Links and Complements of Arrangements of Complex Projective Plane Algebraic Curves

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Abstract

This thesis explores questions regarding complex projective plane algebraic curves. Any such specific curve determines three pieces of information. It determines the curve itself, the link of the curve and the complement of the curve. This thesis establishes strong connections between these quantities.

We use methods from 3-manifold topology to show that the complement always determines the link. We also show that the link does not always determine the curve itself, however under certain restrictions it does determine the curve. Furthermore in the case that the link does not determine the curve, we find an algorithm in some cases which generates all curves with a specific link. This is the central part of this thesis and the method uses the combinatorics of plumbed 3-manifolds which enables the computing of finitely many associated graphs. We also explore a weaker invariant that the link may determine. Finally we briefly explore when a curve which consists of a union of lines determines its complement.

Certification

This is to certify that

1. this thesis comprises only my original work,

2. due acknowledgment has been made in the text to all other materials used,

3. the thesis is less than 100,000 words in length, exclusive of tables, maps, bibliographies, appendices and footnotes.

Angelo Di Pasquale

Acknowledgments

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Chapter 1

Introduction

A specific complex projective plane algebraic curve C embedded in $\mathbb{CP}^2$ determines three pieces of information. Firstly it determines the topology of C itself, also the topology of the complement $\mathbb{CP}^2 \setminus C$ of C in $\mathbb{CP}^2$ and finally the topology of the link $L(C)$. The link is the boundary of a closed regular neighbourhood of C in $\mathbb{CP}^2$. (See definition 2.1.1 .)

By a specific curve C, we mean that C can be described as the zero set of a particular homogeneous polynomial in three variables. Of course C can be decomposed into its irreducible components $C_1, C_2, \ldots, C_n$ so that C can be thought of as a specific arrangement A of specific irreducible curves $C_1, C_2, \ldots, C_n$. We shall assume that C has no multiple components so that $C_i \neq C_j$ if $i \neq j$.

Now when we think of an arrangement A of curves, we do not have any specific equations in mind, but only the structure of how A is put together. For example the arrangement A_1 being the union of the two lines $y = 0$ and $y = x$ can be considered to be equivalent to the arrangement A_2 which is the union of the two lines $y = 2x$ and $y = 3x$. Both arrangements are line pairs. Thus we can speak of two specific arrangements A_1 and A_2 as being equivalent. This motivates the following definition.

Given two specific arrangements A_1, A_2 of curves in $\mathbb{CP}^2$, construct the immediate plumbing diagrams, $Pl(A_1)$, $Pl(A_2)$ (see definition 2.1.2) obtained by minimal resolution. For each specific arrangement A we also have the marked plumbing diagram $Mark(A)$, which is $Pl(A)$ with exceptional divisors marked. We say two specific arrangements are equivalent if they have isomorphic marked plumbing diagrams. Henceforth we shall speak of an (abstract) *arrangement* A to mean an equivalence class of equivalent specific arrangements. We similarly say that the complements (respectively the links) of two specific arrangements are equivalent if they are homeomorphic and we shall speak of a *complement* (respectively a *link*) to mean an equivalence class of equivalent complements (respectively links).

Thus given a specific arrangement A, we have three pieces of information:

- The (abstract) arrangement A.
- The complement of A.
- The link of A.

The study of curves has been of long interest. Conic sections had been studied and classified over two millenia ago and Newton classified real cubic curves. When complex projective curves began to be studied, results like Bezout's theorem and the well known genus formula showed that understanding singularities of curves is very important. Methods for understanding singularities include the study of the complement in a small ball about the singularity and also the resolution of singularities by blowing up. Hirzebruch [6] and Jung [12] have shown that any complex surface singularity may be resolved after finitely many blow ups and Hironaka [5] has succeeded in extending this to all algebraic varieties over a field of characteristic zero.

Zariski [21] was interested in the complement and its fundamental group, in particular when the fundamental group of the complement is abelian.

More recently Libgober [13], [14], Falk [4] and Cohen [3] have studied the complement via its fundamental group and its homotopy type. They have found examples of arrangements which are different but have same fundamental group of the complement and some even having same homotopy type of the complement.

The relationship between an arrangement and its complement has been of interest in other contexts, particularly where the arrangement consists solely of hyperplanes. For example in studying the motion of points in the complex plane $\mathbb{C}$ one is quickly led to studying the complement of the hyperplanes $\prod_{1\leq i<j\leq n}(x_i - x_j) = 0$ in $\mathbb{C}^n$ and its fundamental group. The book *Arrangements of Hyperplanes* [17] describes many of the directions in which the study of hyperplanes has been.

In 1991, Jiang and Yau, in studying the problem of arrangements of lines in $\mathbb{CP}^2$, asked if it were possible that two different arrangements could have the same complement. They announced [10] the following result in 1993 and a proof appeared [9] in 1995.

Theorem *Within the class of arrangements of complex lines in* $\mathbb{CP}^2$*, the topology of the complement of the arrangement determines the topology of the arrangement itself.*

(This result showed that the two nonequivalent line arrangements of Falk [4] which had homotopy equivalent complements, actually have topologically inequivalent complements.)

However, they nearly proved something stronger, namely for the case of arrangements of lines, the complement determines the link and the link determines the arrangement. This stronger result is part of what we prove and our proofs are somewhat simpler than theirs. Our methods also lend themselves more easily to investigating arrangements of more general type of curves.

Section 2 revises the machinery which is needed to characterise the link.

The link is actually a 3-manifold which can be described by a plumbing diagram. For this we require elements of the Plumbing Calculus [16].

In section 2.4 we prove the following theorem:

Theorem 1.0.1 (Corollary 2.4.4) *The topology of the complement of any (possibly singular or even reducible) complex curve in a two complex dimensional complex surface determines the topology of the link of the curve.*

Although theorems of this type have been floating around for a while, it seems that a proof of this general statement has not been realised until now.

Sections 3 and 4 deal with regular arrangements, that is arrangements of non-singular curves with only pairwise transversal intersections (definition 3.1.2). We investigate under what circumstances the link may determine the arrangement. We prove the following results:

Theorem 1.0.2 *The topology of the link determines the arrangement for the following classes:*

1. *Arrangements of lines. (Corollary 3.4.6.)*
2. *Arrangements of nonreducible conics in which all curves only pairwise intersect transversally. (Corollary 3.4.7.)*
3. *Arrangements of non-singular curves in which all curves only pairwise intersect transversally and in which the arrangement contains a curve which is not a line nor a conic. (Corollary 3.4.2.)*

For general regular arrangements we find that if the link does not determine the arrangement, then an axiom system results (see Theorem 3.1.3, Axiom 4.3.2, Axiom 4.3.3 and Axiom 4.3.4) and these lead directly to examples of different arrangements with the same complement and the same link. We thus prove theorem 1.0.3, but see algorithm 3.4.5 and section 4.3.9 where the algorithm is explicitly described.

Theorem 1.0.3 *The topology of the link does not always determine the topology of the arrangement, but for the class of arrangements in which all curves pairwise intersect transversally and are nonsingular, there is a simple algorithm which discovers firstly if there is any other arrangement in this class having the same link and then how to find all possible arrangements within this class which have the same link. Furthermore there are only finitely many arrangements within this class which can have a particular link.*

In section 5 we notice that once we pass over to the most general types of arrangements, that is where the constituent curves may be singular and their intersections may be very complicated, then the phenomenon that the link may not determine the arrangement can occur very frequently and it does not seem immediately possible to quantify under what circumstances this may happen. Yet in all known cases it happens that if two arrangements have same link, then they are "birationally topologically equivalent". (In the sense that the exceptional curves are also part of the arrangement. See definition 5.1.2). We thus form a new equivalence on arrangements, BTE (Birational Topological Equivalence), which seems to be the more natural concept to investigate anyway, since two arrangements which are BTE have the same link. We conjecture that BTE is determined by the link. In section 5.2 we do prove this for a very large class of arrangements.

Theorem 1.0.4 (Theorem 5.2.1) *Let $\mathcal{C}$ be the class of arrangements consisting of arrangements A, where A consists of the distinct irreducible curves $C_1, C_2, \ldots, C_n$, and such that any $A \in \mathcal{C}$ fulfills at least one of the following two conditions.*

1. *$\cup_{i=1}^{n} C_i$ is a curve whose singularities are all ordinary.*

2. *The initial plumbing diagram for A contains a cycle and the passage to normal form can be achieved using only (-1)-blow-ups or (-1)-blow-downs.*

Then BTE holds for the class $\mathcal{C}$.

Sections 5.3 and 5.4 are devoted to discussing difficulties and strategies in aiming for a general proof.

In section 6 we revisit the definition of an arrangement and then investigate the question of an arrangement determining its complement. This is very difficult and we only deal with line arrangements. In section 6.2 we define *brittle* arrangements (see definition 6.2.2) and prove that non-brittle arrangements determine their complement. We prove (theorem 6.2.5) that all arrangements of less than eight lines are non-brittle. We also find all reduced brittle arrangements of less than ten lines. Section 6.3 is a short intuitive discussion of why an arrangement does not necessarily determine its complement. Section 6.4 concludes with questions.

Chapter 2

Characterising the Link

2.1 Preliminary Definitions and Machinery

Definition 2.1.1 *The* link *of an arrangement A of curves in $\mathbb{CP}^2$ is the boundary of a closed* regular neighbourhood *of A in $\mathbb{CP}^2$.*

Intuitively this means that we "thicken up" the components of A and then take the boundary. For example three closed neighbourhoods of a circle $\mathbf{S}^1$ in $\mathbb{R}^2$ are shown in figure 2.1, but only the first is a regular neighbourhood. The second has an unnecessary "hole" in it and the third includes too much. Note also that the regular neighbourhood and hence the link depend on the ambient space and also the embedding. For example a regular neighbourhood of a circle on a sphere is an annulus and its boundary consists of two disjoint circles, however a circle embedded in the real projective plane $\mathbb{RP}^2$ can have an annulus as its regular neighbourhood and hence two disjoint circles as its link if the circle represents an orientation preserving path, or a Möbius band as its regular neighbourhood and hence one circle as its link if the circle represents an orientation reversing path.

There is a way of abstractly constructing a regular neighbourhood and its link for an arrangement which we describe now. See Walter Neumann's

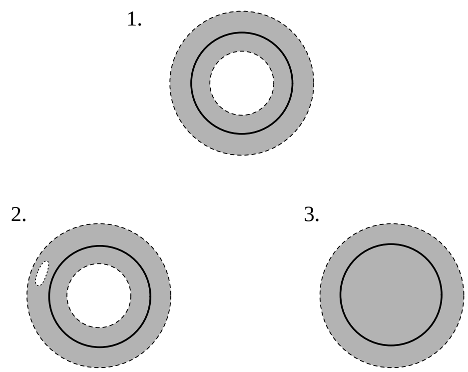


Figure 2.1:

"Calculus for Plumbing" [16] for the results of this section.

Let F be a 2-dimensional compact surface embedded in a 4-manifold M. Let $T(M)$ be the tangent bundle of M and consider its restriction $T(M)|_F$ to F. Consider also the tangent bundle $T(F)$ of F as a smooth manifold. We have a natural embedding

$$\iota : T(F) \hookrightarrow T(M)|_F$$

and this induces the quotient bundle $T(M)|_F/T(F)$ which is also the normal bundle of F in M. We can assume that we have endowed the tangent bundle with a Riemannian metric $<,>$ and the normal bundle can be identified with the orthogonal complement of $T(F)$. We denote this by $T(F)^\perp$. Let F_0 denote the zero section of F in $T(F)^\perp$. The following theorem provides a regular neighbourhood of F in M.

Theorem 2.1.1 (Tubular Neighbourhood Theorem) *F has a neighbourhood in M diffeomorphic to a closed tubular neighbourhood of F_0 in $T(F)^\perp$.*

The tubular neighbourhood of F_0 in $T(F)^\perp$ is simply the set of vectors of norm at most 1 using the Riemannian metric (a disc bundle). The boundary of this is a circle bundle consisting of the set of vectors of norm equal to 1. See pp. 37–39 of [2].

Theorem 2.1.2 *The self intersection number $F \cdot F$ classifies the disc bundle of F up to diffeomorphism.*

A consequence of this is that we can use any abstract disc bundle of F to construct a regular neighbourhood of F provided that the zero section of F has the same self intersection number in the bundle as F itself does in M. This number e is called the Euler number of the bundle. For example consider a line l in $\mathbb{CP}^2$ and a conic γ in $\mathbb{CP}^2$. Now l and γ are topologically equivalent, both being homeomorphic to the sphere $\mathbf{S}^2$, however $l \cdot l = +1$ while $\gamma \cdot \gamma = +4$ so that they have different self intersection numbers. Thus they have different regular neighbourhoods and different links. Hence $\mathbb{CP}^2 \setminus l$ and $\mathbb{CP}^2 \setminus \gamma$ are topologically inequivalent as we will see later.

We still need to describe a regular neighbourhood of an arrangement of surfaces.

Assume F_1, F_2 are two surfaces in M. We construct the regular neighbourhood of $F_1 \cup F_2$ as follows (plumb F_1 and F_2). First we have disc bundles $\pi_1 : E_1 \to F_1$, $\pi_2 : E_2 \to F_2$. For each intersection point P of F_1 and F_2 of positive orientation choose disjoint small discs $D_1 \subset F_1$, $D_2 \subset F_2$. Then we have

$$\pi_1^{-1}(D_1) \simeq D_1 \times D \subset E_1$$
$$\pi_2^{-1}(D_2) \simeq D_2 \times D \subset E_2$$

and form $E_1 \cup E_2$ but identify $D_1 \times D$ with $D_2 \times D$ under the map $(x, y) \mapsto (y, x)$ which identifies D_1 in $D_1 \times D$ with D in $D_2 \times D$ and D in $D_1 \times D$ with D_2 in $D_2 \times D$. (If the intersection is of negative orientation then identify D_1 with $-D$ and D with $-D_2$ via the map $(x, y) \mapsto (-y, x)$). This yields the required regular neighbourhood of $F_1 \cup F_2$. See figure 2.2.

In general we have the following result:

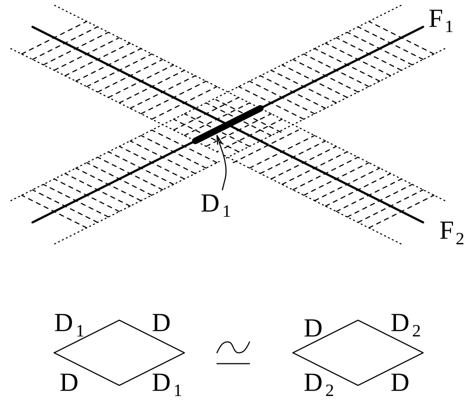


Figure 2.2:

Theorem 2.1.3 *If $F_1, F_2, \ldots, F_n$ are closed surfaces in a 4-manifold intersecting each other pairwise transversally and all intersection points are double points, then a regular neighbourhood of $F_1 \cup F_2 \cup \ldots \cup F_n$ in M, and hence its boundary, can be reconstructed from the following data which is also called the plumbing diagram.*

1. *For each F_i we have a vertex v_i with a pair of numbers $e_i, [g_i]$ attached where g_i is the genus of F_i and e_i is the self intersection number of F_i in M. This is written as in figure 2.3. We take the convention that if $g_i = 0$ we omit it. We also have the convention that the surface of genus $g_i < 0$ is the non orientable surface being the connect sum of $-g_i$ copies of $\mathbb{RP}^2$. We shall also need the following notation. If g_i, g_j are integers, let F_{g_i}, F_{g_j} be surfaces of the corresponding genus, then $g_i \# g_j$ is the integer which is the genus of the connect sum $F_{g_i} \# F_{g_j}$.*

2. *There are edges between v_i and v_j. For each intersection point of F_i and F_j there is an edge between v_i and v_j which is weighted with a "+" if the intersection is in the positive sense or a "$-$" if the intersection is in the negative sense.*

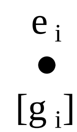


Figure 2.3:

We will usually only have to deal with positively oriented intersections. Hence the edge signs will generally be omitted but included if there exist some negatively oriented intersections. An example of a plumbing diagram is shown in figure 2.4.

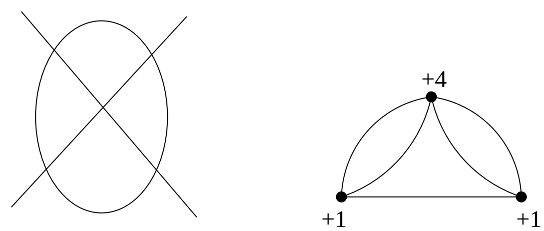


Figure 2.4: Line Pair and Conic

There is another way of describing a regular neighbourhood of an arrangement. The standard construction is firstly to give the ambient space a finite triangulation such that each curve in the arrangement is a subcomplex and each point of intersection of curves is part of the 0-skeleton. Now make two successive barycentric subdivisions and take the closed star of those simplices which intersect the arrangement. This is the required regular neighbourhood. Of course this construction is dependent on the triangulation but the following result ensures uniqueness.

Theorem 2.1.4 *Any two regular neighbourhoods are homeomorphic whether constructed by triangulation or by vector bundles and plumbing. Furthermore* *if U, V are regular neighbourhoods of an arrangement A with $V \subseteq U$ then the inclusions $A \hookrightarrow V \hookrightarrow U$ are homotopy equivalences via contraction.*

We can reconstruct the boundary of the regular neighbourhood of $F_1 \cup F_2 \cup \ldots \cup F_n$ from the plumbing graph as follows. For a vertex v_i we have a surface F_i of genus g_i and a unique up to diffeomorphism disc bundle $\pi_i : E_i \to F_i$ of Euler number e_i. If v_i has valence d_i, cut out d_i closed small disjoint discs from F_i. Whenever v_i is joined to v_j we paste together the circle bundles over F_i and F_j along the corresponding boundaries of the corresponding discs using the map described earlier on page 20. If Γ is the plumbing graph we denote by $M(\Gamma)$ the boundary of the closed neighbourhood just constructed. Note that $M(\Gamma)$ is a compact 3-manifold without boundary. We also allow the disjoint union of plumbing graphs $\Gamma = \Gamma_1 + \Gamma_2$ and the corresponding 3-manifold to be the connected sum $M(\Gamma) = M(\Gamma_1) \# M(\Gamma_2)$. Also allow $M(\phi) = \mathbf{S}^3$. The result that we will need is:

Theorem 2.1.5 *The plumbing diagram of an arrangement codes the topology of the link.*

Note that often in an arrangement A we may have some intersection points which are not double points and the construction does not apply. In this case we need to blow up these points until we arrive at a (blown up) arrangement of curves in a compact manifold M. Recall that blowing up a point P replaces P by a copy of $\mathbb{CP}^1$ and separates curves or branches of curves of different gradients through P. Furthermore blowing up a point on a curve reduces the curve's self intersection number by n^2, where n is the order of the curve at P (theorem 5.1.7). The $\mathbb{CP}^1$ produced replacing P is called an exceptional curve. It has self intersection number -1 and genus 0. For an arrangement A, we blow up all singular points on singular curves of A and also all points of order ≥ 3. We denote by $\overline{A}$ the corresponding blown up arrangement.

Blowing up is useful in view of the following result.

Theorem 2.1.6 *The link of the blown up arrangement $\overline{A}$ in the blown up manifold is homeomorphic to the link of the arrangement A.*

Thus the plumbing diagram for this blown up arrangement $\overline{A}$ codes the link of $\overline{A}$ and hence the link of A. We now build up a brief example.

Consider an arrangement A_0 consisting of a triangle of lines l_1, l_2 and l_3 and also other curves some of which pass through the triangle's vertices P_1, P_2 and P_3. (This is so we feel justified in blowing up these three vertices.) To obtain $Pl(A_0)$, blow up the vertices P_1, P_2, P_3 to obtain exceptional divisors E_1, E_2, E_3. Each E_i has self-intersection number -1. Since two points have been blown up on each l_i, the self-intersection number of l_i has dropped from $+1$ to -1. See figures 2.5 and 2.6 for the resolution and marked plumbing diagrams respectively.

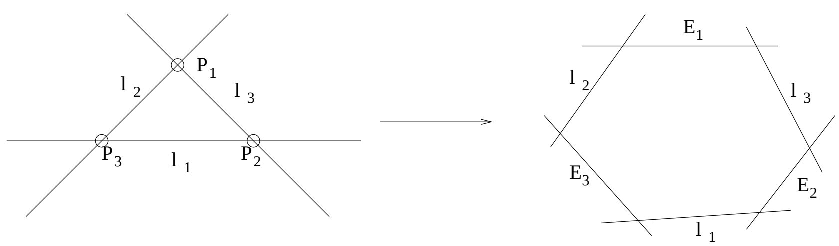


Figure 2.5:

Note that since there are other -1-curves it is important to remember which ones are the marked ones. This becomes a central issue in section 3.1.

Next let A_1 be the arrangement A_0 where the other curves are specifically a cubic through the vertices P_1, P_2, P_3 but otherwise in general position, and a further line in general position with respect to everything else. Figure 2.7 shows the marked plumbing diagram.

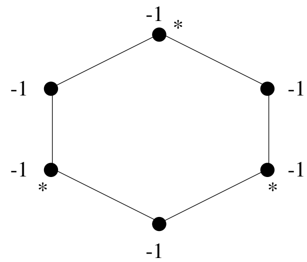


Figure 2.6:

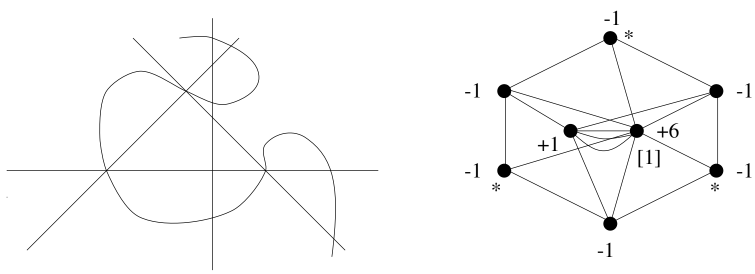


Figure 2.7:

Our strategy shall be to blow up arrangements where necessary to get plumbing diagrams so as to compare their links.

One notices that "redundant" blowing up (that is, blowing up when there is no need to, for example at a simple or double point), does not change the topology of the link, but does change the plumbing diagram.

Definition 2.1.2 *The* immediate plumbing diagram *for an arrangement A is obtained by performing only the necessary blow-ups needed to resolve singularities and complicated intersections and no more. It is denoted by $Pl(A)$.*

Note that under this definition even a double point is to be blow up if it is a singular point of some curve. Thus $Pl(A)$ can have no loops.

(Recall the definition of an arrangement A alluded to in the introduction. We can recover an arrangement from $Pl(A)$ provided we remember which curves are to be blown down.) Hence even though the plumbing diagram uniquely determines the topology, the converse is far from true. There are certain operations which one can apply to the plumbing diagram which do not change the topology. This is described in "A Calculus for Plumbing ..." by W.D. Neumann and the relevant parts are reviewed here. See section 2 of [16] for full details.

2.2 Plumbing Calculus

Theorem 2.2.1 *Applying any of the operations R0 to R7 to a plumbing graph* Γ *does not change the oriented diffeomorphism type of* $M(\Gamma)$.

R0(a). If v_i is a vertex with $g_i \geq 0$, reverse the signs on all edges other than loops at this vertex.

R0(b). If v_i is a vertex with $g_i < 0$ reverse the sign on any edge at this vertex.

R1. (Blowing down). In any of the following four situations, replace the graph on the left by the one on the right. Here $\varepsilon = \pm 1$ and $\varepsilon_0, \varepsilon_1, \varepsilon_2$ are the edge signs and are related by $\varepsilon_0 = -\varepsilon\varepsilon_1\varepsilon_2$.

R2. ($\mathbb{RP}^2$-absorption). Here $\delta_1 = \pm 1$, $\delta_2 = \pm 1$ and $\delta = \frac{\delta_1 + \delta_2}{2}$.

R3. (0-chain absorption). Here the edge signs ε_i' are given by $\varepsilon_i' = -\varepsilon\bar{\varepsilon}\varepsilon_i$ if the edge in question is not a loop and $\varepsilon_i' = \varepsilon_i$ if it is a loop.

R4. (unoriented handle absorption). In the diagram below if g_i is the genus of F_i and g_j is the genus of F_j then $g_i \# g_j$ is the genus of $F_i \# F_j$.

