same area as the triangle from the other polygon. (We can do this by the intermediate value theorem.) Now cut the two equal area trangles into identical pieces. This has reduced the problem. In the first case the total number of triangles is reduced by two and in the second case the number is reduced by one.

Scissors congruence is an equivalence relation since we can cut a polygon further to prove transitivity. Thus to solve the equal area triangle problem it is enough to show that any triangle is scissors congruent to the square of equal area. From the diagram we see how to go from a triangle to a rectangle.

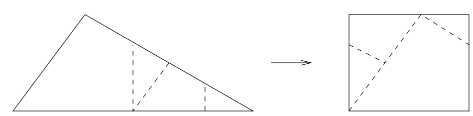

FIGURE 5

To go from the rectangle to the square we cut the rectangle as shown until the ratio of the two sides is not greater than two.

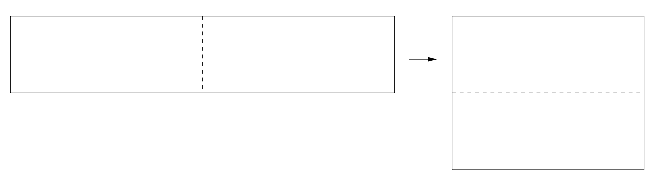

FIGURE 6

Then the following diagram gives the equivalence with the square.

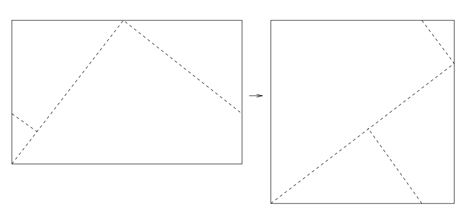

FIGURE 7

APPENDIX B

Problems

The following set of problems were used during the Turán Workshop. They are sometimes answered in the course notes and the worked examples.

1. Geometries

QUICK

(i) Take a family of non-conjugate representations of $\pi_1\Sigma$ into $SL(2,\mathbb{R})$. Does Mostow-Prasad rigidity imply these representations are conjugate inside $SL(2,\mathbb{C})$?

(ii) Find all integer solutions $\{p_1, ..., p_s\}$ of

$$2 - \Sigma_1^s(1 - 1/p_i) = 0 \ \ (= \chi(F)).$$

(These give all flat compact genus 0 orbifolds, where the $\{p_i\}$ give the mulplicities of the orbifold points.)

(iii) Put a hyperbolic structure on the surface of genus g.

UNDERSTANDING

(i) Describe the complement of the trefoil in terms of the link of the singularity $y^2 = x^3$ and hence describe the holonomy.

(ii) Put geometries on each of: the trefoil complement; figure-8 complement.

(iii) Why is the type of the geometry that can be put on a closed manifold unique? (Hint: look for topological invariants.)

(iv) Describe the isometry groups of the 8 geometries in dimension 3.

(v) Describe the space of geodesics for each geometry.

THOUGHTFUL

(i) Given any two Euclidean polygons of the same area, show that you can cut one of them into a finite number of Euclidean polygons and reassemble the pieces to get the other polygon.

(ii) We have seen that the trefoil complement admits a geometric structure of type $H^2 \times \mathbb{R}$ and also of type $\mathbb{P}SL$. How can we see that it does *not* admit a hyperbolic structure?

2. Decomposition

QUICK

(i) What does an incompressible torus in Y^3 imply about the fundamental group $\pi_1 Y^3$? If the incompressible torus is boundary parallel what does this say in terms of the fundamental group?

(ii) Give an example to show what is wrong with this definition:

an embedded annulus $(A, \partial A) \hookrightarrow (Y, \partial Y)$ is *boundary parallel* if there exists an annulus $\tilde{A} \subset \partial Y$ with $\partial A = \partial\tilde{A}$ and $A \cup \tilde{A} = \partial H$ for an embedded solid torus $H \hookrightarrow Y$ with $H \cap \partial Y = \tilde{A}$.

(iii) Describe canonical circles in two dimensions.

Understanding

(i) Find the maximal set of canonical tori, and hence the JSJ-decomposition, of the following 3-manifolds: trefoil complement; figure-8 complement; trefoil cabled on figure-8 complement; link of the singularity

$$f = (x^2 + y^3)^2 + xy^5 + z^5.$$

(Hint: use the algorithm described in Nemethi's course to verify that the resolution of the singularity has plumbing graph

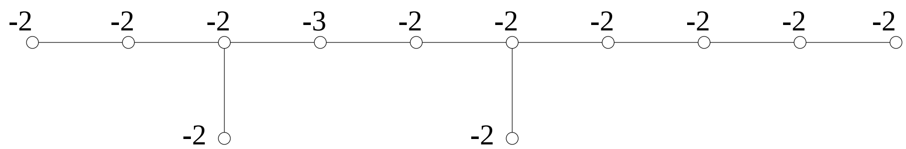

Figure 1

where each curve is a $\mathbb{P}^1$. Since the link is given by the boundary of a neighbourhood of the divisor and hence a circle bundle over the smooth points of the divisor any embedded circle in the divisor pulls back to a torus.)

Here are examples of decompositions where uniqueness fails:

(ii) Connect sum in two dimensions: prove that $\mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2 \cong T^2\#\mathbb{RP}^2$.