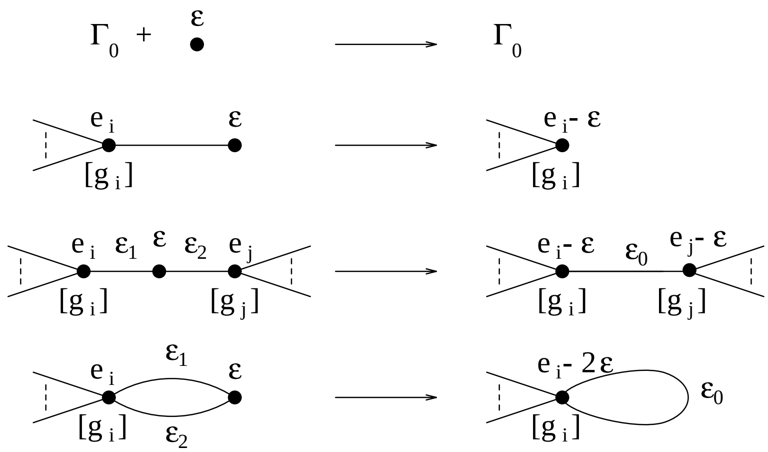


Figure 2.8: R1

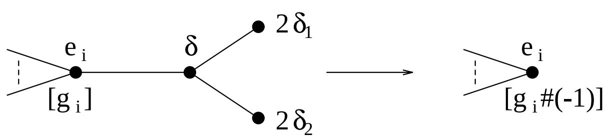


Figure 2.9: R2

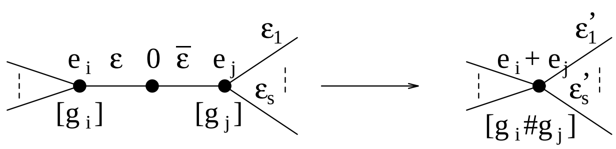


Figure 2.10: R3

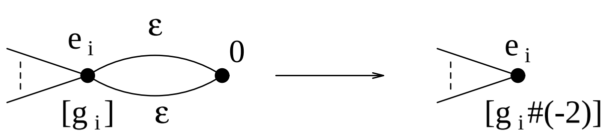


Figure 2.11: R4

R5. (oriented handle absorption).

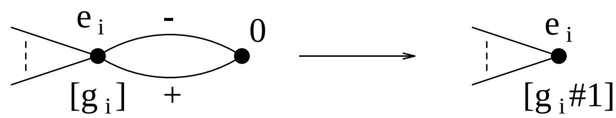


Figure 2.12: R5

R6. (splitting). If any component of Γ has the form as in figure 2.13, where

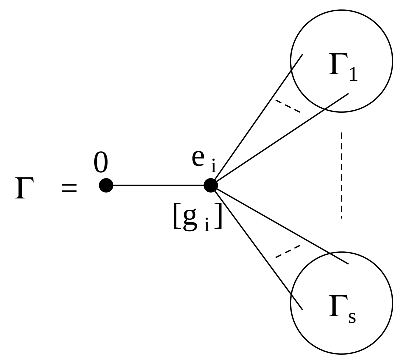


Figure 2.13: R6

each Γ_j is connected and for each j, Γ_j is joined to a vertex v_i by k_j edges, replace Γ by the disjoint union of $\Gamma_1, \ldots, \Gamma_s$ and k copies of figure 2.14 where

$$k = \begin{cases} 2g + \sum_{j=1}^{s}(k_j - 1) & \text{if } g \geq 0 \\ -g + \sum_{j=1}^{s}(k_j - 1) & \text{if } g < 0. \end{cases}$$

R7. (Seifert graph exchange). If any component of Γ is one of the graphs on the left of the following list, replace it by the corresponding graph on the right.

Figure 2.14: R6

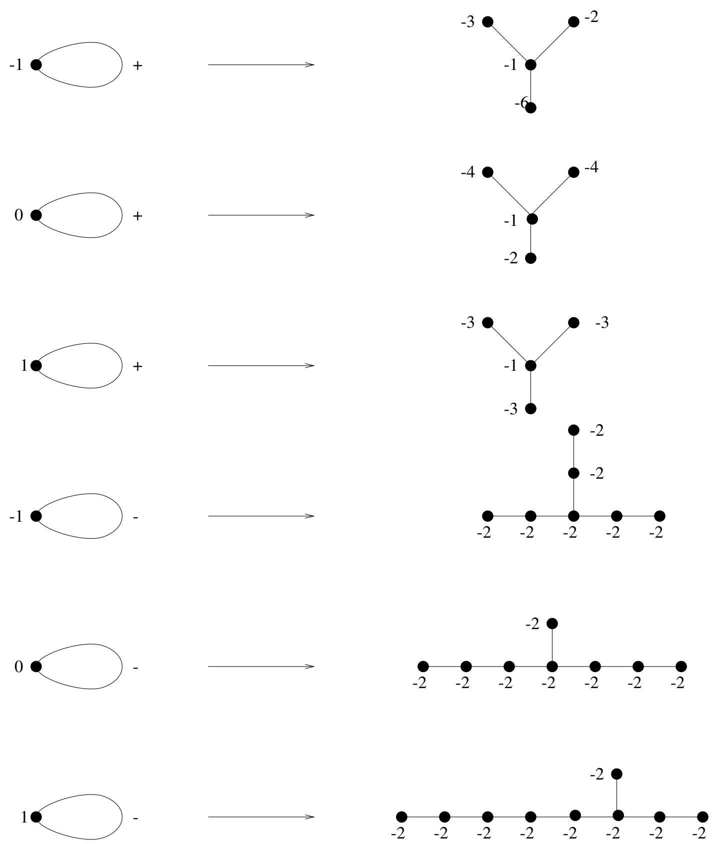


Figure 2.15: R7

2.3 Normal Form

Two plumbing diagrams can describe the same 3-manifold and the calculus of the preceding section describes ways in which we can alter the plumbing diagram but still leave the 3-manifold described unaltered. For each 3-manifold described by a plumbing diagram there is a minimal such diagram called the *normal form* for the plumbing diagram. This normal form can be obtained by applying the moves R0-R7 and is unique up to application of R0. Thus if we would like to know when two plumbing diagrams describe the same 3-manifold, all we need to do is to convert them both to their respective normal forms and then see if they are equivalent modulo R0. We now describe the normal form and also the algorithm needed to convert a plumbing diagram into normal form.

Definition 2.3.1 *We define a* chain of length k *in a plumbing diagram* Γ *to be any part of the diagram (after reindexing vertices) with distinct vertices* $e_1, e_2, \ldots, e_k$ *such that there is exactly one edge of any sign joining* e_i *to* e_{i+1} *for* $i = 1, \ldots, k-1$ *and also such that* $e_1, e_2, \ldots, e_{k-1}$ *all have valence* 2 *and* e_k *has valence* 1 *or* 2*. We further require that the genus [g] of each* e_i *be zero. See figure 2.16.*

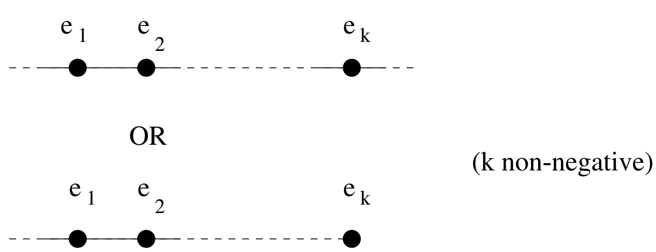


Figure 2.16:

Definition 2.3.2 *The chain is* maximal *if it can be included in no larger chain.*

Definition 2.3.3 (Normal Form) *A plumbing diagram* Γ *is said to be in* normal form *if the following conditions N1-N6 hold.*

N1. None of the operations R1-R7 can be applied to Γ, except that Γ may have components of the form in figure 2.17 with $k \geq 1$ and $e_i \leq -2$ for

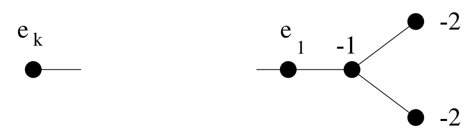


Figure 2.17:

$i = 1, \ldots, k$.

N2. The weights e_i on all chains of Γ satisfy $e_i \leq -2$.

N3. No part of Γ has the form as in figure 2.18 unless it is a component of

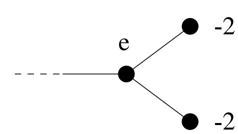


Figure 2.18:

Γ of the form in figure 2.19 with $k \geq 1$ and $e_i \leq -2$ for $i = 1, \ldots, k$.

N4. No portion of Γ has the form in figure 2.20 unless vertex i is an interior vertex of a chain.

N5. No component of Γ has the form in figure 2.21.

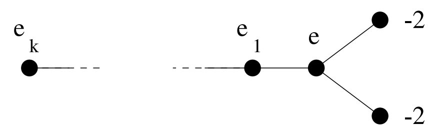


Figure 2.19:

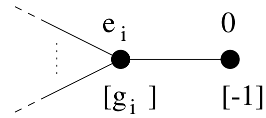


Figure 2.20:

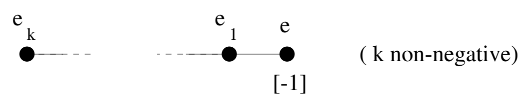


Figure 2.21:

N6. No component of Γ is isomorphic to one of the diagrams in figure 2.22.

Theorem 2.3.1 *Any plumbing diagram can be reduced to normal form using operations R1-R7 and their inverses.*

The proof of this can be found in [16] . However the proof is in the form of an algorithm and parts of this algorithm are to be used frequently. The first step is to apply operations R1-R7 to Γ until no more are applicable. This process will clearly terminate. The second step is to convert a maximal chain into one with all weights ≤ -2. By the first step we cannot have any weights in the chain equal to $-1, 0, 1$. Thus choose the rightmost weight e_i which is ≥ 2 and do sufficiently many (that is $e_i - 1$) (-1)-blow-ups directly to the left of this vertex until vertex i has weight 1. Now do a $(+1)$-blow-down of vertex i. Thus provided $e_{i-1} \neq 2$, we get a chain like the one with which we started except it has fewer positive weights. If $e_{i-1} = 2$, then after

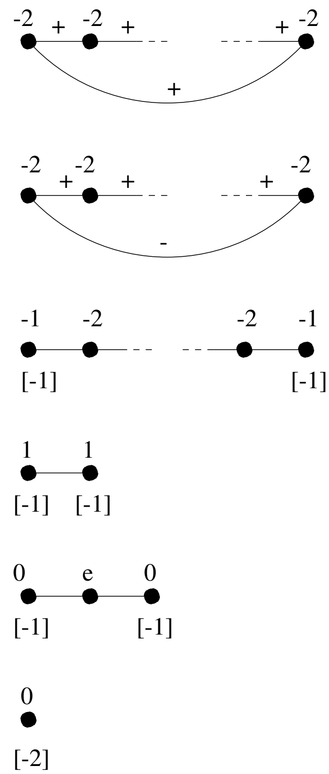


Figure 2.22:

this process vertex $i-1$ will have weight $+1$. If this is the case, blow this vertex down. This will solve all our problem provided vertex $i-2$ didn't start out with weight $+2$, because now it will have weight $+1$. So now we have to blow this vertex down also. It is easy to see that we can keep doing this if we have a whole row of $+2$ weighted vertices and this will eventually remedy the situation provided the chain is not a cycle. However the cycle case if dealt with in the same way as outlined will also be transformed into normal form by this procedure unless all the vertices have weight $+2$. But if all vertices of the cycle have weights $+2$, then this procedure will reduce the diagram to a two vertex graph on which we can to a handle absorption (R4 or R5) and the resulting single vertex graph will be in normal form.

Some examples of reduction to normal form can be found in figures 3.51 to 3.57 on pages 85 to 90. There are further examples in section 4.1.

Definition 2.3.4 *We define the* normal form plumbing diagram *for an arrangement A, $NPl(A)$, to be the normal form of the (immediate) plumbing diagram $Pl(A)$ of the arrangement A.*

2.4 Complement Determines Link

The fundamental reason for studying the link of an arrangement is because for arrangements of curves, the complement determines the link. We now establish this result.

Recall two compact oriented manifolds M, N are h-cobordant if there exists a compact oriented manifold X such that $\partial X = M \cup (-N)$ and M, N have the property that the natural inclusions into X are homotopy equivalences.

Theorem 2.4.1 *If X, Y are compact manifolds with boundaries M, N respectively such that $X \setminus M$ is homeomorphic to $Y \setminus N$, then M, N are h-*

cobordant.

Proof: By the collaring theorem we know there is a collar neighbourhood of M in X so that we can see $X \equiv X \cup ([0,1] \times M)$ where $[0,1] \times M$ is a collar neighbourhood, in fact also a regular neighbourhood for M. Thus we have $X \setminus M \equiv X \cup ([0,1) \times M)$ and similarly $Y \setminus N \equiv Y \cup ([0,1) \times N)$. Now we have

$$X \cup ([0,1) \times M) \overset{\varphi}{\equiv} Y \cup ([0,1) \times N)$$

Now $\{0\} \times M$ disconnects $X \cup ([0,1) \times M)$ into two pieces. One piece is $X \cup \{0\} \times M$ which is compact and the other is $[0,1) \times M$. Hence the image $\varphi(\{0\} \times M)$ must disconnect $Y \cup ([0,1) \times N)$ into two pieces. One is compact namely $\varphi(X \cup \{0\} \times M)$ and the other $\varphi([0,1) \times M)$ is not. Now if $t_1, t_2, \ldots$ is a sequence in $[0,1)$ with $t_i \to 1$ then we certainly have that $\varphi(X \cup \{0\} \times M) \subseteq \cup_{i=1}^{\infty}(Y \cup [0,t_i) \times N) = Y \cup [0,1) \times N$. Thus since $\varphi(X \cup \{0\} \times M)$ is compact we see that there is a $t < 1$ such that $\varphi(X \cup \{0\} \times M) \subseteq Y \cup [0,t) \times N$. However since φ is a bijection we then must also have that $[t,1) \times N \subseteq \varphi([0,1) \times M)$. Thus $\varphi([0,1) \times M)$ is a neighbourhood of $[t,1) \times N$ for some t with $0 < t < 1$.

Also $\varphi^{-1}(\{t\} \times N)$ disconnects $[0,1) \times M$ and we can similarly find t_1 such that $\varphi^{-1}([t,1) \times N)$ is a neighbourhood of $[t_1,1) \times M$. and again find t_2 such that $\varphi[t_1,1) \times M)$ is a neighbourhood of $[t_2,1) \times N$ and t_3 such that $\varphi^{-1}([t_2,1) \times N)$ is a neighbourhood of $[t_3,1) \times M$.

Now consider the manifold Z which is the region between $t_1 \times M$ and $\varphi^{-1}(\{t_2\} \times N)$. (See figure 2.23.) It is clear that $\partial Z \equiv M \cup (-N)$. Set $M_s = [s,1) \times M, N_s = [s,1) \times N$ and $M'_s = \varphi([s,1) \times M), N'_s = \varphi^{-1}([s,1) \times N)$. Now consider the following sequence of inclusions:

$$M_{t_3} \overset{i}{\hookrightarrow} N'_{t_2} \overset{j}{\hookrightarrow} M_{t_1} \overset{k}{\hookrightarrow} N'_t$$

Now because of the product structure in $[0,1) \times M$ we have that the composition $j \circ i : M_{t_3} \hookrightarrow M_{t_1}$ is a homotopy equivalence. Furthermore if i', j', k' are the the maps induced by φ between subspaces of Y, then we also see that $k' \circ j' : N_{t_2} \hookrightarrow N_t$ is a homotopy equivalence. Since φ is a homeomorphism we conclude that $k \circ j : N'_{t_2} \hookrightarrow N'_t$ is a homotopy equivalence. Thus j, having a left inverse and a right inverse, is thus a homotopy equivalence. Similarly k is a homotopy equivalence.

Consider now the following maps:

$$\varphi^{-1}(\{t_2\} \times N) \overset{\alpha}{\hookrightarrow} N'_{t_2} \overset{j}{\hookrightarrow} M_{t_1} \overset{\beta}{\to} Z$$

We can choose β to be the map such that β' is the collapsing map $\beta' : Z' \cup N_{t_2} \to Z'$, being the identity map on Z' and projecting $[t_2, 1) \times N$ onto $\{t_2\} \times N$. One sees that α is a homotopy equivalence because α' is. Also β is a homotopy equivalence fixing $\varphi^{-1}(\{t_2\} \times N)$. We already have seen that j is a homotopy equivalence. Thus we see that the composition $\beta \circ j \circ \alpha$ is a homotopy equivalence and is also the inclusion map $\varphi^{-1}(\{t_2\} \times N) \hookrightarrow Z$. Additionally we have the maps:

$$\{t_1\} \times M \overset{\gamma}{\hookrightarrow} M_{t_1} \overset{\beta}{\to} Z$$

These maps once again are homotopy equivalences and whose composition is the inclusion map $\{t_1\} \times M \hookrightarrow Z$. See figure 2.23.

□

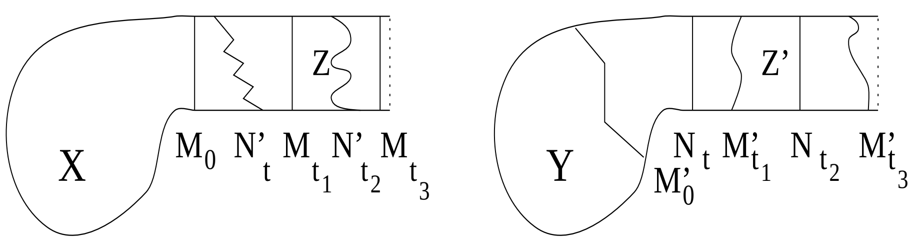


Figure 2.23:

Corollary 2.4.2 *Let C_1, C_2 be homeomorphic complements of arrangements of algebraic curves A_1, A_2. Then their links are h-cobordant.*

Proof: Let L_i be the link for A_i. Then one sees that C_i is homeomorphic to $C_i \cup ([0,1) \times L_i)$ so that we can cap C_i with a boundary L_i to get a compact manifold X_i with boundary L_i. Applying theorem 2.4.1 yields the result.

□

Incidentally this result is much more general than stated. For example if C_1, C_2 are homeomorphic complements of subsets A_1, A_2 of any manifolds such that both A_1, A_2 have regular neighbourhoods. Then their links are h-cobordant. A particularly interesting version of this is when we have immersed surfaces in a 4-manifold.

Theorem 2.4.3 (Turaev [20]) *The following three conditions for a pair of geometric 3-manifolds are equivalent.*

1. M is homeomorphic to N.

2. M is simple homotopy equivalent to N.

3. M is h-cobordant to N.

Here geometric means those 3-manifolds which after splitting along incompressible spheres (see [15]) and tori (JSJ decomposition, see [7] and [11]), have geometric pieces i.e. are Seifert fibred, Haken or Hyperbolic. We have such a decomposition in our case. Firstly separating out disjoint components of a plumbing graph corresponds to splitting along incompressible spheres. Secondly, in the connected components that remain, each edge represents a gluing along a torus, however the set of tori here does not necessarily form a minimal family for the JSJ splitting, but a subset of them does. The pieces remaining are certainly geometric, all of them being Seifert fibred.

Corollary 2.4.4 *For arrangements of complex projective curves, be they singular or having non-transversal intersections, the complement determines the link.*

Proof: Let C_1, C_2 be two homeomorphic complements of such arrangements. Then the respective links L_1, L_2 are h-cobordant. Yet L_1, L_2 are possibly reducible graph manifolds and are thus geometric. Applying Turaev's theorem yields the result.

□

Again this result would also hold for arrangements of immersed surfaces in a 4-manifold whose singular points can be resolved by blowing up of points.

Chapter 3

Links and Arrangements : The Normal Nonsingular Case

3.1 Axiomatizing Ambiguity

We begin by asking the following question: Is it possible that two different arrangements A_1, A_2 could have homeomorphic links? Equivalently, is it possible that the link does not determine the arrangement? The answer is yes.

Let A_1 consist of a triangle of lines, a cubic passing through the triangle's vertices and a line in general position. Let A_2 consist of a triangle of lines, a cubic passing through the triangle's vertices and a conic passing through the triangle's vertices. Then the plumbing diagrams are shown in figure 3.1.

Both plumbing diagrams are in normal form already and are clearly isomorphic, (although they are not isomorphic as marked plumbing diagrams), hence have same link.

We would like to quantify when this phenomenon occurs.

Definition 3.1.1 *An arrangement is said to be* normal *if* $Pl(A) = NPl(A)$, *that is the immediate plumbing diagram is already in normal form.*

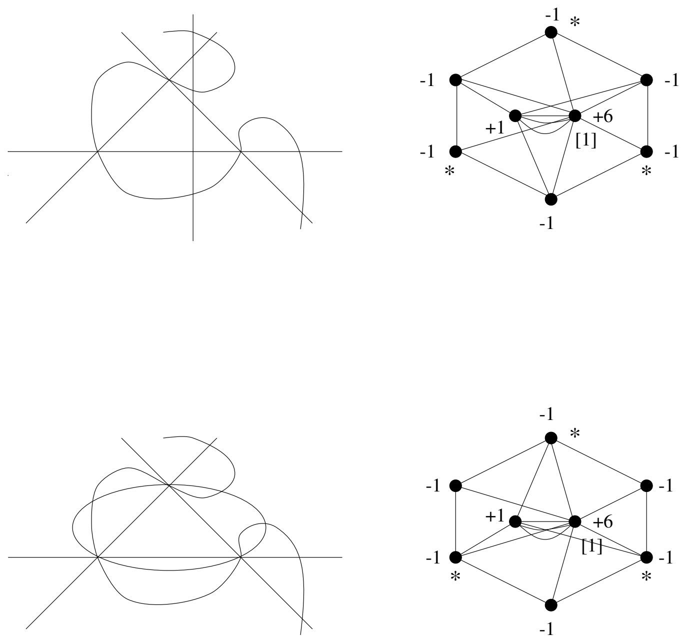


Figure 3.1:

Definition 3.1.2 *An arrangement is said to be* regular *if all curves in the arrangement are nonsingular and the pairwise intersections are all transverse.*

The two arrangements shown in figure 3.1 are both regular and normal. We begin by studying the class of normal, regular arrangements. In section 4 we will extend this to include all regular arrangements.

Many regular arrangements can be immediately seen to be normal as the following shows.

Theorem 3.1.1 *Let A be a regular arrangement of curves in $\mathbb{CP}^2$ such that each curve has at least three intersection points on it. Then every vertex of $Pl(A)$, the plumbing graph of A, has valence at least three.*

Proof: If $E \in \overline{A}$ is an exceptional divisor, then E must have arisen by blowing up a point with n curves $C_1, C_2, ..., C_n$ $(n \geq 3)$ passing through it. Hence E intersects the n curves $\overline{C}_1, \overline{C}_2, ..., \overline{C}_n$ in $\overline{A}$ and hence has valence $n \geq 3$ in $Pl(A)$.

If $\overline{C} \in \overline{A}$ is not an exceptional divisor, then C had $n \geq 3$ intersection points in A. Those points which have exactly one other curve D_i passing through them do not get blown up so that $\overline{C}$ is joined to $\overline{D}_i$ in $\overline{A}$. Those points which have at least two other curves passing through them do get blown up to give one exceptional divisor $E_j \in \overline{A}$ corresponding to each such point, and each of the E_j intersects $\overline{C}$.

Hence $\overline{C}$ is joined to exactly n curves in $\overline{A}$ and so has valence at least three in $Pl(A)$ □

Corollary 3.1.2 *Let A be a regular arrangement of curves in $\mathbb{CP}^2$ such that each curve has at least three intersection points on it. Then A is also a normal arrangement.*

Proof: All the moves of the plumbing calculus used in reduction to normal form involve vertices of valence at most two. □

Let A_1, A_2 be two normal, regular arrangements in $\mathbb{CP}^2$. Furthermore assume that their immediate plumbing graphs $Pl(A_1)$ and $Pl(A_2)$ are both isomorphic to say $Pl(A)$ so that the links of A_1 and A_2 are both (abstractly) homeomorphic to the link of the abstract blown-up arrangement $\overline{A}$ in a blown-up surface, say where (see figure 3.2) the α_i are the isomorphisms

$$\alpha_i : A \to \overline{A_i} \quad (i = 1, 2)$$

and ϕ_i are the projections (blowing downs)

$$\phi_i : \overline{A_i} \to A_i (i = 1, 2)$$

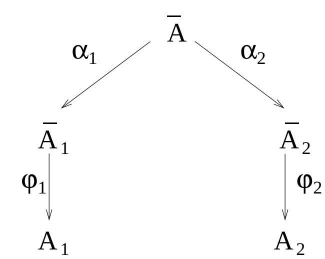


Figure 3.2:

Also define $\pi_i = \phi_i \circ \alpha_i$ (for $i = 1, 2$).

Definition 3.1.3 *A curve is said to be* regular *with respect to* π_i *if it is not blown down.*

Definition 3.1.4 *A curve is said to be* exceptional *with respect to* π_i *if it is blown down.*

Let R_i be those curves regular with respect to π_i and let E_i be those curves exceptional with respect to π_i.

Definition 3.1.5 *A curve* $\overline{C} \in \overline{A}$ *is said to be* ambiguous *if it is regular with respect to one* π_i *and exceptional with respect to the other. Equivalently* $\overline{C} \in \overline{A}$ *is ambiguous means that* $\overline{C} \in (R_1 \cap E_2) \cup (R_2 \cap E_1)$.

Note that we could also see these definitions as relating to a plumbing diagram that can be marked in two different ways. Two arrangements A_1 and A_2 which are different but have equivalent links amounts to $Mark(A_1)$ and $Mark(A_2)$ being non-isomorphic as marked plumbing diagrams, but isomorphic if we ignore the markings. Now let G be an unmarked plumbing diagram which is isomorphic to $Pl(A_1)$ and $Pl(A_2)$ so that we have maps $\alpha_i : Mark(A_i) \to G$, which are the maps that forget the markings. We see that the regular vertices of G are by definition 3.1.3 those vertices which correspond to being unmarked in both $Mark(A_1)$ and $Mark(A_2)$. Similarly the exceptional vertices of G are by definition 3.1.4 those vertices which correspond to being marked in both $Mark(A_1)$ and $Mark(A_2)$. The ambiguous vertices of G are those vertices which correspond to being marked in one $Mark(A_i)$ but not in the other.

Let K be the subgraph of $Pl(\overline{A})$ on the vertices representing the set of ambiguous curves. Differences in A_1, A_2 may arise because we may have to blow down different curves to go from A to A_1 than in going from A to A_2 as was demonstrated in the example in figure 3.1.

To go from A to A_1, one must blow down the curves corresponding to the dark (blue) coloured vertices (see figure 3.3).

But to go from A to A_2 one must blow down the curves corresponding to the light (red) coloured vertices. In our case K is the graph shown in figure 3.4.

Now if K is nonempty, colour each vertex of K which is regular with respect to π_1 red and colour each vertex that is regular with respect to π_2 blue.

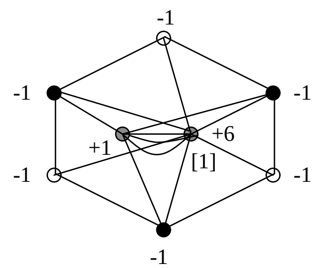


Figure 3.3:

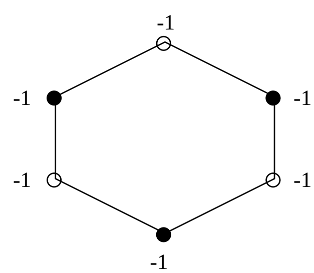


Figure 3.4:

Theorem 3.1.3 *K has the following properties:*

K1 *$K \neq \phi$.*

K2 *All vertices of K have genus zero and weights (self intersection numbers)minus one.*

K3 *Each vertex of K is coloured either red or blue.*

K4 *No vertices of the same colour are joined by an edge.*

K5 *There are no double edges or loops.*

K6 *Every vertex has valence either two or five.*

K7 *For any pair (u, v) of vertices of the same colour, denote by $n(u, v)$ the number of vertices $w \in K$ such that both u and v are joined to w. Then*

$$\begin{aligned} n(u(2), v(2)) &= 1 \\ n(u(2), v(5)) &= 2 \\ n(u(5), v(5)) &= 4 \end{aligned}$$

where the notation $u(i)$ means that u is a vertex of valence i.

K8 *K is connected.*

Proof:

K1 This is part of the definition of K.

K2 If v is a red vertex, then v is exceptional with respect to π_2 and hence has genus zero and weight minus one. Similarly if v is a blue vertex.

K3 This is part of the definition of K.

K4 Assume that v and w are two red vertices joined by an edge. But then v and w are both exceptional with respect to π_2 and two exceptional divisors cannot intersect (contradiction).

K5 Loops cannot occur because all curves are assumed to be nonsingular. Also if v and w have two edges between them, without loss of generality we can assume by (K4) that v is red and w is blue, then with respect to π_2, v is exceptional and after blowing v down w has two points identified and hence must be singular (contradiction).

K6 Let v be without loss of generality a red vertex of K. Now since all vertices of K represent nonsingular curves of genus zero, we see that with respect to π_1, v must be either a line or a non-degenerate conic. Assume that v has valence n where n is a non-negative integer, hence in view of (K4) we see that v is joined to n blue vertices $w_1, w_2, ..., w_n$ and these must all be exceptional with respect to π_1. Hence after blowing down via π_1 we see that $\pi_1(v)$ has self-intersection number $n-1$ (The result of doing n blow-ups on a -1-curve). However since v is either a line or a non-degenerate conic which have self-intersection numbers $+1$ and $+4$ respectively we see that $n=2$ or $n=5$.

K7 Let u and v be without loss of generality red vertices of K (so u has valence 2 if u is a line with respect to π_1 or valence 5 if u is a conic with respect to π_1). Now by (K4) u and v are not joined, hence any intersection points of $\pi_1(u)$ and $\pi_1(v)$ in A_1 must be blown up under π_1^{-1}. Hence u and v are joined to a vertex w in $Pl(\overline{A})$ of weight minus one. Now assume that $w \in Pl(\overline{A}) \setminus K$, so then w is exceptional with respect to both π_1 and π_2. Hence in particular with respect to π_2 all of u, v, w are exceptional. This is a contradiction because u is joined to w and no two exceptional curves can intersect. Hence $w \in K$ and by (K4), w is blue. So for each intersection point of $\pi_1(u)$ and $\pi_1(v)$ in

A_1 there is an associated blue vertex to which u and v are both joined. The number of such vertices is of course the product of the degrees of the curves represented by u and v by Bezout's theorem. Conversely suppose that u and v are both joined to a red w. Then after applying π_1 we find u and v intersecting. Hence the correspondence between blue vertices w to which both u and v are joined and the intersection points of $\pi_1(u)$ and $\pi_1(v)$ in A_1 is bijective. Now since all crossings are normal, apply Bezout's theorem, we see that $|\text{line} \cap \text{line}| = 1$, $|\text{line} \cap \text{conic}| = 2$ and $|\text{conic} \cap \text{conic}| = 4$, thus proving (K7).

K8 (i) If u and v are the same colour (without loss of generality red), then applying (K7) and noting that $n(u, v) \geq 1$ yields a path of length two joining them. (ii) If u and v are different colours say without loss of generality u is red and v is blue, then by (K6) v is joined to at least two vertices so say v is joined to w. By (K4) w is red. But now by (i) u and v have a path between them. Note that in all cases the length of the path is at most three.

Hence we have verified (K1)-(K8). □

3.2 Determining Ambiguity - Classifying K

It so happens that the properties K1-K8 that K has, force K to be one of four possible graphs. We now try to classify combinatorially all possible graphs K satisfying (K1)-(K8). Clearly the critical elements which will determine K are the properties (K3)-(K7).

Now since $K \neq \phi$ by (K1), K has a vertex. By (K4) and (K6) K has a vertex of each colour and in fact by (K4) and (K6) again K has at least two vertices of each colour.

Case 1 : K has no blue vertices of valence 5.

Hence it has at least two blue vertices of valence 2. Now it also has at least two red vertices say r_1, r_2. Now if r_1 had valence 5, then $n(r_1, r_2) \geq 2$ by (K7), hence there would exist two blue vertices B_1, B_2 such that both r_1 and r_2 are joined to both $B_1$and B_2 as in figure 3.5.

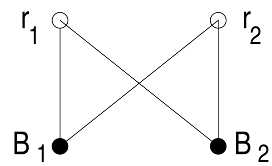


Figure 3.5:

But then $n(B_1(2), B_2(2)) \geq 2$. This contradicts (K7). Hence all red vertices have valence 2. However all blue vertices have valence 2. This together with the fact that K is connected (K(8)) implies that K is a cycle. furthermore by (K4), the vertices alternated in colour. Hence K is a 2n-gon $n \geq 2$ with vertices $b_1, b_2, \ldots, b_n, r_1, r_2, \ldots, r_n$. If K is a 4-gon we get figure 3.6 which we have seen contradicts (K7).

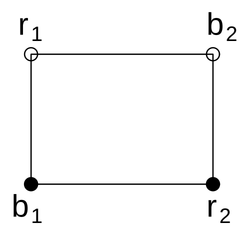


Figure 3.6:

If K is a 2n-gon with $n \geq 4$, then part of K looks like figure 3.7.

This contradicts (K7) since we require $n(b_1, b_3) = 1$, however b_1 and b_3 have no further edges extending from them. This only leaves the possibility

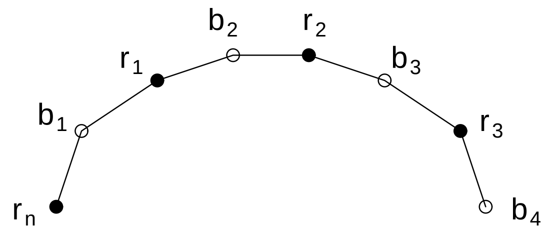


Figure 3.7:

that K is a hexagon. Thus we get figure 3.8 which is easily seen to satisfy (K1)-(K8).

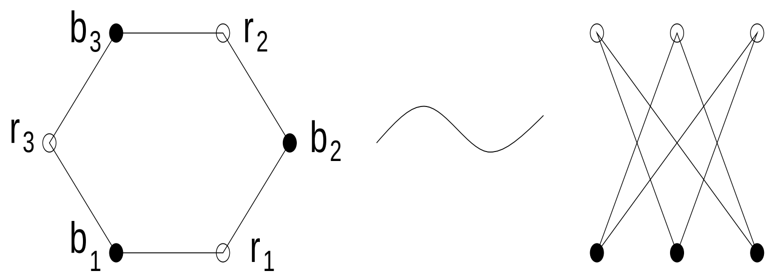


Figure 3.8:

Case 2 : K has no blue vertices of valence 2.

Lemma: K has no red vertex of valence 2.

Proof: Assume that r_1 is a red vertex of valence 2. Now r_1 is joined to two blue vertices say b_1, b_2 (of valence 5). Then by (K7) $n(b_1(5), b_2(5)) = 4$. However b_1 and b_2 are already both joined to r_1, hence let them also be joined to r_2, r_3, r_4.

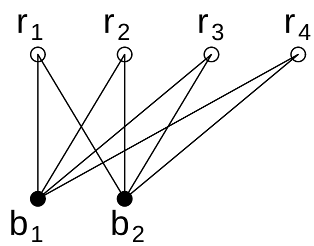


Figure 3.9:

Now we can easily see from figure 3.9 that $n(r_1(2), r_2) \geq 2$, hence by (K7) we see that r_2 must have valence 5 and similarly so do r_3 and r_4. Remembering that the valence of b_1 is 5, let r_5 be the fifth vertex joined to b_1. Now r_5 has valence 2 or 5, hence the number of edges emanating from r_1, r_2, r_3, r_4, r_5 is either $2+5+5+5+2 = 19$ or $2+5+5+5+5 = 22$. However we can count these edges in a different way. Let there be n other blue vertices $b_2, b_3, \ldots, b_{n+1}$ besides b_1. However by (K7) $(n(b_1(5), b_i(5) = 4)$ each of $b_2, b_3, \ldots, b_{n+1}$ must be joined to four of the r_1, r_2, r_3, r_4, r_5 and when we remember that b_1 is joined to all five of r_1, r_2, r_3, r_4, r_5 we see that this accounts for all the edges between r_1, r_2, r_3, r_4, r_5 and the b_i. This yields the equation $4n + 5 = 19$ or 22. Hence $4n = 14$ or 17 which is clearly impossible for non-negative integers n thus establishing the lemma. □

Hence all vertices have valence 5. Now again let b_1 be a blue vertex and $r_1, \ldots, r_5$ the five red vertices joined to it. Once again assume there are n other blue vertices $b_2, \ldots, b_{n+1}$ besides b_1 and again count the number of edges emanating from the $r_1, \ldots, r_5$ in two ways. The direct sum of all the valences of the $r_1, \dots, r_5$ yields the number of edges to be 25 and using (K7) each $b_i (i \geq 2)$ satisfies $n(b_1, b_i) = 4$,hence each b_i is joined to four of the $r_1, \dots, r_5$ and adding on the five edges from b_1 to $r_1, \dots, r_5$ yields $4n+5 = 25$. Thus $n = 5$. Hence if K has no blue vertex of valence 2, then K has precisely six blue vertices of valence 5. Similarly because we have already established in the lemma that K has no red vertices of valence 2, then K has precisely six red vertices of valence 5. From the combinatorics we can easily draw as much of the graph as in figure 3.10 and the remaining edges are forced, that is all $b_2, \dots, b_6$ are joined to r_6. Rearranging and relabeling yields figure 3.11

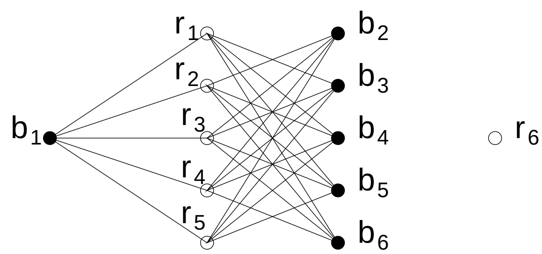


Figure 3.10:

It is easy to see that this satisfies (K1)-(K8).

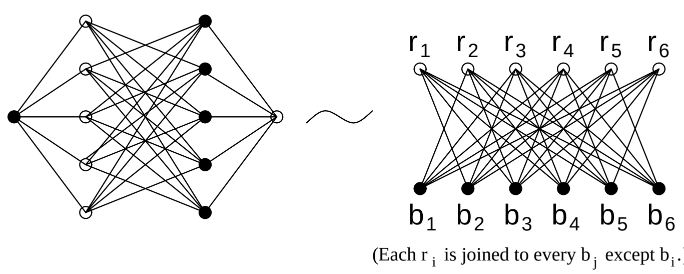


Figure 3.11:

Now what remains is if K has a blue vertex say b_1 of valence 5 and a blue vertex say b_2 of valence 2. Now by (K7) $n(b_1(5), b_2(2)) = 2$, hence we have the diagram as in figure 3.12. There are four possible cases by (K6) for the

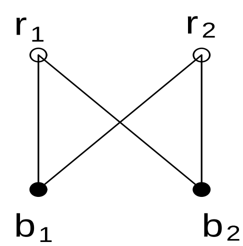


Figure 3.12:

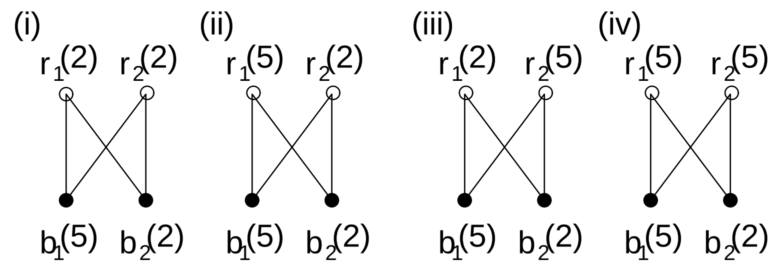


Figure 3.13:

valences of r_1 and r_2, namely those shown in figure 3.13

Now (i) is impossible since the diagram shows that $n(r_1(2), r_2(2)) \geq 2$ contradicting (K7). Also (ii) is equivalent to (iii), hence only (ii) needs discussing. We shall discuss (iv) separately. In any case we have two further cases to study.

Case 3 : K has a subdiagram of the form in figure 3.14.

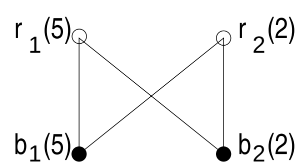


Figure 3.14:

Now let b_3 be any other blue vertex. If b_3 has valence 5, then by (K7) $n(b_2(2), b_3(5)) = 2$. Hence b_2 and b_3 are both joined to two red vertices.

However b_2 is already joined to r_1 and r_2 and has valence 2, hence this forces b_3 to be joined to both r_1 and r_2. This contradicts that the valence of r_2 is 2. See figure 3.15.

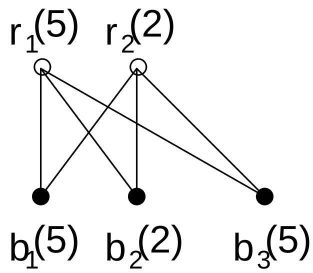


Figure 3.15:

Now r_1 has valence 5, hence we know that there must exist at least three other blue vertices b_3, b_4, b_5, all of which by the preceding remarks must have valence 2. By exactly the same reasoning b_1 is joined to three further red vertices r_3, r_4, r_5 all of valence 2 and so we have as much of the diagram as shown in figure 3.16.

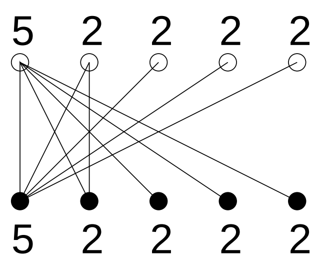


Figure 3.16:

Now if there were another blue vertex b_6, then by (K7) $n(b_2(2), b_6) = 1$ or 2. But b_2 has valence 2 and is already joined to r_1 and r_2, hence b_6 must be joined to at least one of r_1, r_2. However r_1 and r_2 already have their full quota of valences and neither can afford to be joined to b_6. Hence we have all the blue vertices and their valences. Similarly we have all the red vertices and their valences. Now each of b_3, b_4, b_5 is already joined to r_1. However b_3, b_4, b_5 each have valence 2 and r_2 cannot have any more edges. Hence it is easy to see that b_3, b_4, b_5 must be joined in one to one correspondence with r_3, r_4, r_5 (without loss of generality b_i is joined to r_i for $i = 3, 4, 5$), and the final configuration as shown in figure 3.17 satisfies (K1)-(K8).

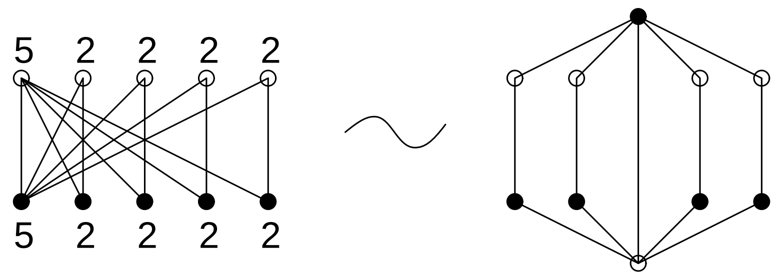


Figure 3.17:

Case 4 : K has a subdiagram of the form in figure 3.18.

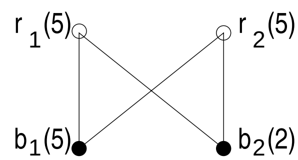


Figure 3.18:

Now by (K7) $n(r_1(5), r_2(5)) = 4$, so there are two more blue vertices b_3, b_4 such that both r_1 and r_2 are joined to b_3 and b_4. Now consider the subdiagram defined by r_1, r_2, b_2, b_3 (see figure 3.19).

We see that $n(b_2(2), b_3) \geq 2$, hence by (K7) b_3 cannot have valence 2 and hence b_3 has valence 5. Similarly b_4 has valence 5. Also r_1 has valence 5 hence there is a fifth blue vertex b_5 such that r_1 is joined to b_5 as shown in figure 3.20.

If r_2 were joined to b_5, then from the diagram r_1 and r_2 would be joined to all of b_1, b_2, b_3, b_4, b_5, thus $n(r_1(5), r_2(5)) \geq 5$ which contradicts (K7). Hence

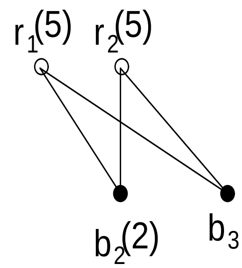


Figure 3.19:

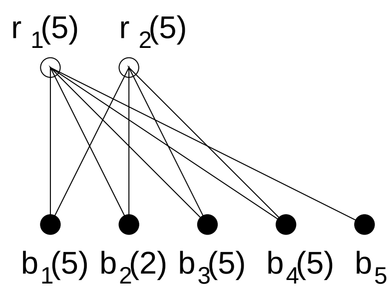


Figure 3.20:

r_2 is not joined to b_5. However because r_2 has valence 5, a further blue vertex b_6 must exist such that r_2 is joined to b_6.

Now if b_5 had valence 5, then (K7) forces $n(b_2(2), b_5) = 2$ and since b_2 has valence 2, b_5 must be joined to both r_1 and r_2, but this causes r_2 to have valence at least 6 contradicting (K6). Hence b_5 has valence 2. Similarly b_6 has valence 2.(See figure 3.21.)

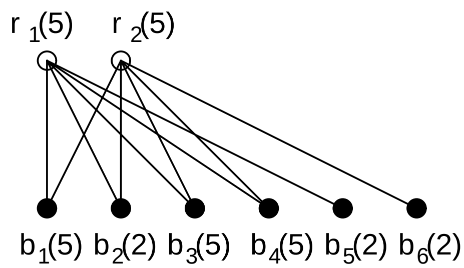


Figure 3.21:

Now assume a further blue vertex b_7 exists. Then by (K7) $n(b_2(2), b_7) \geq 1$, hence b_7 is joined to at least one of r_1, r_2. But this would cause either r_1 or r_2 to have valence at least six which is impossible from (K6). Hence we have all the blue vertices of K - three are of valence 2 and three are of valence 5. (See figure 3.22.)

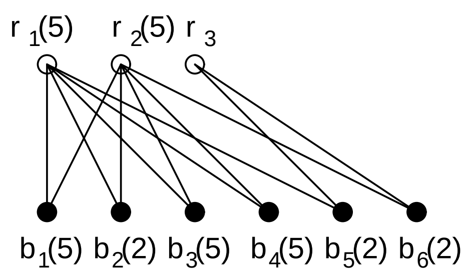


Figure 3.22:

Now by (K7) $n(b_5(2), b_6(2)) = 1$, hence both b_5 and b_6 are joined to a red vertex r_3 (since b_6 cannot be joined to r_1 nor b_5 to r_2). Now if r_3 had valence 2 then by (K7) $n(r_2(5), r_3) = 2$ and so r_2 would have to be joined to b_5 which has already been ruled out. Hence r_3 has valence 5. Since we have established that $b_1, \ldots, b_6$ are all the blue vertices we see that $r_3(5)$ is joined to five of them. However r_3 cannot be joined to $b_2(2)$ because b_2 already has its full quota of edges. Thus r_3 is joined to b_1, b_3, b_4, b_5, b_6. Tidying up by rearranging and relabeling yields the picture so far in figure 3.23.

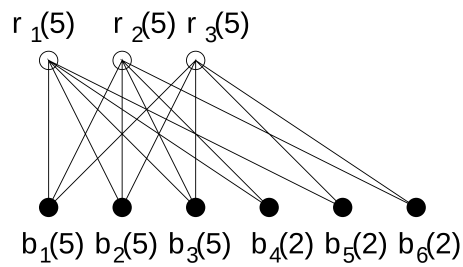


Figure 3.23:

Now by (K7) $n(b_1(5), b_2(5)) = 4$, hence there exists a further red vertex r_4 with both b_1 and b_2 joined to r_4. Now if r_4 had valence 5, then since we already have all the blue vertices, by the pigeonhole principle b_4 is joined to one of b_4, b_5, b_6 which contradicts the fact that b_4, b_5, b_6 all have valence 2. Hence r_4 has valence 2. A similar argument shows that any further r_i $(i \geq 4)$ must have valence 2. Also by (K7) b_1 and b_3 must both be joined to an r_5 (of valence 2) and b_2 and b_3 must both be joined to an r_6 (of valence 2).
We thus get the completed picture in figure 3.24 which satisfies properties (K1)-(K8).