(iii) Connect sum in three dimensions: $Y \cong M_1\#M_2\#...\#M_k$ decomposes an orientable 3-manifold Y into prime manifolds M_i, unique up to order. What is not unique here?

(iv) Connect sum of non-orientable 3-manifolds: give an example to show non-uniqueness.

(v) Connect sum in four dimensions: give an example to show non-uniqueness.

(vi) What is the key point in the proof of uniqueness of JSJ-decompositions that fails in the previous three cases. Show this explicitly with the example of the blown up projective plane and torus (in (ii) above).

(vii) Show that every Seifert fibred space has a normal $\mathbb{Z}$ in its fundamental group and thus a Seifert fibred space does not admit a hyperbolic structure.

(viii) Put a geometry on the torus with one orbifold point.

Thoughtful

(i) Canonical decomposition of maps of a surface: given a diffeomorphism $f : \Sigma \to \Sigma$, we say an embedded circle $S^1 \hookrightarrow \Sigma$ has finite order if some iterate of f takes the circle to an istopic circle. Define a *canonical* circle to be a finite order circle with the property that all other finite order circles can be isotoped off the canonical circle. Use this to get a canonical decomposition of (Σ, f) and show that it is unique.

(ii) Given any two hyperbolic or spherical polygons of the same area, show that you can cut one of them into a finite number of hyperbolic, respectively spherical, polygons and reassemble the pieces to get the other polygon.

(iii) Prove that if both $\cos\theta$ and θ/π are rational then θ is a multiple of $\pi/2$ or $\pi/3$.

Scissors congruence

Quick

(i) Show that scissors congruence is an equivalence relation.

(ii) Show $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \cong \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}/\mathbb{Q}$.

(iii) Calculate the Dehn invariant of the Euclidean polyhedron $\delta(\Sigma \times I)$ for any 2-dimensional polygon Σ.

(iv) What is the Dehn invariant in dimension 2 (Euclidean, hyperbolic and spherical) and why doesn't it obstruct scissors congruence?

Understanding

(i) Prove that Euclidean polygons of the same area are scissors congruent:

(a) show that it is enough to prove this for two triangles;

(b) show that a triangle is scissors congruent to a rectangle;

(c) show that any rectangle is scissors congruent to a square;

(d) conclude the equivalence of any two polygons of equal area.

(ii) Prove that the Dehn invariant doesn't vanish on a regular tetrahedron and create your own example with $\delta \neq 0$.

(iii) Prove: $\mathrm{im}(vol|\ker\delta) = \mathbb{R}$ and $\mathrm{im}(\delta|\ker vol) = \mathrm{im}(\delta)$.

(iv) Recall that scissors congruence for finite area hyperbolic polygons with ideal vertices is allowed. Show that stable scissors congruence is not the same as scissors congruence. Show that two polygons are scissors congruent if and only if they have the same area and they are either both compact or both non-compact and stably scissors congruent if and only if they have the same area.

Thoughtful

(i) Where does the argument that Euclidean polygons of the same area are scissors congruent (in Understanding(i) above) fail for hyperbolic and spherical geometry? Can it be fixed?

(ii) Where does the argument that Euclidean polygons of the same area are scissors congruent (in Understanding(i) above) fail in dimension 3? Does it succeed on a limited class of polyhedra?

(iii) Show that for polygons P and Q, $2P \sim 2Q$ implies $P \sim Q$. Since this is a general fact in any dimension and geometry, one can argue without using the equality of area.

(iv) Define a Dehn invariant in dimension 4. Is it unique?

Commensurability

Quick

(i) Show that all ideal triangles in $\mathbb{H}^2$ are isometric.

(ii) Show that not all ideal tetrahedra in $\mathbb{H}^3$ are isometric.

(iii) Construct a rotation of $\mathbb{R}^2$ that acts on $\mathbb{Z}[i]$ giving a commensurable lattice. Show that a dense set of rotations has this property.

Understanding

(i) Show that the group of symmetries of the tessellation of $\mathbb{H}^3$ by ideal tetrahedra is arithmetic.

(ii) Show that $Comm^+(PSL(2,\mathbb{Z}[i])) = PGL(2,\mathbb{Q}[i])$.

(iii) Prove that there are exactly two quaternion algebras over $\mathbb{R}$.

(iv) Show that the figure 8 knot and the Whitehead link are arithmetic.

Thoughtful

(i) Does arithmeticity exist in dimension 2?

Bloch group.

Quick

(i) Show explicitly that the figure 8 knot lies in the kernel of the complex Dehn invariant. (Hint: the parameter for the figure 8 knot is given by $2[\zeta]$ where $\zeta^6 = 1$.)

(ii) Does hyperbolic Dehn surgery on the figure 8 knot change its scissors congruence class in the Bloch group, $\beta_0(Y) \in \mathcal{B}(\mathbb{C})$?

(iii) Are the hyperbolic Dehn surgeries on the figure 8 knot arithmetic?

Understanding

(i) Show that the invariant trace field of an ideally triangulated hyperbolic manifold with cusps is generated by its tetrahedra parameters.

(ii) Show that two hyperbolic 3-manifolds with imaginary quadratic trace field are scissors congruent commensurable if and only if they have the same invariant trace fields.

(iii) Prove that the five-term scissors congruence relation implies that $[z] = [1 - 1/z]$.

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