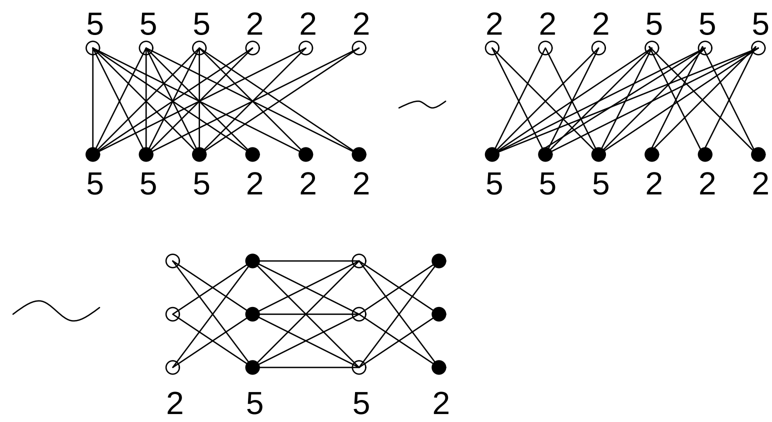


Figure 3.24:

We now exhibit the subarrangements of ambiguous curves in A_1 corresponding to the four cases to verify they all exist geometrically.

Case 1. We get a triangle of lines. See figure 3.25.

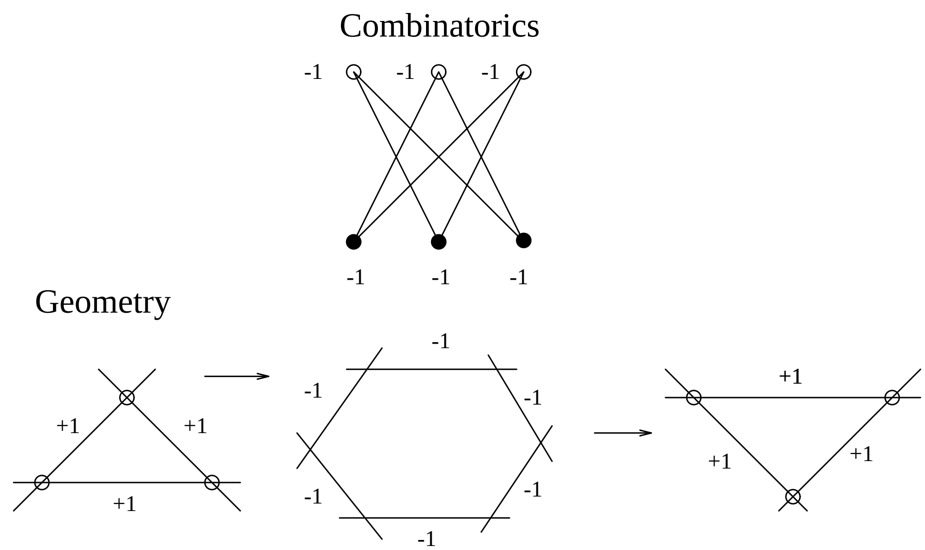


Figure 3.25: Case 1

N.B. The circles represent points to be blown up (curves in $\overline{A}$ that have been blown down via π_1).

Case 2. We get in A_1 six points which are in sufficiently general position with respect to conics (that is all six points are not conconic and no three are collinear) such that a conic passes through each 5-tuple of points (and not the sixth). Note that this forces each pair of conics to have four of the six points in common and these are all their intersection points by Bezout's theorem. See figure 3.26.

Case 3. It is easy to see that this represents four lines and a conic all passing through a single point with the conic intersecting each line at one other place. See figure 3.27.

Case 4. It is easy to see that the three valence 2 blue vertices could only possibly have come from three lines (since they pairwise intersect in only one point i.e. have only one red vertex in common). Furthermore the three valence 5 blue vertices must have come from three conics (this comes from considering their self intersection numbers after blowing down the red vertices), all passing through the three points of intersection of the lines, but also intersect pairwise in one other point. As before the six intersection points must be in general position with respect to conics with the possible exception that the three intersection points which do not lie on the three lines may be collinear. See figure 3.28.

Hence all combinatorially possible situations for K occur geometrically. Now let us remember that K was only a subgraph of $Pl(A)$. Hence we must see also how other curves could intersect with the four cases just shown. We will do this from the perspective of π_1. But first we need a few preliminary results.

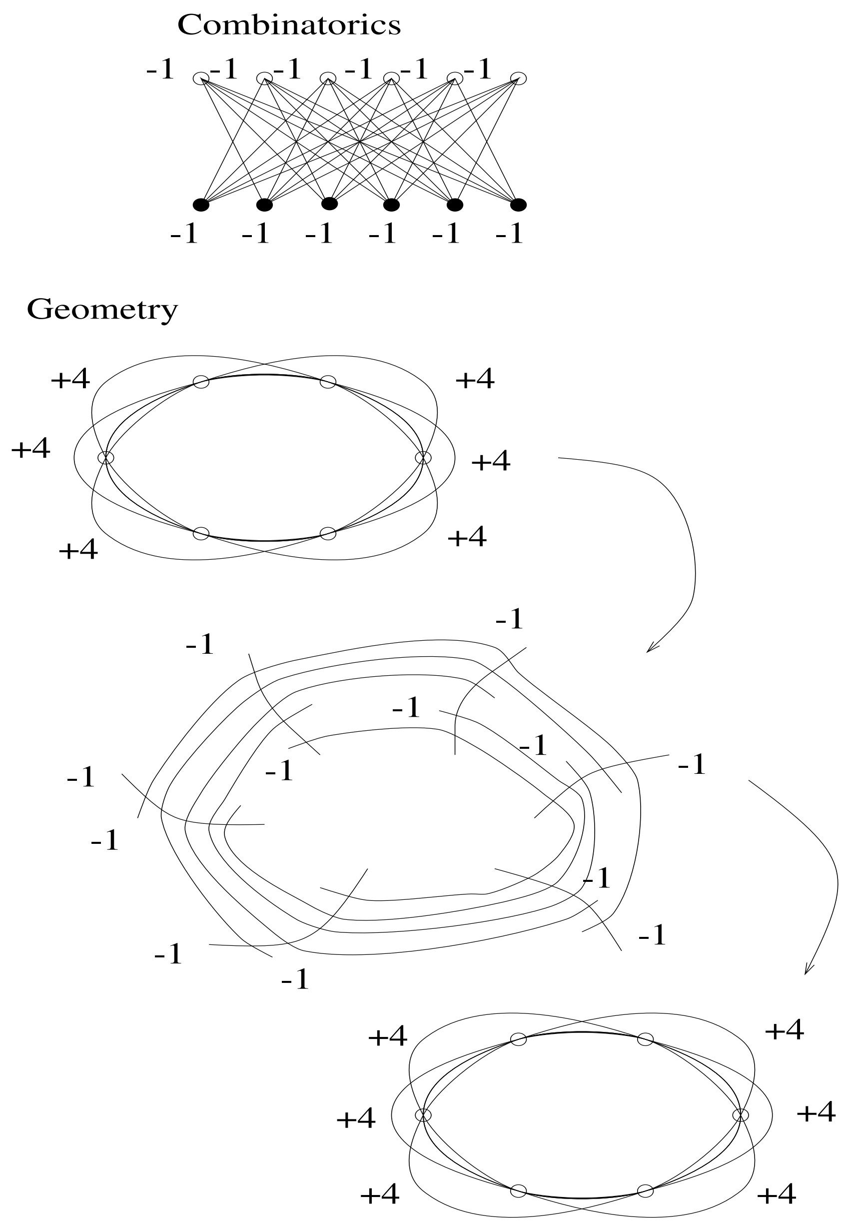


Figure 3.26: Case 2

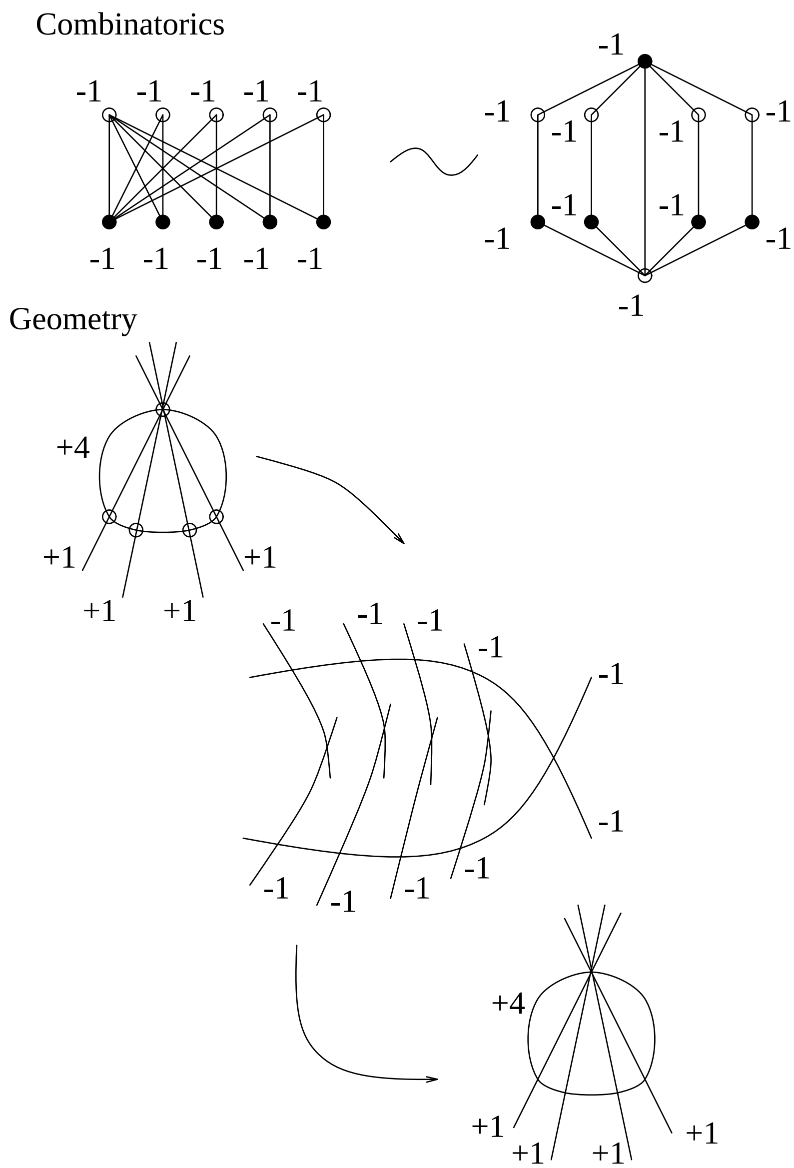
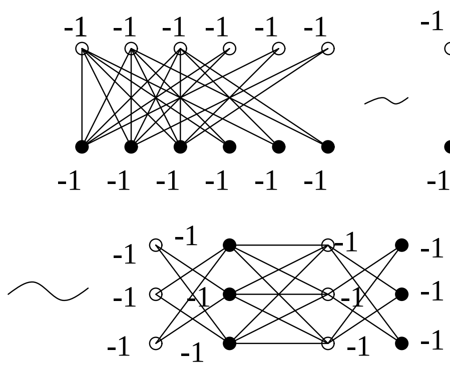


Figure 3.27: Case 3

Combinatorics



Geometry

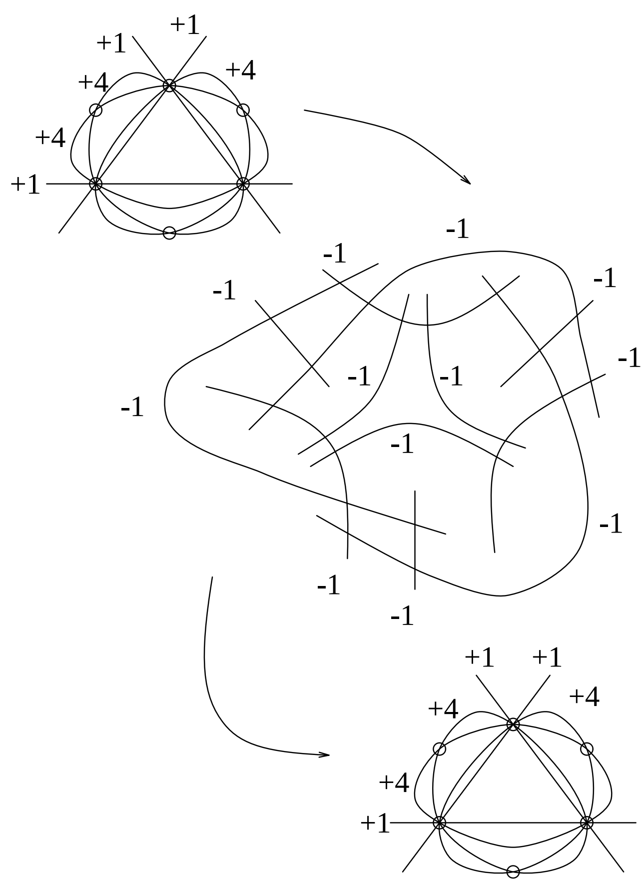


Figure 3.28: Case 4

3.3 How Other Curves may Intersect the Ambiguous set

Lemma 3.3.1 *No vertex of K can be joined in $Pl(A)$ to a vertex in $Pl(A)\backslash K$ which is exceptional with respect to either π_1 or π_2.*

Proof: If this were so, say $k \in K$ were joined to $l \in Pl(A) \setminus K$ where k is without loss of generality exceptional with respect to π_1. But l is exceptional with respect to either π_1 or π_2 and since l is not in K, it is not ambiguous. Thus l must be exceptional with respect to both π_1 and π_2. Hence with respect to π_1 both k and l are exceptional divisors and hence cannot intersect. □

Corollary 3.3.2 *The subarrangement of ambiguous curves with respect to (without loss of generality) π_1 has the property that apart from their natural intersection points, no other points on them can be blown up when seen in A_1. Hence in particular if three curves in A_1 intersect at a point P, then this point is either one of the intersection points of the subarrangement of ambiguous curves, or else is not on any ambiguous curve.*

Hence when investigating how a curve C intersects with the ambiguous subarrangement, if P is a point blown up on C, then either P is not part of the ambiguous subarrangement and hence under π_1^{-1}, P is blown up, but is blown straight down again under π_2, of else P is one of the intersection points of the ambiguous subarrangement. Also note that any intersections of C with the ambiguous subarrangement not at the special intersection points are part of a curve which gets blown down. In particular we have

Lemma 3.3.3 *A curve C cannot intersect one of the ambiguous curves in two places P_1 and P_2 such that neither P_1 nor P_2 are intersection points of the subarrangement of ambiguous curves.*

Proof: If this were so then the image of C under $\pi_2 \circ \pi_1^{-1}$ would identify P_1 and P_2 to a single point thus causing the image to be singular - contradiction. □

We can now begin discussing the various cases.

Case 1(i): A line l intersects the configuration without passing through any ambiguous intersection points. See figure 3.29.

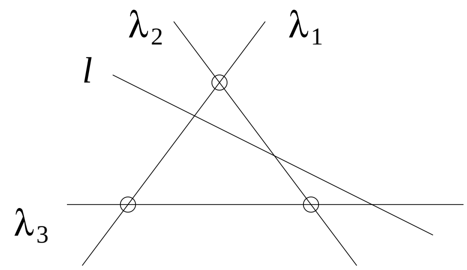


Figure 3.29: Case 1(i)

Blowing up the three intersection points leaves the self intersection number of l at $+1$ and blowing down the three lines increases the self intersection number of l to $+4$. Note that any other points getting blown up by π_1^{-1} get blown straight back down by π_2 since the vertices of K constituted the ambiguous set, and hence any other points play no part in this type of discussion (compare corollary 3.3.2).

The only candidate for $l' = \pi_2(\pi_1^{-1}(l))$ is thus a conic which passes through the three points $\lambda_i' = \pi_2(\pi_1^{-1}(\lambda_i))$ $i = 1, 2, 3$. This is clearly possible in essentially only one way as shown in figure 3.30.

Note that (of course) the two configurations have the same immediate plumbing diagram as in figure 3.31.

Strictly speaking the first has plumbing diagram as in figure 3.32 (which is of course equivalent to the plumbing diagram in figure 3.31), however we had to blow up the three intersection points, reminding us that there were other curves passing through these intersection points.

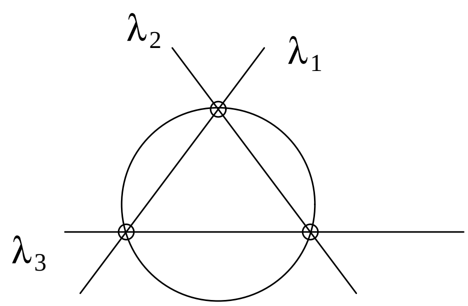


Figure 3.30: Case 1(i)

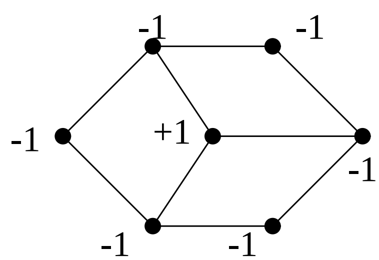


Figure 3.31: Case 1(i)

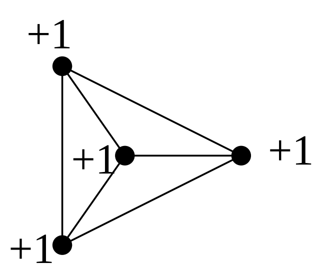


Figure 3.32: Case 1(i)

Comment: This type of phenomenon where the configurations in A_1 and A_2 are different is an example of a *non-self-inverse* scenario. This can be detected in the asymmetry in the plumbing diagram of the ambiguous set together with the curve l with respect to blue and red vertices. See figure 3.33.

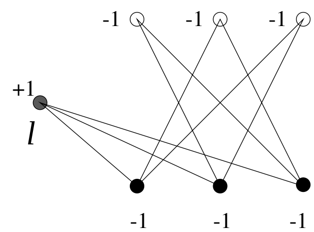


Figure 3.33: Case 1(i)

Henceforth such lengthy explanation will be omitted since the same goes for all of the following discussion.

Case 1(ii): A line l intersects the configuration and passes through exactly one ambiguous intersection point.

Blowing up decreases the self intersection number of l to 0 and blowing down then increases it back to $+1$, so that l' can only be a line. See figure 3.34.

Note that the first diagram looks like the last. This can be detected in the symmetry of the plumbing diagram (figure 3.35) with respect to blue and red vertices.

This is an example of a *self-inverse* scenario.

Case 1(iii): A conic γ intersects the configuration without passing through any ambiguous points.

This cannot occur by lemma 3.3.3 since a conic intersects a line in two points and so γ' would have to be singular. See figure 3.36 which demonstrates this.

Case 1(iv): A conic γ passes through exactly one ambiguous point.

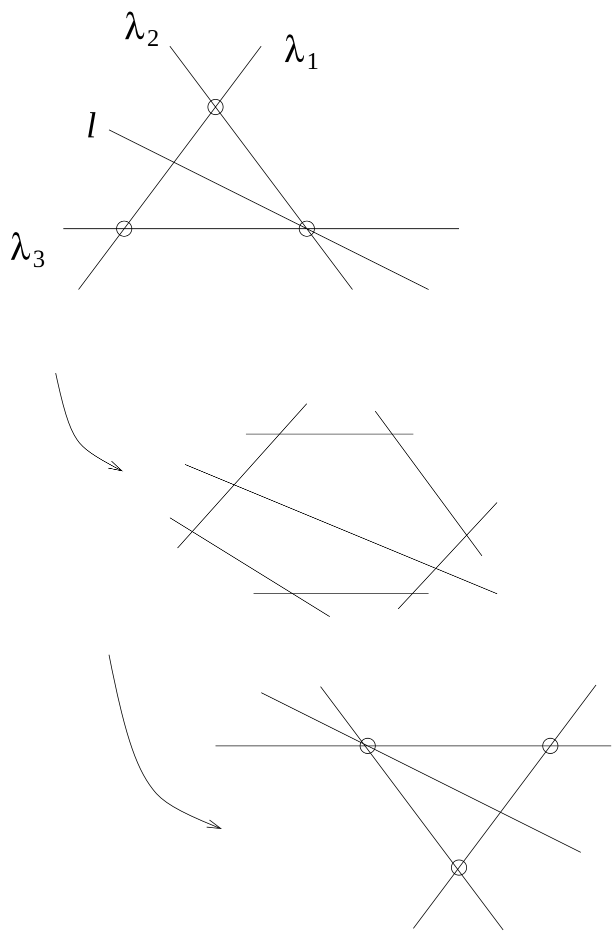


Figure 3.34: Case 1(ii)

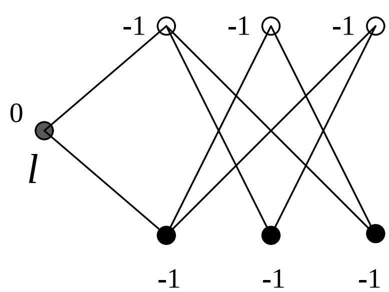


Figure 3.35: Case 1(ii)

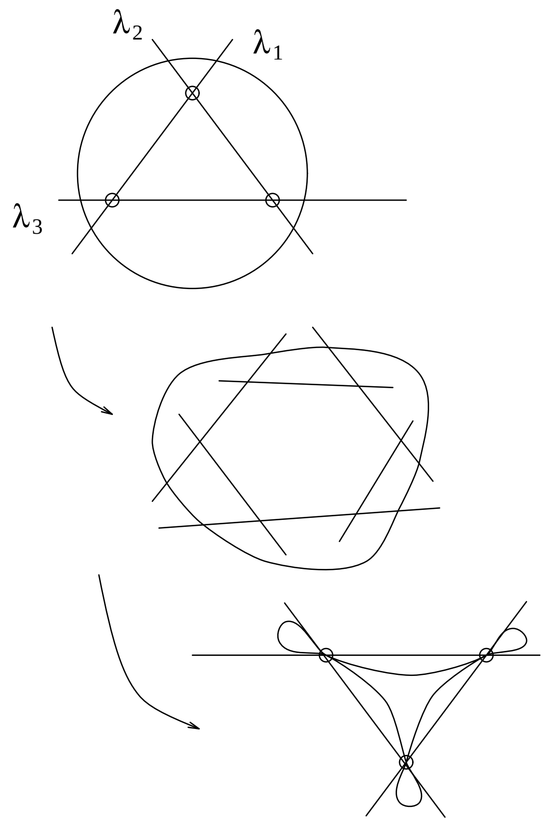


Figure 3.36: Case 1(iii)

This cannot occur by lemma 3.3.3 because then it intersects an ambiguous line in two non-ambiguous points. Hence this case does not occur.

Case 1(v): A conic γ passes through exactly two ambiguous intersection points.

Blowing up decreases the self intersection number of γ from $+4$ to $+2$ and blowing down increases it back to $+4$. This is possible in essentially only one way as shown in figure 3.37.

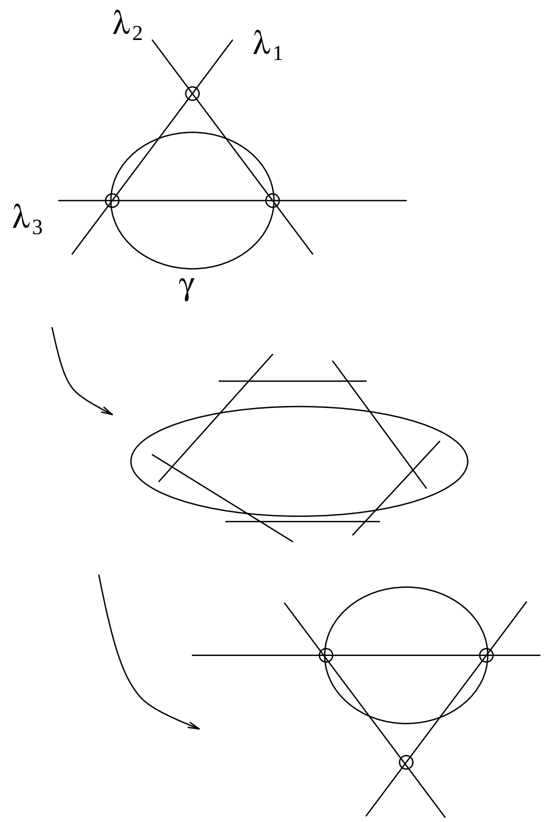


Figure 3.37: Case 1(v)

This is another self-inverse scenario since the plumbing diagram (figure 3.38) is symmetric with respect to blue and red vertices.

Case 1(vi): A conic γ passes through all three ambiguous intersection

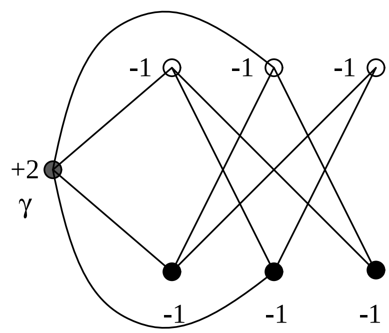


Figure 3.38: Case 1(v)

points as in figure 3.39.

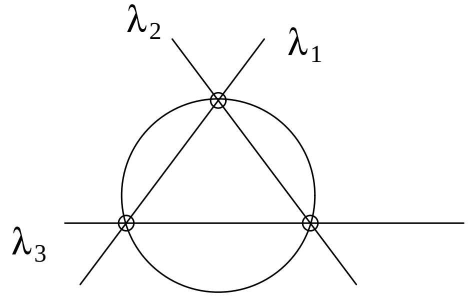


Figure 3.39: Case 1(vi)

This case is *inverse* to case 1(i) so the reverse argument applies.

Case 1(vii): A (nonsingular) cubic δ passes through all three ambiguous intersection points.

Blowing up decreases the self intersection number of δ from $+9$ to $+6$, then blowing down increases it again to $+9$. See figure 3.40.

This is possible in essentially only one way and is another self-inverse scenario. The plumbing diagram (figure 3.41) is symmetric with respect to blue and red vertices.

Case 1(viii): A cubic δ passes through at most two ambiguous intersection points. See figure 3.42.

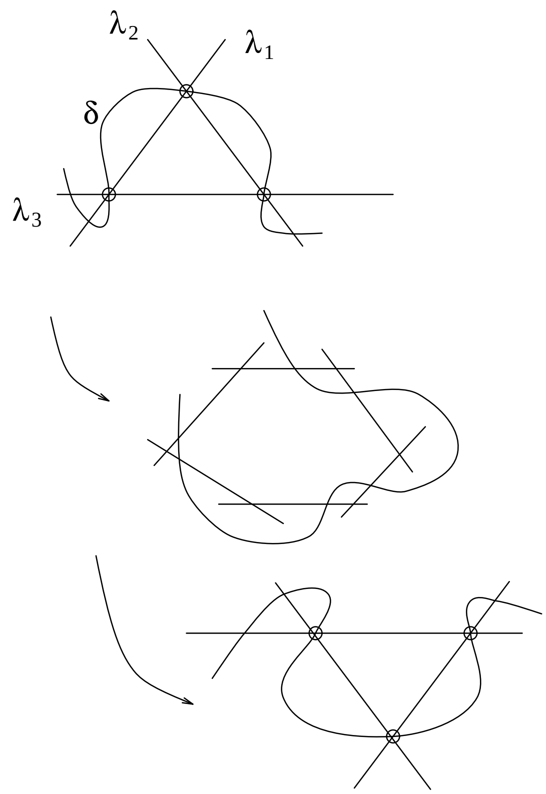


Figure 3.40: Case 1(vii)

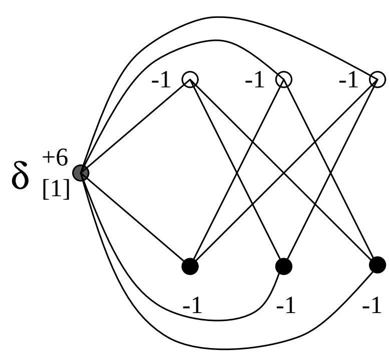


Figure 3.41: Case 1(vii)

Let P be an ambiguous point through which δ does not pass and λ one of the ambiguous lines passing through P so that λ and δ have at most one ambiguous intersection point in common. However by Bezout's theorem δ must intersect λ in three points, and so at least two of these intersection points are non-ambiguous. This contradicts lemma 3.3.3.
Hence this case does not occur.

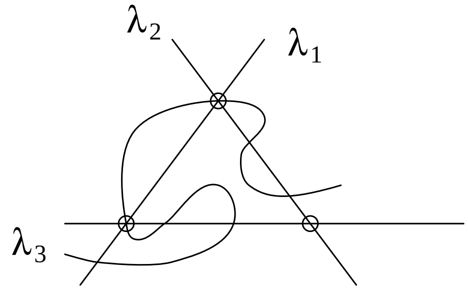


Figure 3.42: Case 1(viii)

Case 1(ix): A degree n (nonsingular) curve δ ($n \geq 4$) is in the configuration. See figure 3.43.

Then the total number of intersection points of δ with one of the lines is n by Bezout's theorem. Since $n \geq 4$ at least two of these intersection points are non-ambiguous intersection points. This contradicts lemma 3.3.3.
Hence this case does not occur.

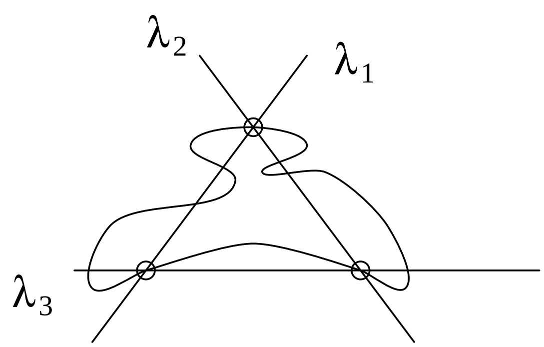


Figure 3.43: Case 1(ix)

Now for the remaining three cases there are many subcases to consider. The arguments as to whether or not a curve is allowed to be in the configuration, and if it is, how it transforms under the blowing up and blowing down, are much the same as in the treatment of Case 1. Hence we just give the summary of what curves and how they lie are allowable, and if allowable how they transform. The full treatment is given in Appendix A. First we fix notation in figure 3.45 and then list what can happen. See figures 3.46, 3.47, 3.48 and 3.49.

Comment: In listings 3 and 5 in figure 3.48 it appears that we have a self inverse scenario. This is true in the sense that the two diagrams have the same plumbing diagram after a small amount of twisting (that is, relabeling). However if more curves enter the picture then the net result may not be self inverse even though each individual curve appears to be a twisted self inverse. For example see figure 3.44.

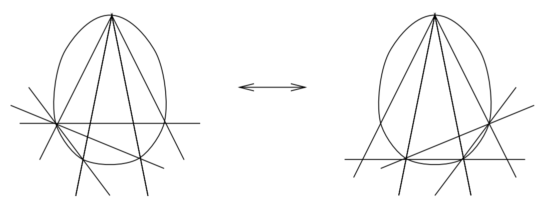


Figure 3.44:

From the classification of ambiguous cases we immediately have the following.

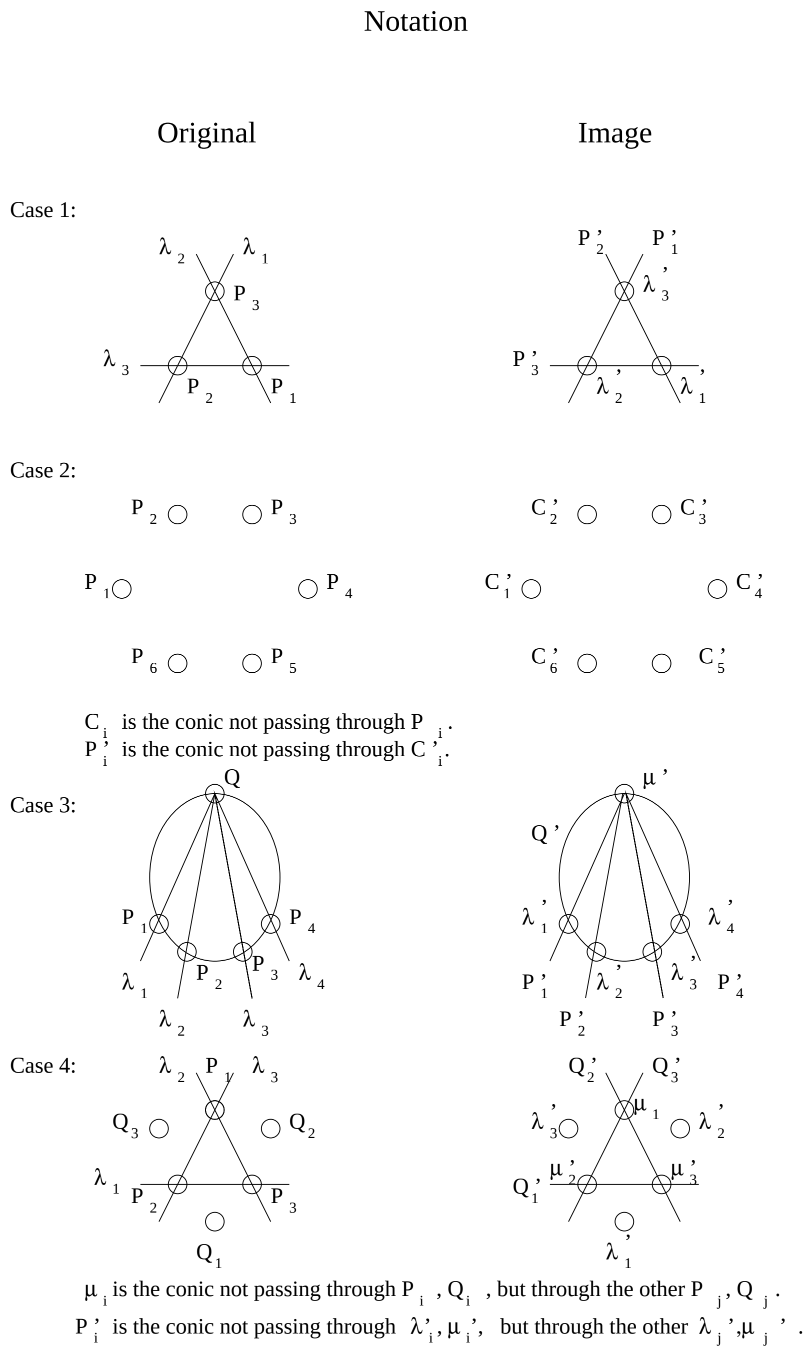


Figure 3.45: Notation

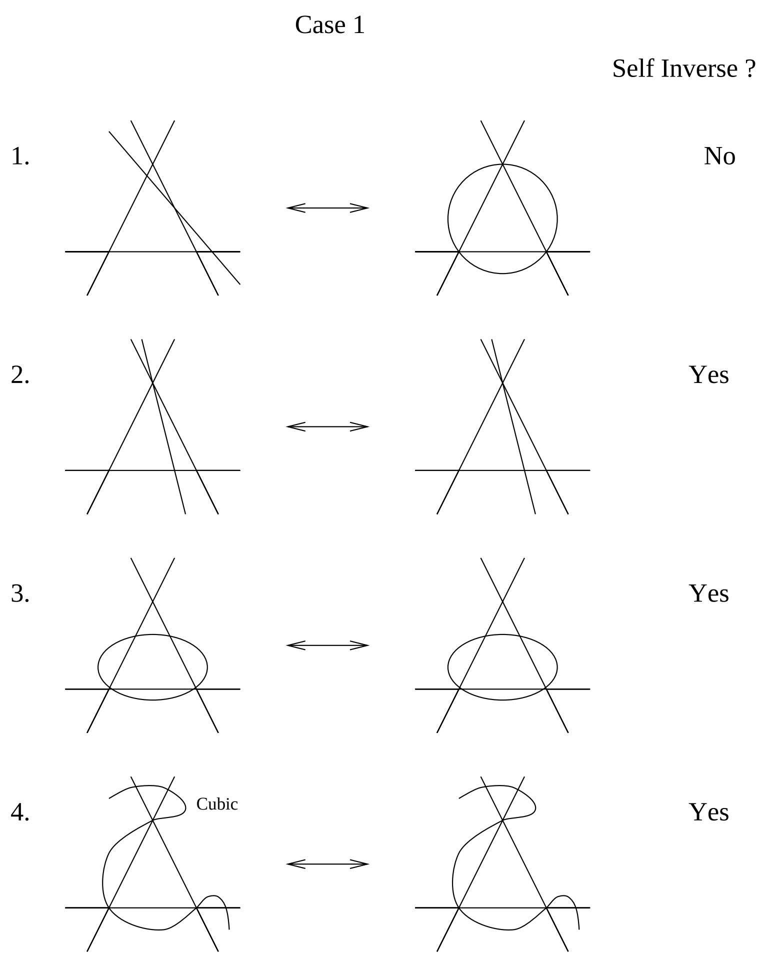


Figure 3.46: Case 1

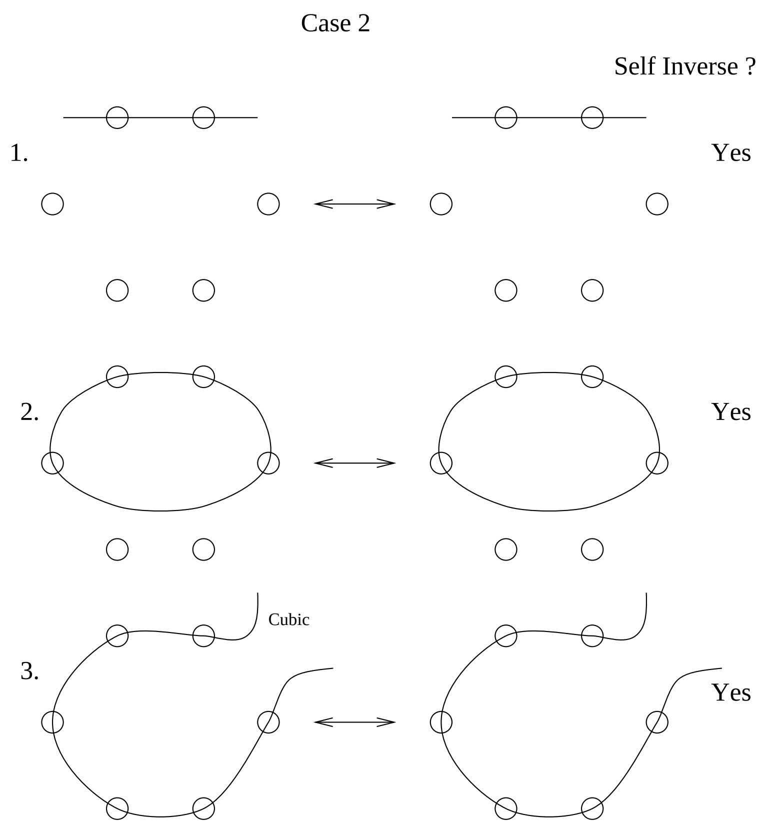


Figure 3.47: Case 2

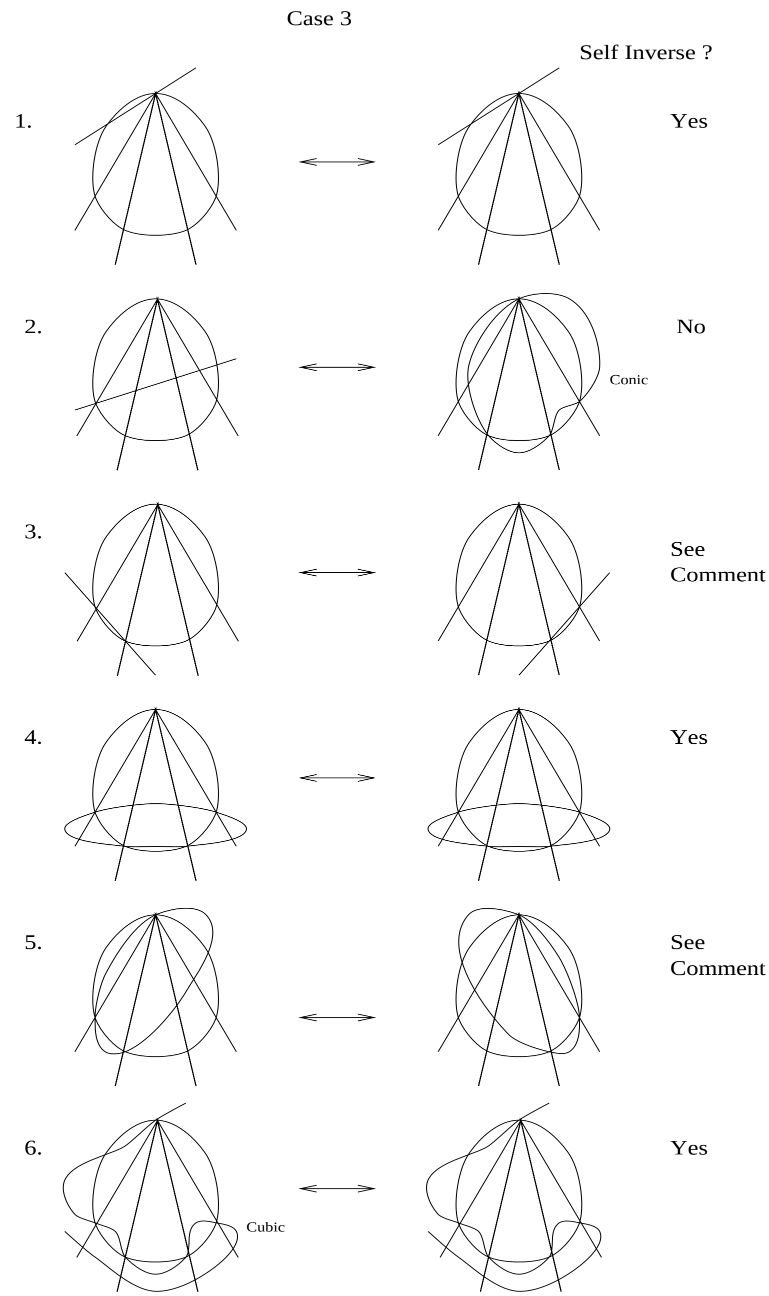


Figure 3.48: Case 3

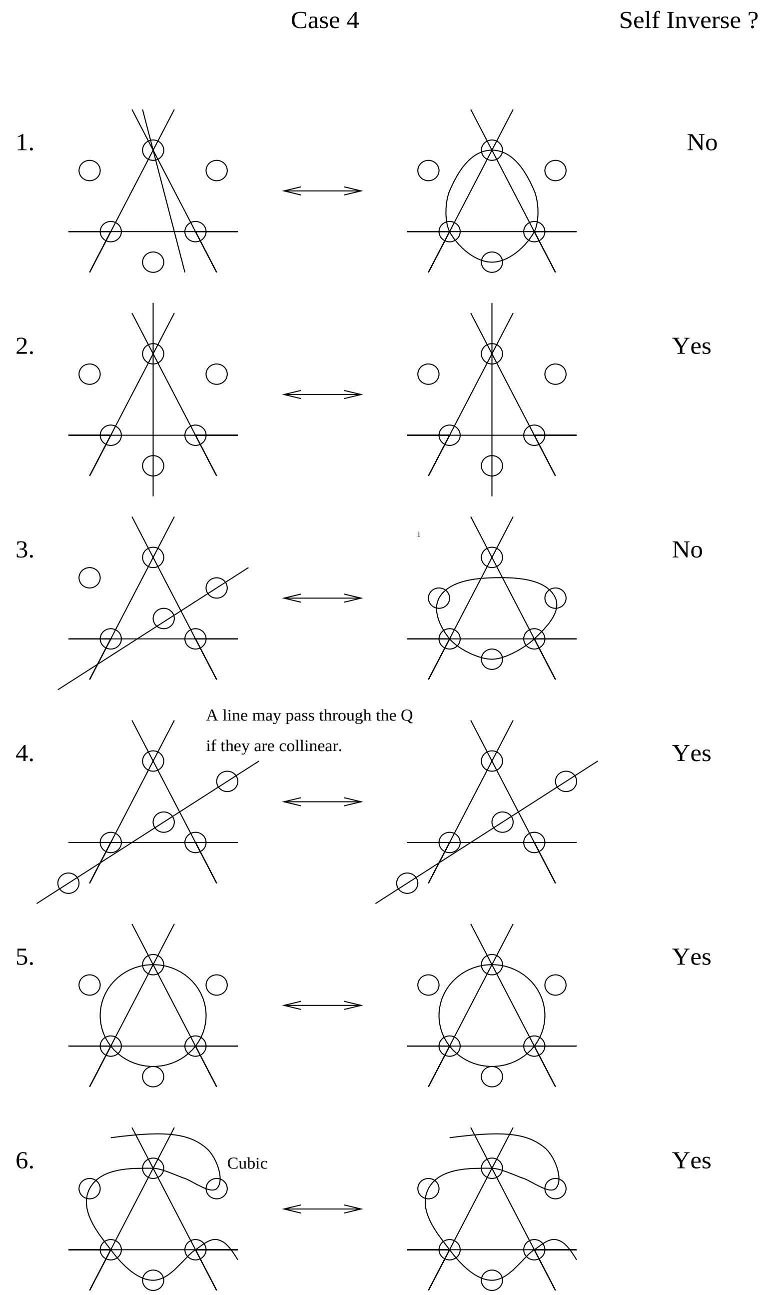


Figure 3.49: Case 4

3.4 Main Results

Corollary 3.4.1 *Let A_1 and A_2 be two normal regular arrangements having homeomorphic links, then A_1 and A_2 contain the same number of curves.*

Proof: Firstly, since they have homeomorphic links, the map $\pi_2 \circ \pi_1^{-1} : A_1 \to A_2$ is a bijection between curves of A_1 and A_2 outside of the ambiguous set. In the ambiguous set we see from the four cases that the number of ambiguous curves blown down equals the number of ambiguous points blown up thus maintaining equal cardinalities of curves. □

Corollary 3.4.2 *Let A_1 and A_2 be two regular arrangements with homeomorphic links. If A_1 contains a curve of genus > 1 (that is of degree > 3), then A_1 and A_2 are equivalent arrangements.*

Proof: Firstly we see that if the arrangements contain at least two curves then the plumbing diagram for each is already in normal form because the presence of a curve of degree greater than three forces each curve to have at least three intersection points on it. (Corollary 3.1.2.) Of course if the arrangement consists of a single curve then the plumbing diagram is in normal form. From the classification just completed we see that in the four cases it is not possible to have a curve of genus greater than one in the arrangement. Thus the ambiguous set must be empty and the map $\pi_2 \circ \pi_1^{-1}$ furnishes the required isomorphism between the arrangements.

□

Corollary 3.4.3 *Let A_1 and A_2 be two arrangements with homeomorphic complements. If A_1 contains a curve of genus > 1 (that is of degree > 3), then A_1 and A_2 are equivalent arrangements.*

It is now an appropriate time to quantify exactly what we mean by a symmetric plumbing diagram.

Define a map σ on each of the graphs in figure 3.50 to be the map which interchanges a vertex with the vertex opposite to it that is reflection in the horizontal.

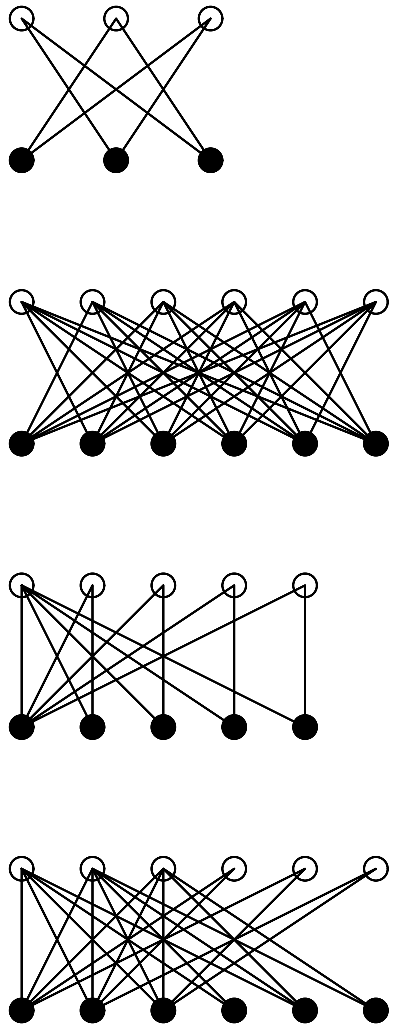


Figure 3.50:

Definition 3.4.1 *Let G be a plumbing diagram which arises from two arrangements having the same link and with a distinguished labeled subset of* *regular, exceptional and ambiguous vertices. Let K be the subgraph on the ambiguous set of vertices, so that K defines a subgraph which is one of the diagrams in figure 3.50. We say that G is* semi-symmetric *with respect to K if the map σ defined on K can be extended to an automorphism of G which maps regular vertices to regular vertices and exceptional vertices to exceptional vertices. We say that G is* symmetric *with respect to K if σ can be extended to an automorphism of G such that the restriction of σ to $G \setminus K$ is the identity map.*

Lemma 3.4.4 *Let A_1, A_2 be two arrangements with same plumbing diagram $Pl(A) = G$. Let K be the corresponding labeled ambiguous set with map σ. If the plumbing diagram is semi-symmetric with respect to the ambiguous set then A_1 and A_2 are equivalent arrangements.*

Proof: Let β be the map $\pi_2 \circ \sigma \circ \pi_1^{-1} : A_1 \to A_2$. We claim that β is an isomorphism of marked plumbing diagrams.

Firstly if P is a point of intersection outside the ambiguous set, let $C_1, C_2, \ldots, C_n$ be the curves passing through P. Since none of the curves $C_1, C_2, \ldots, C_n$ are ambiguous we easily see that $\beta(C_1), \beta(C_2), \ldots, \beta(C_n)$ are all curves in A_2. Also $\beta(P)$ is a point contained in each of $\beta(C_1), \beta(C_2), \ldots, \beta(C_n)$. (If P gets blown up, then it gets blown straight back down).

If P is a point of intersection on one of the ambiguous curves but not at an ambiguous vertex, then by lemma 3.3.3 P cannot be blown up and hence only two curves C, D pass through P where D is ambiguous and C is not. Now C gets mapped to another curve $\beta(C)$. D gets mapped to the curve $\pi_1^{-1}(D)$, then to another curve $\sigma \circ \pi_1^{-1}(D)$, and then to another curve $\pi_2 \circ \sigma \circ \pi_1^{-1}(D)$. The effect of σ stops $\pi_1^{-1}(D)$ from being blown down to a point by π_2. Also P gets mapped to the point $\pi_1^{-1}(P)$ which is the intersection of $\pi_1^{-1}(C)$ and $\pi_1^{-1}(D)$, then to the point $\sigma \circ \pi_1^{-1}(P)$ which is the intersection of $\sigma \circ \pi_1^{-1}(C)$ and $\sigma \circ \pi_1^{-1}(D)$ by symmetry, then to the point $\beta(P)$ which is the intersection of $\beta(C)$ and $\beta(D)$.

If P is one of the ambiguous points of intersection with nonambiguous curves C_i and ambiguous curves D_j passing through P, then the C_i are mapped to curves $\beta(C_i)$ and the D_i to curves $\beta(D_i)$ and P is mapped to the exceptional curve $\pi_1^{-1}(P)$ where $\pi_1^{-1}(C_i)$ and $\pi_1^{-1}(D_i)$ pass through, then to the exceptional curve $\sigma \circ \pi_1^{-1}(P)$ where $\sigma \circ \pi_1^{-1}(C_i)$ and $\sigma \circ \pi_1^{-1}(D_j)$ pass through by symmetry then to the point $\beta(P)$ where $\beta(C_i)$ and $\beta(D_i)$ pass through. The effect of σ ensures that $\pi_1^{-1}(P)$ eventually gets blown down.

Thus we have proven that β is a lattice homomorphism. Similarly β^{-1} is a lattice homomorphism. Thus β is an isomorphism.

□

Notice that this result also yields a way to detect for a given normal regular arrangement, if there is another normal regular arrangement with the same link.

Algorithm 3.4.5 *For a given arrangement A_1, firstly form the plumbing diagram (which will already be in normal form for what we are considering). Next systematically search the plumbing diagram for each of the four diagrams in figure 3.50. If no such diagram is found, then there is no other arrangement giving the same link. If such a diagram is found, say for example the third one in figure 3.50, then look up its summary (in this example it is found in figure 3.48). If there is a curve in the arrangement not fitting into the scheme of things found in the summary then again there is no other arrangement giving the same link for the chosen subdiagram. If all curves in the arrangement fit into the scheme, then replace each curve in the scheme with its inverse partner and then check if this is a new arrangement. By using lemma 3.4.4 we can quickly preclude that this is a new arrangement with same link if the plumbing diagram is semi-symmetric with respect to the chosen subset in figure 3.50. If it is not a semi-symmetric scenario then the new arrangement is still a candidate.*

Incidentally when checking (for example, the case 3 type diagrams in figure 3.48), for the property of being semi-symmetric, all self inverse type curves can be ignored. For a strictly non-selfinverse situation such as in the second listing of figure 3.48, a curve and its image provided they both occur, can be ignored in pairs.

As an example, consider the arrangement of seven curves which is in fact the fourth listing of the summary to Case 4 found on page 78. It is a triangle of lines with three conics all passing through the triangles vertices, and they pairwise intersect in three further points which are collinear and through which pass another line. Searching systematically for the subdiagrams quickly shows that only Cases 1 and 4 could possibly occur, if at all. Case 4 obviously occurs but leads to nothing new because the situation is symmetric. Case 1 occurs four times but only one of them yields a new arrangement with same link. It is a triangle of lines with a conic c passing through the vertices and also three other lines which intersect pairwise on the conic c but are in general position with respect to the triangle of lines.

Corollary 3.4.6 *Consider the class of arrangements of lines. Then there is a bijective correspondence between the combinatorics of the arrangements and the topology of the links.*

Proof: Assume firstly that there exist two arrangements A_1 and A_2 such that they have same plumbing graph but nonisomorphic lattices. It is easy to see that the following cases partition all possible arrangements of lines in $\mathbb{CP}^2$.

Case 1 : Every line has at least three intersection points on it.

Then as we have seen, the plumbing diagram must be in normal form. In this case if two arrangements A_1, A_2 have the same plumbing diagram then from the classification of possible cases we see that because only lines are involved, that only case 1(ii) is applicable here (see page 67) and in this case the plumbing diagram is symmetric with respect to the exceptional set and hence lemma 3.4.4 is applicable so that A_1 and A_2 are isomorphic arrangements.. Furthermore, if there is an intersection point which requires blowing up, then we see that the plumbing diagram must contain a vertex of weight -1. If no point requires blowing up, then the lines are in generic position and the plumbing diagram is a complete graph with all weights $+1$. In either case the normal form plumbing diagram of this case contains a vertex of weight $+1$ or -1.

Case 2 : Double Pencil. See figure 3.51.

Note that the final configuration is in normal form since $k, l \geq 2$ implies that all valences are ≥ 3.

Case 3 : Line and Pencil. See figure 3.52.

The final configuration is obviously in normal form and the link is homeomorphic to $\mathbf{S}^1 \times F_{k-1}$ where F_{k-1} is a surface of genus $k-1$.

Case 4 : Pencil consisting of at least three lines. See figure 3.53.

The final configuration is in normal form. The link is $\#_1^{k-1}(S^1 \times S^2)$.

Case 5 : Triangle. See figure 3.54.

Note that this is a special case of Case 3. The link is T^3.

Case 6 : Line pair. See figure 3.55.

Note that this is a special case of case 4. The link is $S^1 \times S^2$.

Case 7 : Single line. See figure 3.56. (Special case of case 4.)

The link is S^3.

Now it remains to show why if A_1 is not isomorphic to A_2 then the corresponding plumbing diagrams $NPl(A_1)$ and $NPl(A_2)$ are not the same.

Let A_1 be in case 1. Then if A_2 is also in case 1 then as discussed, their plumbing diagrams cannot be isomorphic. If A_2 is not in case 1, then $NPl(A_1) \ncong NPl(A_2)$ because $NPl(A_1)$ has a vertex of weight ± 1 but all vertices in cases 2 - 7 are weighted with 0. Furthermore all other combinations of cases 2-7 for A_1 and A_2 are trivial to check by inspection that they are all

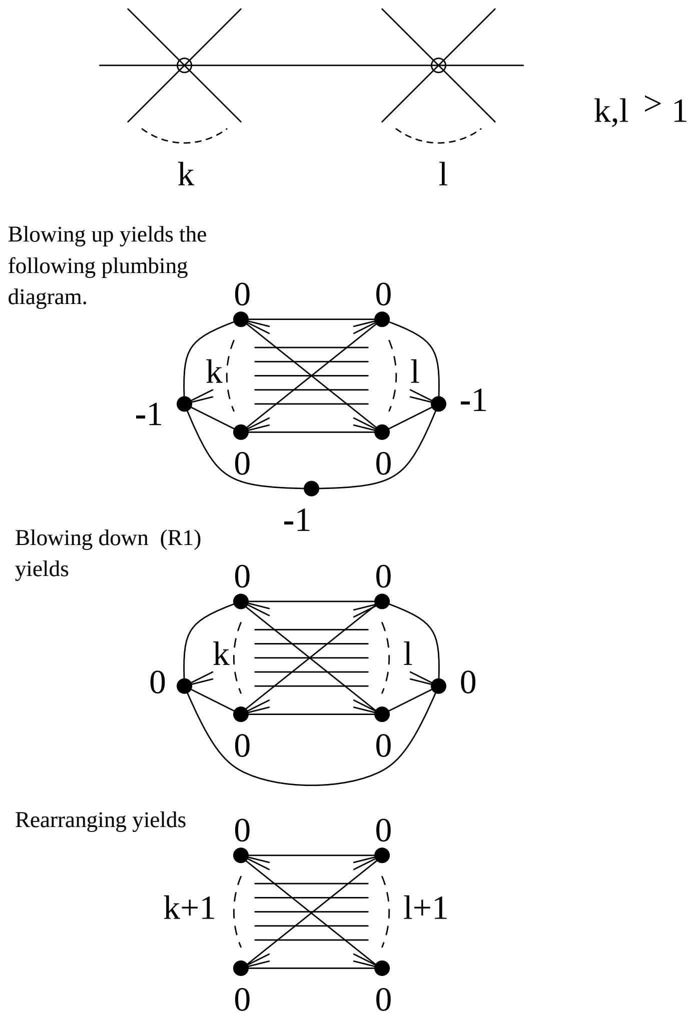


Figure 3.51: Double Pencil

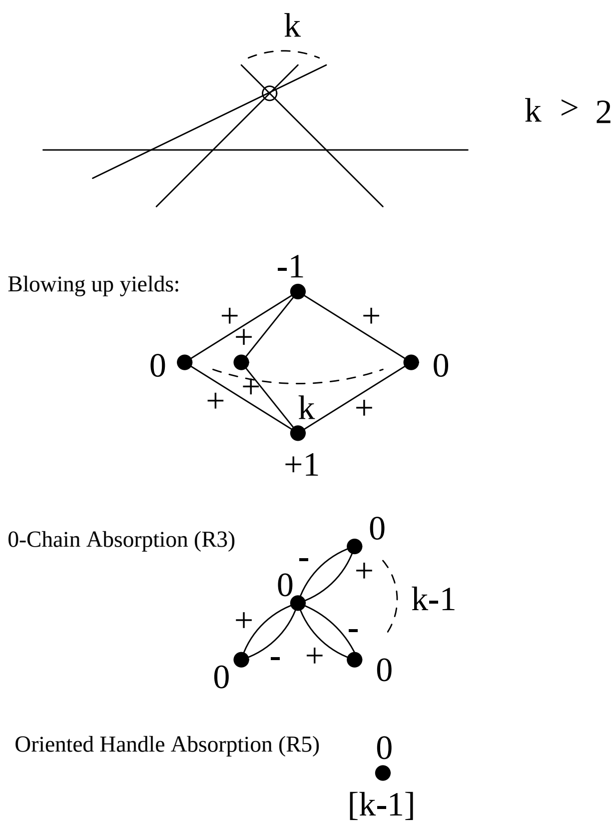


Figure 3.52: Line and Pencil

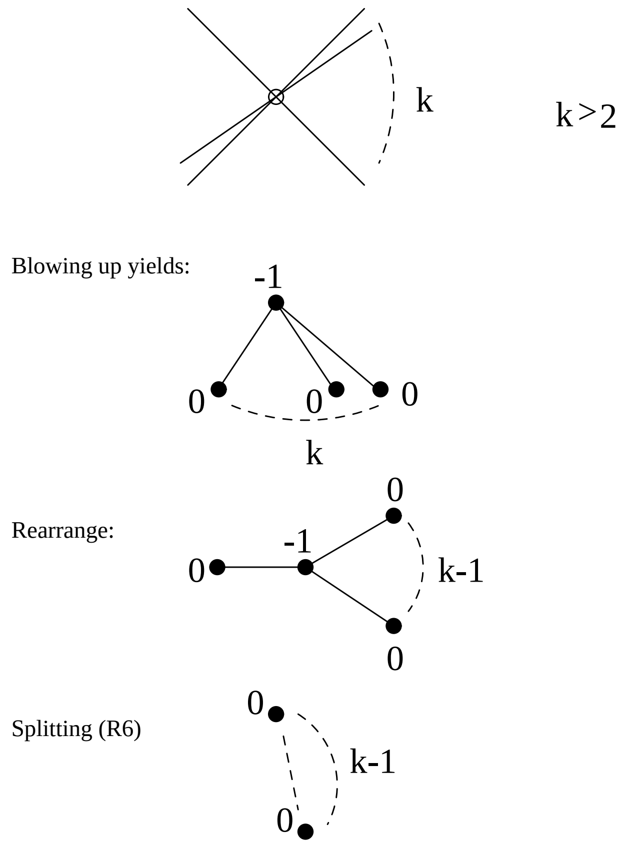


Figure 3.53: Pencil

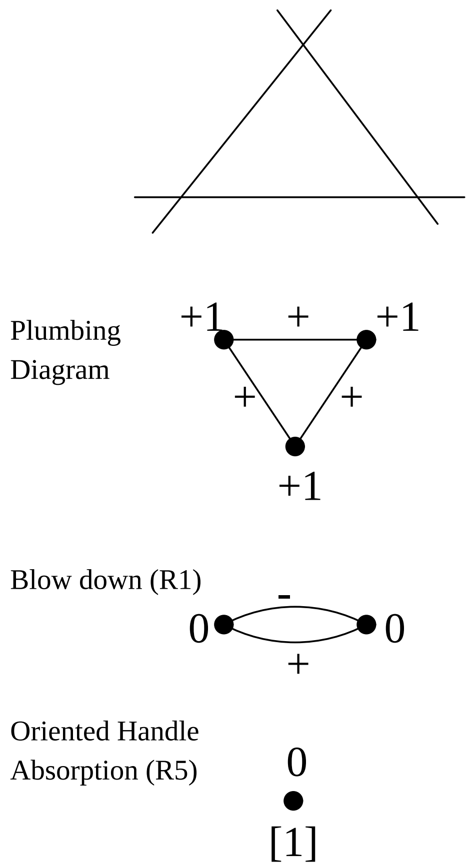


Figure 3.54: Triangle

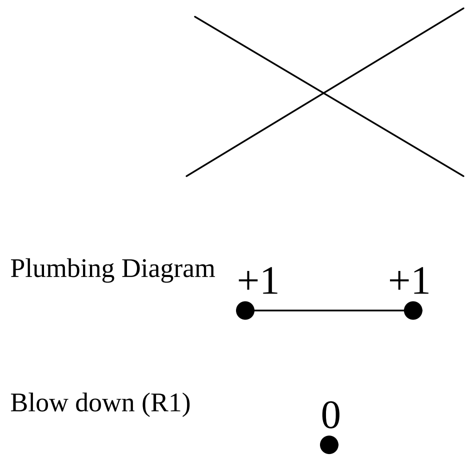


Figure 3.55: Line Pair

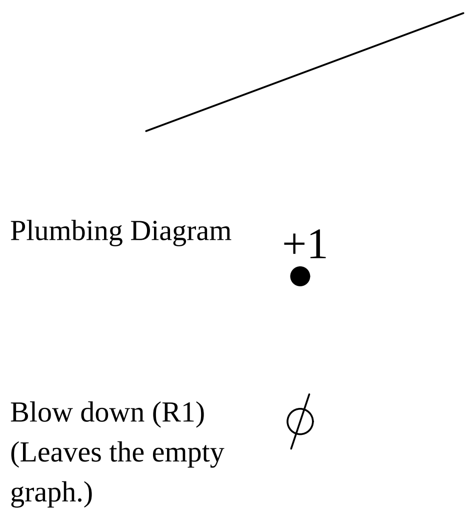


Figure 3.56: Single Line

different. Thus we have shown that the topology of the link determines the arrangement in the case of lines. The converse is trivial.

□

Corollary 3.4.7 *Consider the class of conics. There is a bijective correspondence between the the arrangements and the links.*

Proof: Firstly the plumbing diagram for an arrangement of conics is already in normal form because if an arrangement contains at least two conics, then each conic intersects the first conic in four places, thus we can apply theorem 3.1.2 because each curve has at least three (four) intersection points on it. If the arrangement consists of only one conic, then its plumbing diagram is not in normal form (see figure 3.57), however its normal form is certainly different to the plumbing diagrams of arrangements of conics with at least two curves.

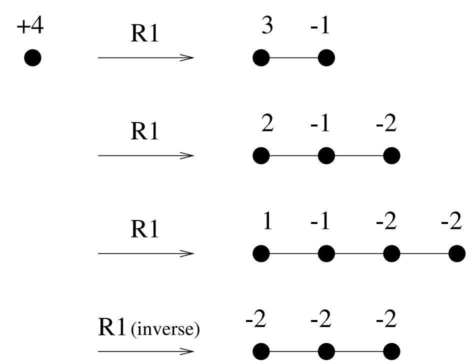


Figure 3.57: Normal Form for single conic

Now suppose we have two arrangements A_1 and A_2 of conics which have isomorphic plumbing diagrams. Since only conics are involved we see that only the second listing in figure 3.47 is applicable here and since this yields a symmetric scenario, we see that lemma 3.4.4 is applicable again and allows us to conclude that the arrangements are isomorphic. Once again the converse is trivial.

□

Corollary 3.4.8 *Consider the class of arrangements of lines. Then the topology of the complement determines the topology of the arrangement.*

Note that this result implies that although the complements of two arrangements may be homotopy equivalent, they are not necessarily topologically equivalent. (See the two inequivalent arrangements of Falk [4] which have homotopy equivalent complements. See also [3] section 7.4 p310.)

Corollary 3.4.9 *Consider the class of arrangements of conics. Then the topology of the complement determines the topology of the arrangement.*

Chapter 4

The Regular Case in Full Generality

4.1 Classification of Non-Normal Cases

The purpose of this section is to classify all possible non-normal cases into families and then divide and conquer by comparing families. The approach will be similar to that taken in the classifying into families arrangements of lines in the previous section.

Case 1: The plumbing diagram is already immediately in normal form.

Note that this is precisely the case when all curves have at least three intersection points on them or the arrangement is a single curve of genus ≥ 1. Consequently we can assume henceforth that our arrangement consists only of lines and conics.

Case 2: The arrangement consists of lines only such that the plumbing diagram is not in normal form. The normal forms for this class of arrangements were computed in section 3.4.

Henceforth we can assume that there is at least one conic.

Case 3: Our arrangement consists of a single conic. The normal form is shown in figure 4.1.

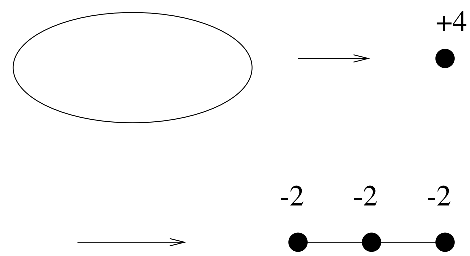


Figure 4.1:

Case 4: Our arrangement consists of a single conic and a single line. The normal form is shown in figure 4.2.

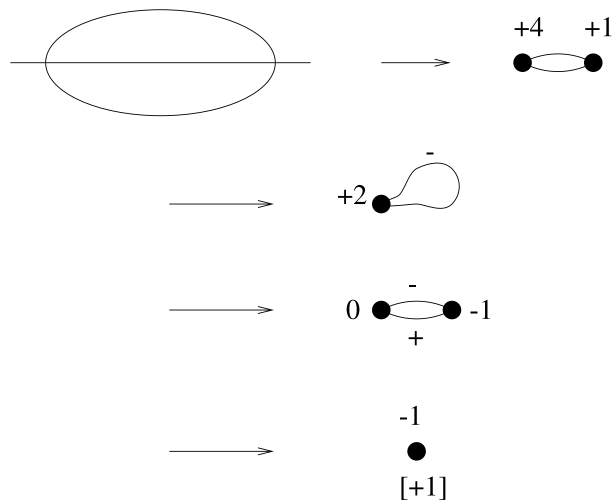


Figure 4.2:

Henceforth we can assume that all conics will have at least three intersection points on them. Thus lines are now the only possible curves that could have at most two intersection points on them. Note that no line could possibly have exactly one intersection point on it since there is at least one conic. We now classify as to the number of lines with exactly two intersection points on them.

Case 5: Exactly one line l has only two intersection points on it, and the arrangement is not case 4.

Now say l intersects the conic c at points P and Q. We claim that both P and Q have branches of curves passing through them other than l and c. Indeed if say P had only two branches passing through it then c must be the only conic. (If d were another conic then d intersects l twice. Since l only has intersection points at P and Q, we see that d intersects l at P and Q.) Now since we are not in case 4 there are further lines in the arrangement. By assumption these must all pass through Q. However then these lines would also have the property of having exactly two intersection points on them which contradicts the uniqueness of l for this case.

The point of this discussion is that the normal form for arrangements in this case is only one step away, namely we only need to do a single (-1)-blow-down. See figure 4.3.

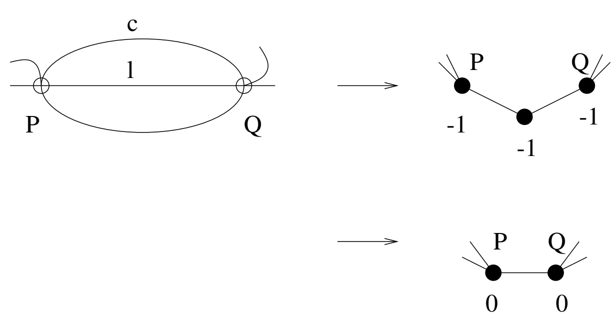


Figure 4.3:

Case 6: The arrangement consists of one conic and two lines, all three curves passing through a common point. The normal form is shown in figure 4.4

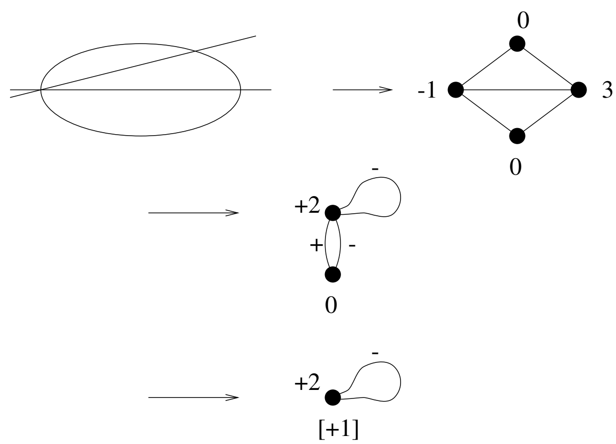


Figure 4.4:

Case 7: Exactly two lines say l and m have exactly two intersection points on them, and the arrangement is not case 6.

This forces l, m and c to concur at a point say P. Now let Q be the other intersection point of l and c. Let R be the other intersection point of m and c. Now P has three curves passing through it. We claim that both R and Q have at least three branches of curves passing through each of them. Indeed assume that Q say has no other curves passing through it other than l and c. As in case 5 we see that there can be no other conics. However since we are not in case 6, there are further lines in the arrangement and these all intersect l, thus they all pass through P. This contradicts the uniqueness of l and m having only two intersection points.

Again the point of this discussion is that the normal form for arrangements in case 7 is found by blowing down the curves l and m. See figure 4.5.

Case 8: Exactly $n \geq 3$ lines which are not all concurrent have exactly two intersection points on them.

This immediately implies there are three non-concurrent lines l, m and n with exactly two intersection points on them. Any fourth line would violate

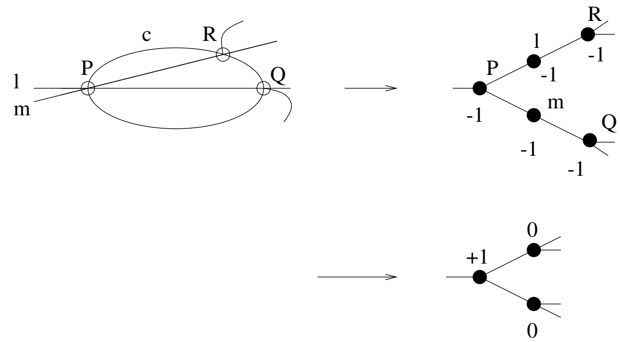


Figure 4.5:

the two point property. Thus $n = 3$. Figure 4.6 shows the passage to normal form.

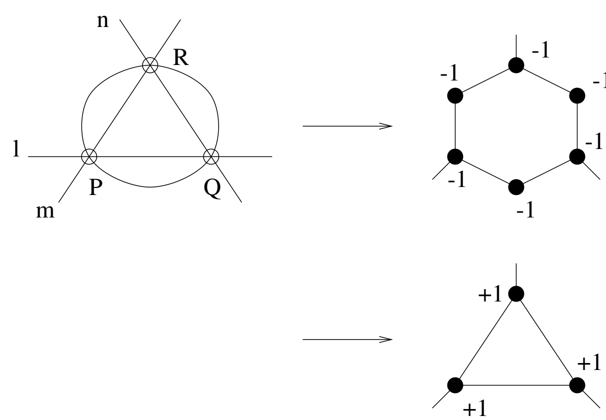


Figure 4.6:

Case 9: The arrangement consists of three lines and a conic all concurrent. See figure 4.7 for normal form.

Case 10: Exactly three concurrent lines say l, m and n have exactly two intersection points on them and we are not in case 9.

Assume the curves l, m, n, c all concur at P and that the other points of intersection of l, m, n with c are Q, R, S respectively. Now if there is any

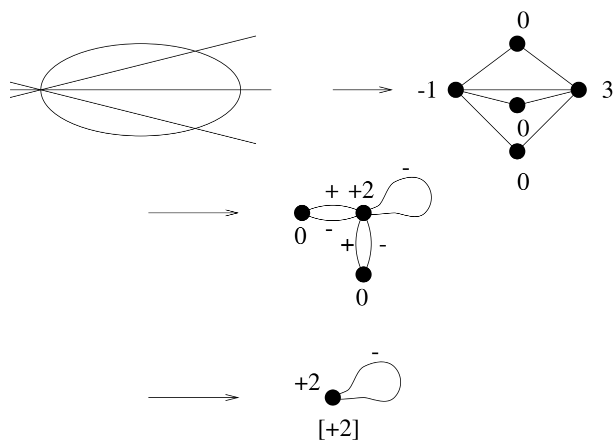


Figure 4.7:

other line say k in the configuration it must pass through P since Q, R, S are not collinear. Thus there must be another conic in the configuration. All such conics must pass through the four points P, Q, R and S. The passage to normal form is shown in figure 4.8.

Case 11: Exactly $n \geq 4$ concurrent lines have exactly two intersection points on them.

Let the lines $l_1, \ldots, l_n$ intersect the conic c in points $P, Q_1, \ldots, Q_n$ with P, Q_i lying on l_i. Any other conic d say would have to pass through all the points $P, Q_1, \ldots, Q_n$ in order not to violate the two point property of all the l_i. But then c and d would intersect in $n + 1 \geq 5$ points which contradicts Bezout's theorem. Thus there are no additional conics and we have all the lines. The passage to normal form is shown in figure 4.9.

Theorem 4.1.1 *The eleven cases described above classify all possible arrangements of nonsingular curves such that all crossings are pairwise transversal.*

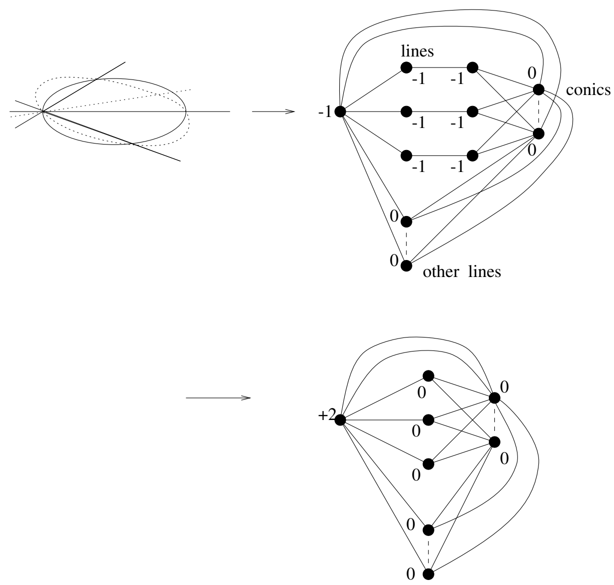


Figure 4.8:

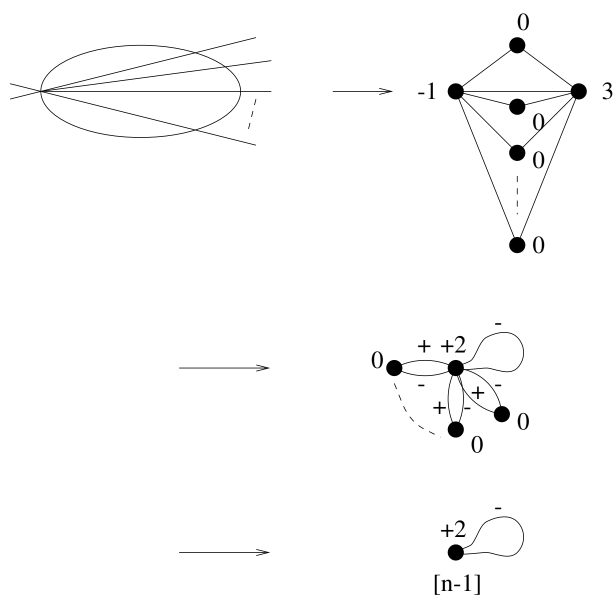


Figure 4.9:

4.2 Divide and Conquer

Given two arrangements A_1 and A_2 we must now check if they can both give rise to the same plumbing diagram. We divide and conquer much in the same way as was done in section 4.2 where we were only dealing with lines. Only in this case there are many more comparisons to do. Actually $\binom{11}{2} + 11 = 65$ comparisons to do. Luckily most are trivial, but some are not and as in section 3.1, we will need to develop an axiom system similar to the one in theorem 3.1.3 to sort some comparisons out.

4.2.1 Case 1 versus Case n

Say A_1 is in case 1 and A_2 is in case n. The cases $n = 1, 2$ were handled in the previous section. The case $n = 3$ cannot occur because $NPl(A_1)$ will either have all vertices with positive weights or will contain a vertex with weight -1. $NPl(A_2)$ does not have this property.

The case $n = 4$ cannot occur because this would force A_1 to have only one curve in it, and no non-blown-up algebraic curve has negative self intersection number.

The case $n = 5$ certainly can occur. Referring to the notation of the previous section, label the vertices of $NPl(A_2)$ corresponding to P, Q as E_1, E_2, and let the vertices to they correspond in $NPl(A_1)$ be R_1, R_2 respectively. Now it is clear that the Euler weights of vertices in $NPl(A_2)$ corresponding to exceptional divisors in A_2 are all -1 with the exception of E_1, E_2, whose weights are 0. It is also clear that the vertices in $NPl(A_1)$ corresponding to exceptional divisors in A_1 all have weights -1. Thus in particular R_1, R_2 being of weight 0 correspond to curves in A_1 and so the edge between them corresponds to an intersection point of the curves. Now do a (-1)-blow up on the plumbing diagrams of A_1, A_2 on the edge joining E_1, E_2 in $NPl(A_2)$

and on the corresponding edge joining R_1, R_2 in $NPl(A_1)$ so as to preserve the isomorphism of diagrams.

This has the effect of blowing up an intersection point of two curves in A_1. In A_2 it has the simultaneous effect of causing all exceptional divisors to have weight (-1) and also recovers the curve l which was blown down.

Now if we follow the method as in section 3.1 to these altered plumbing diagrams by letting K be the subgraph on the set of ambiguous vertices, then it is easy to see that K satisfies the axioms K1-K8 of theorem 3.1.3 and the analysis of how other curves intersect the ambiguous set is unchanged. Thus we get no new phenomena.

The cases $n = 6, 9, 11$ cannot occur because $NPl(A_2)$ will always have an edge with a negative sign while $NPl(A_1)$ never does.

The case $n = 10$ doesn't occur because $NPl(A_2)$ has a vertex of weight 0. However for $NPl(A_1)$ to have a vertex of weight 0 some curve had to have had at least one point blown up on it, thus $NPl(A_1)$ would need to have a vertex of weight -1. Yet $NPl(A_2)$ has no such vertex.

The cases $n = 7, 8$ are similar to the case $n = 5$. For $n = 7$, using the notation of the previous section, label the vertices of $NPl(A_2)$ which correspond to the points P, Q, R in A_2 as E, E_1, E_2 and the corresponding vertices in $NPl(A_1)$ as R, R_1, R_2. Now do (-1)-blow-ups along the edges joining E_1, E and E_2, E in $NPl(A_2)$ and also the corresponding blow-ups in $NPl(A_1)$ so as to maintain isomorphic diagrams. In $NPl(A_1)$ this merely corresponds to blowing up two intersection points. In $NPl(A_2)$ this causes the rogue vertices E, E_1, E_2 corresponding to exceptional divisors to have weights -1 and also recovers vertices corresponding to curves l, m which had gotten blown down. If we follow the method as in section 3.1 to these altered plumbing diagrams we come to the same conclusions as in the case $n = 5$.

Similarly for the $n = 8$ case we alter plumbing diagrams by blowing up along the edges in $NPl(A_2)$ which join the vertices corresponding to points

P, Q, R in A_2 and do the induced thing in $NPl(A_1)$. This again leads to the same conclusion.

4.2.2 Case 2 versus Case n.

The case $n = 2$ has been dealt with in a previous section.

For $n \geq 3$ we note that $NPl(A_1)$ has all vertices of weight 0, yet for $n = 3, 4, 6, 7, 8, 9, 10, 11$ we immediately can see in $NPl(A_2)$ a nonzero weighted vertex. Finally for $n = 5$ if apart from P and Q there are other points in A_2 which need blowing up then this translates into the existence of a vertex of weight -1 in $NPl(A_2)$. But if no further points are to be blown up, then the vertex in $NPl(A_2)$ which corresponds to the conic c in A_2 will have weight $+2$. Thus for $n = 5$ we also have a nonzero weighted vertex.

4.2.3 Case 3 versus Case n.

Case 3 has all vertices of weight -2. Certainly $n = 3$ is vacuous. For $n \geq 4$ we see that $NPl(A_2)$ ostensively always contains a vertex of weight not equal to -2.

4.2.4 Case 4 versus Case n.

Certainly $n = 4$ is vacuous. Also $NPl(A_1)$ consists of a single vertex with no edges, yet for $n \geq 5$, $NPl(A_2)$ always contains edges.

4.2.5 Case 5 versus Case n

The cases $n = 5, 7, 8, 10$ are difficult and will be dealt with separately. One sees that cases $n = 6, 9, 11$ cannot occur because of having negatively signed edges.

4.2.6 Case 6 versus Case n

Immediately the case $n = 6$ is vacuous. For $n \geq 7$ we see that $NPl(A_1)$ has all vertices with genus [+1] (namely a single vertex like this) while $NPl(A_2)$ does not have this property.

4.2.7 Case 7 versus Case n

The cases $n = 7, 8$ are difficult and will be dealt with later. For $n \geq 9$ we see that $NPl(A_1)$ always contains a vertex of weight +1, but $NPl(A_2)$ never does.

4.2.8 Case 8 versus Case n.

The case $n = 8$ is difficult and will be dealt with later, but again for $n \geq 9$ we see that $NPl(A_1)$ always contains a vertex of weight +1, but $NPl(A_2)$ never does.

4.2.9 Remaining Cases.

Cases 9 versus 9 and 11 versus 11 are vacuous, while case 10 versus case 10 is difficult and will be dealt with later. Case 9 versus case 11 does not happen because their normal forms are explicitly different. Finally Case 10 versus case 9 or case 11 does not occur because the normal form for case 10 contains a vertex of weight zero but the normal forms for cases 9 and 11 do not.

4.3 Difficult Non-Normal Cases

The previous section showed that nothing new happens in most cases for the non-normal situation. Yet there were eight situations that could and do lead to new things happening. They were A_1 belonging to case i and A_2

belonging to case j where (i, j) were one of the following pairs: $(5,5)$; $(5,7)$; $(5,8)$; $(5,10)$; $(7,7)$; $(7,8)$; $(8,8)$; $(10,10)$.

Each of these cases requires detailed analysis and will be put under its own subheading. It turns out in most cases that we can adjust the plumbing graphs in each case so as to be able to do the sort of axiomatizing of ambiguous subgraphs as was done in section 3.1.

Convention: In order to avoid clumsiness, for the rest of this section we shall not distinguish between a curve in A_i and the vertex representing it in $Pl(A_i)$.

4.3.1 $i = j = 5$

Let E_1, E_2 be the two exceptional curves of weight zero in $NPl(A_1)$. Let F_1, F_2 be the two exceptional curves of weight zero in $NPl(A_2)$. The question is what happens under the isomorphism between plumbing graphs. We identify E_i with its image under the isomorphism between $NPl(A_1), NPl(A_2)$.
Case 1: If the sets $\{E_1, E_2\}$ and $\{F_1, F_2\}$ are disjoint, then we can blow up along the edge joining E_1, E_2 and do the corresponding thing in $NPl(A_2)$, then blow up the edge joining F_1, F_2 and do the corresponding thing in $NPl(A_1)$ so that all we have done is to restore curves that had been blown down, made all exceptional divisors have weights -1 and blown up a couple of intersection points of curves. One can easily check after these alterations that the subgraph K on ambiguous vertices satisfies the same eight axioms as in theorem 3.1.3. Thus nothing new occurs here.

Case 2: If the sets $\{E_1, E_2\}$,$\{F_1, F_2\}$ coincide, say $E_1 = F_1$ and $E_2 = F_2$, then blowing up along the edged joining E_1, E_2 will also restore the situation to one that can be dealt with as in section 3.1.

Case 3: If the sets $\{E_1, E_2\}$,$\{F_1, F_2\}$ have exactly one element in common say $E_2 = F_2$ then blow up along the edge joining E_1, E_2 and forming a new vertex R_1 also the edge joining F_1, F_2 forming a new vertex S_1. Here we have an unusual situation where E_2 is now a vertex of weight -2 - representing the only exceptional curve whose weight is not -1. Note that E_2 is not an ambiguous vertex being an exceptional divisor for both A_1 and A_2, but that E_1, R_1, F_1, S_1 are ambiguous. Furthermore E_1, S_1 are blue and F_1, R_1 are red. Now the subgraph K on the ambiguous set of vertices one can check satisfy axioms K1-K5. Axiom K6 holds for all vertices except R_1, S_1, E_1, F_1 which satisfy $v(R_1) = v(S_1) = 1$ and $v(E_1), v(F_1) = 1$ or 4. (Recall that E_1 is a line in A_1 and F_1 a line in A_2). Axiom K7 holds for all pairs of vertices except the following situations: $n(R_1(1), F_1(1)) = n(S_1(1), E_1(1)) = 0$ and $n(R_1(1), F_1(4)) = n(S_1(1), E_1(4)) = 1$.

See also figure 4.10.

Let us write down these axioms.

Axiom 4.3.1

K1 $K \neq \phi$

K2 *All vertices have* $[g] = [0]$ *and weights* -1.

K3 *All vertices are either coloured red or blue and we have two distinguished red vertices* R_1, R_2 *and two distinguished blue vertices* B_1, B_2. *Furthermore* R_1 *is joined to* B_2 *and* B_1 *is joined to* R_2.

K4 *No vertices of the same colour are joined by an edge.*

K5 *There are no double edges or loops.*

K6 *Every vertex has valence either* 2 *or* 5 *except that*

1. $v(R_1) = v(B_1) = 1$
2. $v(B_2), v(R_2) = 1$ *or* 4

K7 *If u, v are vertices of the same colour, then $n(u(x), v(y)) = (x-1)(y-1)$ except that*

1. $n(R_1(1), R_2(1)) = n(B_1(1), B_2(1)) = 0$
2. $n(R_1(1), R_2(4)) = n(B_1(1), B_2(4)) = 1$
3. $n(R(2), R_1(1)) = n(B(2), B_1(1)) = 1$
4. $n(R(5), R_1(1)) = n(B(5), B_1(1)) = 2$
5. $n(R(2), R_2(1)) = n(B(2), B_2(1)) = 1$
6. $n(R(5), R_2(1)) = n(B(5), B_2(1)) = 2$
7. $n(R(2), R_2(4)) = n(B(2), B_2(4)) = 2$
8. $n(R(5), R_2(4)) = n(B(5), B_2(4)) = 4$

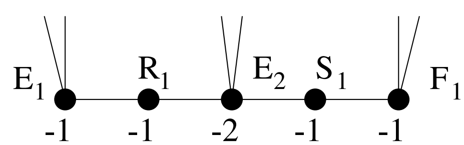


Figure 4.10:

We now need to determine K. One immediate simplification is that $v(R_2) = v(B_2)$. This follows from the axioms because if say $v(R_2) = 1$ and $v(B_2) = 4$, then we have $n(B_2(4), B_1(1)) = 1$ by K7. However B_1 is only joined to R_2 so that B_2 must be joined to R_2 also. Thus R_2 has at least valence at least 2. Thus we really only have two cases to consider. We also have the simplification that there can be no valence 5 vertices, for if say B had valence 5, then $n(B(5), B_1(1)) = 2$ yet $v(B_1) = 1$.

Case 1: $v(B_2) = v(R_2) = 1$**.**

In this setting we immediately see that B_1, B_2, R_1, R_2 are all the vertices of K, for if say B were another blue vertex, then $n(B_1, B) \geq 1$ and this contradicts that B_1 is only joined to R_2. Thus we already have all of the diagram as in figure 4.11.

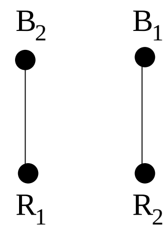


Figure 4.11:

Actually if we augment the graph K to include in addition not only the ambiguous vertices, but also the vertices which are exceptional in both diagrams but have weights not equal to -1 then the axioms for K become a little easier.

Axiom 4.3.2

K1 $K \neq \phi$

K2 *All vertices have* $[g] = [0]$ *and weights* -1 *except vertex* E^* *which has weight* -2.

K3 *All vertices are coloured red or blue except* E^* *which is coloured grey. Furthermore we have distinguished red vertices* R_1, R_2 *and distinguished blue vertices* B_1, B_2. *We further demand that the valences of* E^*, R_1, B_1 *be* 2 *and that* R_1 *is joined to both* B_2 *and* E^* *and that* B_1 *is joined to both* R_2 *and* E^*.

K4 *No vertices of the same colour are joined by an edge.*

K5 *There are no double edges or loops.*

K6 *Every vertex has valence* 2 *or* 5 *except that* $v(R_2), v(B_2) = 1$ *or* 4.

K7 *If u, v are vertices of the same colour then*

$$n(u(x), v(y)) = (x-1)(y-1)$$

except that

1. $n(R_1(2), R_2(1)) = n(B_1(2), B_2(1)) = 0$
2. $n(R_1(2), R_2(4)) = n(B_1(2), B_2(4)) = 1$
3. $n(R(2), R_2(1)) = n(B(2), B_2(1)) = 1$
4. $n(R(5), R_2(1)) = n(B(5), B_2(1)) = 2$
5. $n(R(2), R_2(4)) = n(B(2), B_2(4)) = 2$
6. $n(R(5), R_2(4)) = n(B(5), B_2(4)) = 4$

The presence of E^* in K makes the process of the passage between A_1 and A_2 much easier to understand. See figure 4.12.

When we do an analysis as in section 3.3 of how other curves intersect the ambiguous set we see that firstly only the three situations in figure 4.13 can arise due to the restrictive nature of Case 5, and that these are all self inverse, giving rise to symmetric plumbing diagrams.

Case 2: $n(B_2) = n(R_2) = 4$
It takes little effort to see that the graph K must be the one shown in figure 4.14.

When one does the analysis of how other curves may intersect the ambiguous set the nonself-inverse case does arise. The summary of what may occur is shown in figure 4.15.

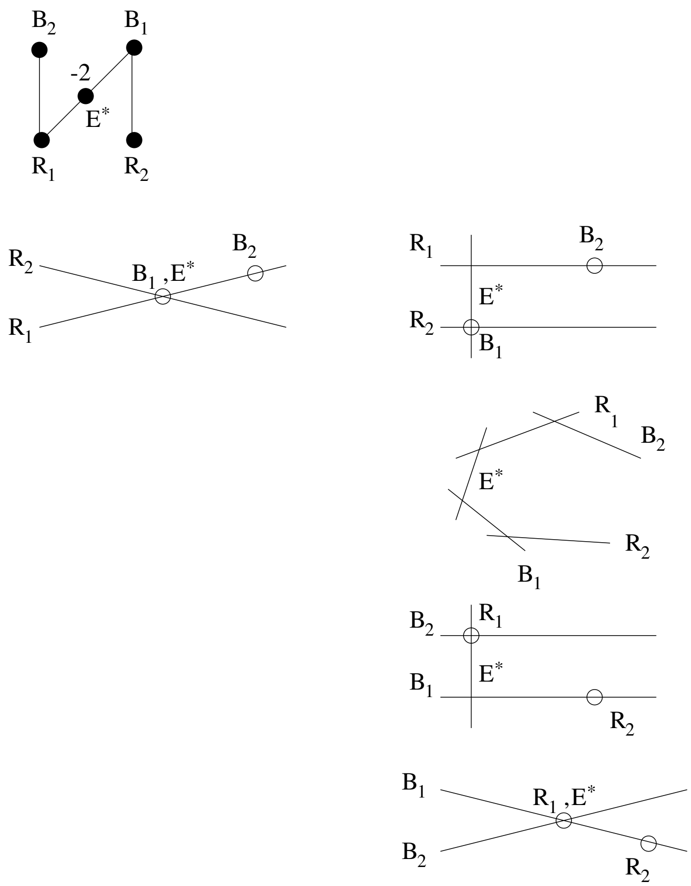

B2
B1
-2
E*
R1
R2
R2
B1 ,E*
B2
R1
R1
B2
E*
R2
B1
R1
B2
E*
R2
B1
B2
R1
E*
B1
R2
B1
R1 ,E*
B2
R2

Figure 4.12:

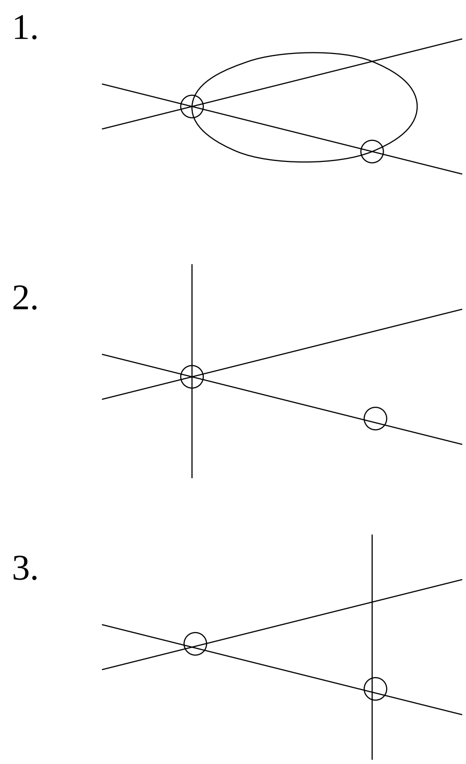


Figure 4.13:

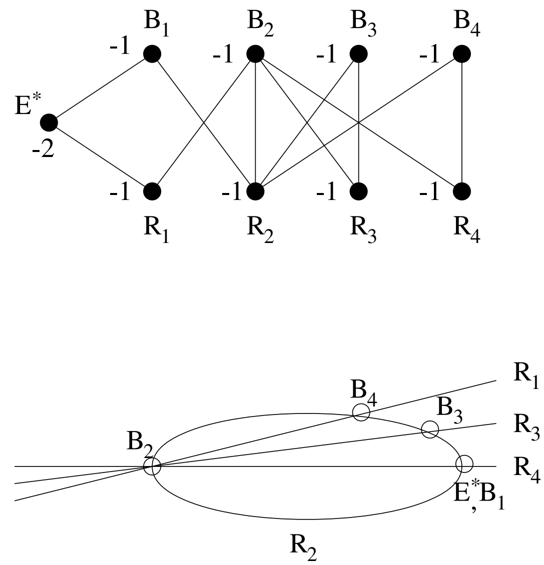


Figure 4.14:

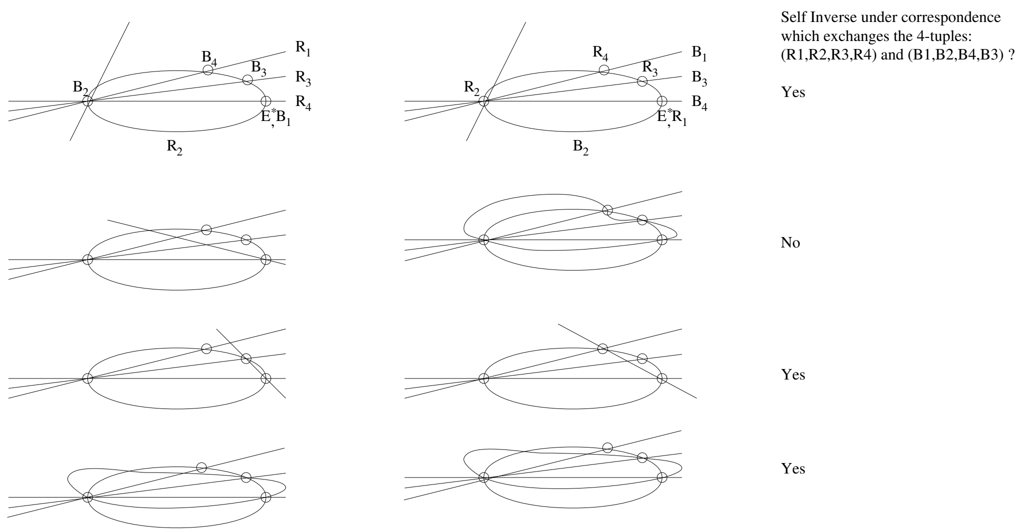


Figure 4.15:

4.3.2 $i = 5, j = 7$

From here on and in succeeding sections, it will be useful to see $NPl(A_1)$ and $NPl(A_2)$ as being the same graph but looked at through different glasses. For example in the previous case of $i = j = 5$ our analysis boiled down to where in the diagram the two figures in figure 4.16 were. Figure 4.17 shows all possible relative positions, and only the third relative position caused something new which had to be investigated.

Figure 4.16:

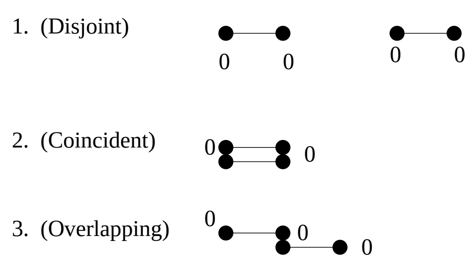


Figure 4.17:

In our present case we have the two figures as shown in figure 4.18 with all their possible relative positions.

The first position where they are disjoint is easiest. Just blow up as was done in case 1 of section 4.3.1 to recover all ambiguous curves which got blown down in the passage to normal form. One can check that ambiguous subgraph K of the altered plumbing diagram will satisfy axioms K1-K7 of theorem 3.1.3 and so nothing new happens.

The second position is impossible because it is illegal from the point of

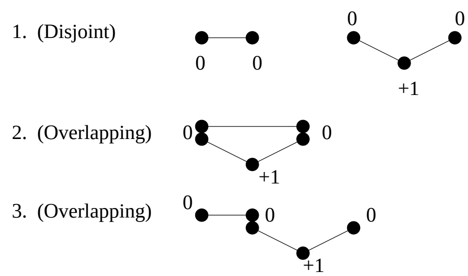


Figure 4.18:

view of $NPl(A_2)$, namely the two exceptional divisors of weight 0 could not have been joined by an edge.

The third position can happen. We use the same process as in the previous subsection by blowing up along relevant edges in order to restore blown down ambiguous curves. One can check that the ambiguous subgraph K of the altered plumbing diagram satisfies the axioms K1-K7 of Axiom 4.3.2 in subsection 4.3.1 where the axioms needed altering. The only difference is that there is one more distinguished vertex and that the ambient non-ambiguous curves satisfy *at least* the same conditions as in Axiom 4.3.2.

Thus as far as $i = 5, j = 7$ goes we don't see anything that we haven't seen before.

4.3.3 $i = 5, j = 8$

There is only one relative position in which the relevant small diagrams (figure 4.19) can occur and this is by being disjoint. We blow up to restore any lost ambiguous curves and see that the resulting ambiguous subgraph K of the altered plumbing diagram satisfies axioms K1-K8 of theorem 3.1.3 so that nothing new happens.

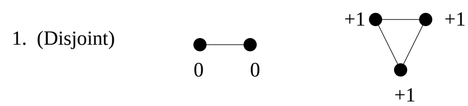


Figure 4.19:

4.3.4 $i = 5, j = 10$

All the relative positions of the relevant small diagrams are shown in figure 4.20.

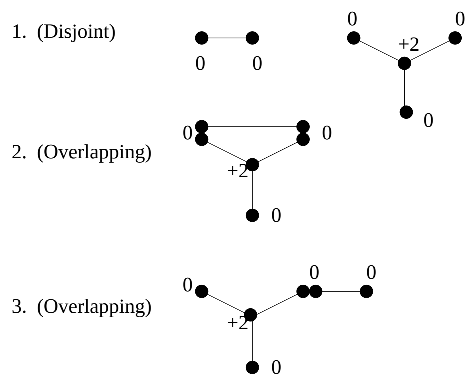


Figure 4.20:

The first position (disjoint) we again easily deal with by blowing up to restore any lost ambiguous curves and see that the resulting subgraph K of the altered plumbing diagram satisfies axioms K1-K8 of theorem 3.1.3 so that nothing new happens.

The second position is impossible being illegal from the point of view of $NPl(A_2)$.

The third position can occur and we deal with it again by blowing up to restore any lost ambiguous curves. The resulting K satisfies axioms K1-K8 of Axiom 4.3.2 so that nothing new happens.

4.3.5 $i = j = 7$

All the relative positions of the relevant small diagrams are shown in figure 4.21.

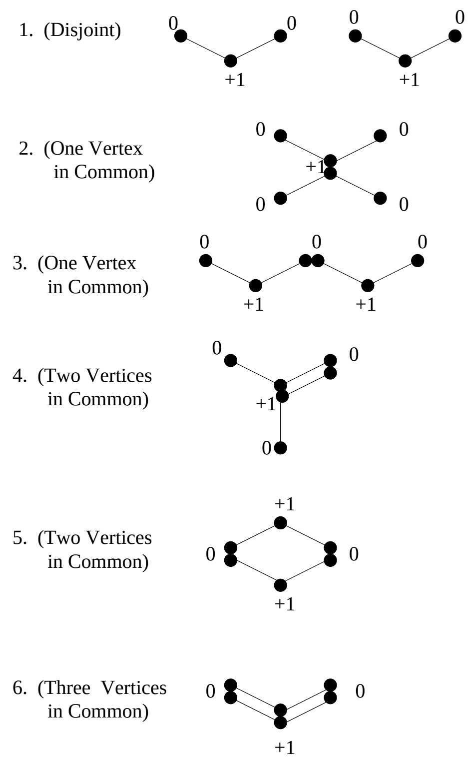


Figure 4.21:

The first and the last positions are easiest. Blow up to restore lost ambiguous curves. The resulting K satisfies axioms K1-K8 of theorem 3.1.3 so that nothing new happens.

The third and the fourth positions are also easy and result in the axioms of Axiom 4.3.2.

The second position is different. Blowing up to restore lost ambiguous curves yields figure 4.22.

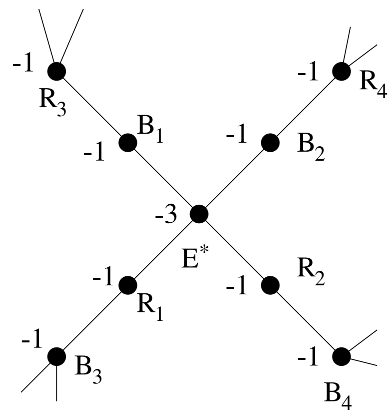


Figure 4.22:

The resulting axioms for the resulting K are as follows:

Axiom 4.3.3

K1 $K \neq \phi$

K2 *All vertices have* $[g] = [0]$ *and weights* -1 *except one special vertex* E^* *which has weight* -3.

K3 *All vertices are coloured red or blue except* E^* *which is coloured grey. Furthermore we have distinguished red vertices* R_1, R_2, R_3, R_4 *and distinguished blue vertices* B_1, B_2, B_3, B_4. *We further demand that the valences of* R_1, R_2, B_3, B_4 *be* 2, *the valence of* E^* *be* 4 *and that* R_1 *is joined to both* B_1 *and* E^*, R_2 *is joined to both* B_2 *and* E^*, B_3 *is joined to both* R_3 *and* E^* *and that* B_4 *is joined to both* R_4 *and* E^*.

K4 *No vertices of the same colour are joined by an edge.*

K5 *There are no double edges or loops.*

K6 *Every vertex has valence* 2 *or* 5 *except that* $v(B_1), v(B_2), v(R_3), v(R_4) =$ 1 *or* 4 *and* $v(E^*) = 4$.

K7 *If u, v are vertices of the same colour then*

$$n(u(x), v(y)) = (x-1)(y-1)$$

except that if C is one of R_1, R_2 and D is one of R_3, R_4 then

1. $n(C(2), D(1)) = 0$
2. $n(C(2), D(4)) = 1$
3. $n(R_3(1), R_4(1)) = 0$
4. $n(R_3(4), R_4(1)) = 1$
5. $n(R_3(1), R_4(4)) = 1$
6. $n(R_3(4), R_4(1)) = 3$
7. $n(R_3(1), R(2)) = n(R_4(1), R(2)) = 1$
8. $n(R_3(1), R(5)) = n(R_4(1), R(5)) = 2$
9. $n(R_3(4), R(2)) = n(R_4(4), R(2)) = 2$
10. $n(R_3(4), R(5)) = n(R_4(4), R(5)) = 4$

Corresponding axioms for the blue vertices holds where R_i corresponds to B_{4-i}. From this one can easily determine that $v(R_3) = v(B_1)$ and $v(R_4) = v(B_4)$. One also easily sees that if one of R_3, R_4 has valence 4 then so does the other. This induces the two cases that follow:

Case 1: $v(R_3) = 1$

From this one quickly determines K to be as in figure 4.23.

When we do the analysis of how other curves may intersect the ambiguous set we see that they are all self inverse situations so that nothing new occurs. See figure 4.24.

Case 2: $v(R_3) = 4$

It is not too hard to determine that K must be as in figure 4.25.

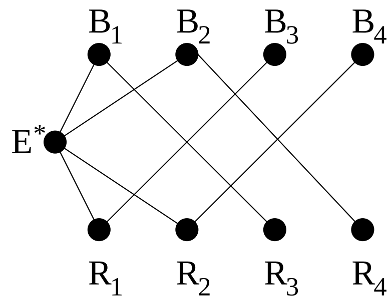


Figure 4.23:

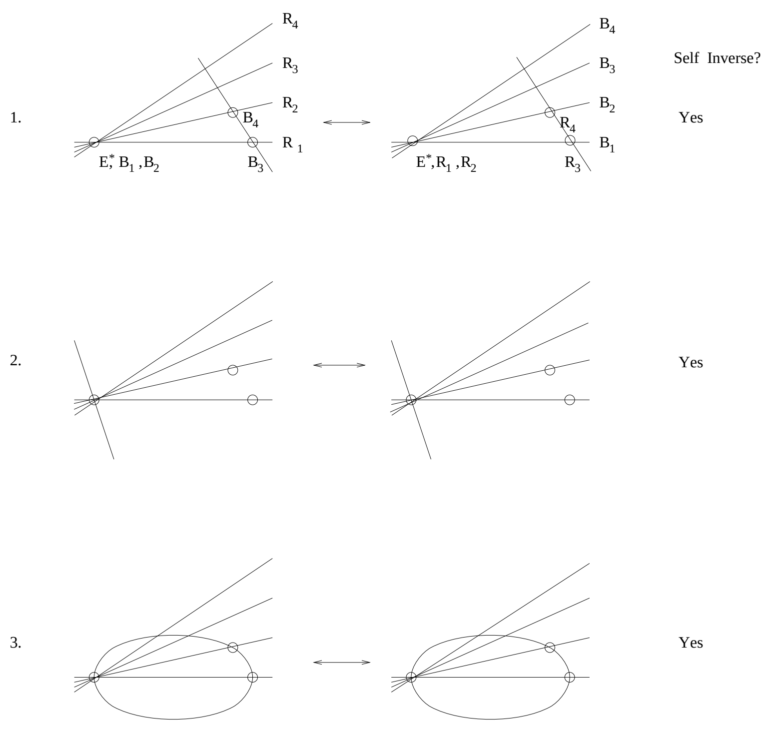


Figure 4.24:

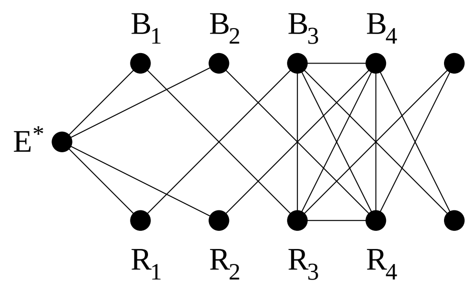


Figure 4.25:

The classification as to how other curves may intersect the ambiguous set is shown in figure 4.26. So something new occurs here because we have a nonself-inverse scenario.

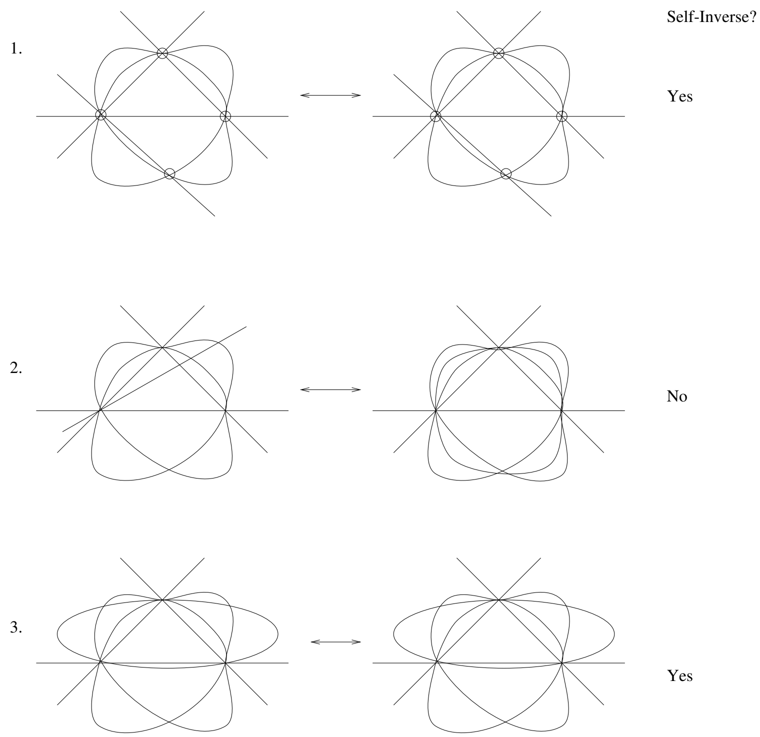


Figure 4.26:

The fifth position is different. Blowing up to restore lost ambiguous curves yields the diagram in figure 4.27.

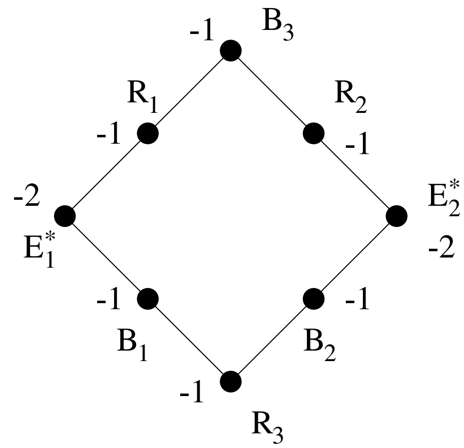


Figure 4.27:

The axioms for K in this instance are as follows.

Axiom 4.3.4

K1 $K \neq \phi$

K2 *All vertices have* $[g] = [0]$ *and weights* -1 *except two special vertices* ${E_1}^*, {E_2}^*$ *which have weights* -2.

K3 *All vertices are coloured red or blue except* ${E_1}^*, {E_2}^*$ *which are coloured grey. Furthermore we have distinguished red vertices* R_1, R_2, R_3 *and distinguished blue vertices* B_1, B_2, B_3. *We further demand that the valences of* R_1, R_2, B_1, B_2 *be* 2, *and that* R_1 *is joined to both* ${E_1}^*$, B_3, *and that* R_2 *is joined to both* ${E_2}^*$ *and* B_3,B_1 *is joined to both* R_3 *and* ${E_1}^*$ *and that* B_2 *is joined to both* R_3 *and* ${E_2}^*$.

K4 *No vertices of the same colour are joined by an edge.*

K5 *There are no double edges or loops.*

K6 *Every vertex has valence* 2 *or* 5 *except that* $v(R_3) = v(B_3) = 3$.

K7 *If* u, v *are vertices of the same colour then* $n(u(x), v(y)) = (x-1)(y-1)$ *except that*

1. $n(R_1(2), R_3(3)) = n(R_2(2), R_3(3)) = 1$
2. $n(R(2), R_3(3) = 2$
3. $n(R(5), R_3(3) = 4$

From these axioms one can very quickly deduce that R_3 must be joined to B_3 and that once this is done then figure 4.27 is already complete and no further vertices are possible. The rearranged completed K is shown in figure 4.28 and the summary of how other curves may intersect the ambiguous set is shown in figure 4.29. Note that nothing new occurs in this case.

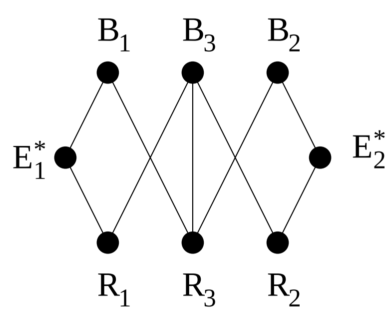


Figure 4.28:

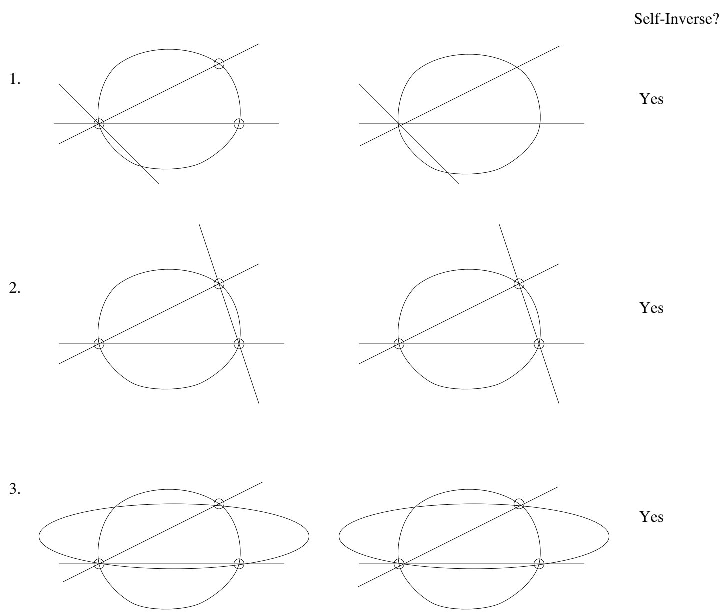


Figure 4.29:

4.3.6 $i = 7, j = 8$

The only possible relative positions for the two small relevant diagrams to occur are shown in figure 4.30.

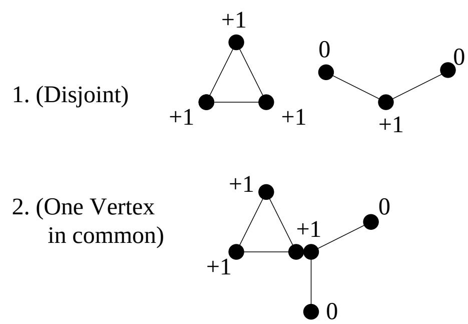


Figure 4.30:

We know how to deal with the disjoint case. The relevant K ends up satisfying the axioms of theorem 3.1.3 and so leads to nothing new.

We also know how to deal with the second case. Blowing up lost ambiguous curves eventually leads to K with axioms as in Axiom 4.3.3. Again nothing new.

4.3.7 $i = j = 8$

The possible relative positions for the two small relevant diagrams are shown in figure 4.31.

Again the first and the last cases are easiest and lead to the axioms in theorem 3.1.3.

The second position leads to the axioms in Axiom 4.3.3.

The third position leads to the axioms in Axiom 4.3.4.

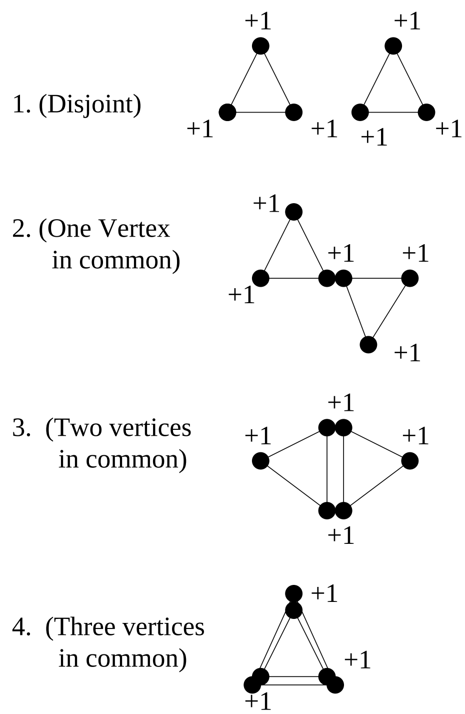

1. (Disjoint)
+1
+1
+1
+1
+1
+1
2. (One Vertex
in common)
+1
+1
+1
+1
+1
+1
3. (Two vertices
in common)
+1
+1
+1
+1
4. (Three vertices
in common)
+1
+1
+1

Figure 4.31:

4.3.8 $i = j = 10$

Recall that a case 10 arrangement consists of exactly three lines l, m, n and a conic c all concurrent at a point say P. The lines also intersect the conic c at Q, R, S respectively. There is at least one other conic which passes through P, Q, R, S, as must all such other conics. There may be in addition some other lines. These would then have to all pass through P. Let x be the total number of lines so that there are $x - 3$ other lines not l, m, n, and let y be the total number of conics. Figure 4.8 shows that the ordered pair (x, y) completely determines the plumbing graph which is a single weighted $+2$ vertex with edges to every vertex of the complete bipartite graph $K(x, y)$, thus showing that only (x, y) and (y, x) can lead to the same case 10 graph.

4.3.9 Algorithm for Regular Arrangements

We can now collect together all the information to derive an algorithm which for a given arrangement A_1 finds all arrangements A_2 which are different from A_1 but have same link. It basically means to systematically search A_1 for the subarrangements found in figures 3.46, 3.47, 3.48, 3.49 and also the subarrangements found in figures 4.15 and 4.26. Provided all curves present fit within the scheme we can just do the replacement by the inverse curve method as in Algorithm 3.4.5, and see whether or not a new arrangement arises. Of course if A_1 is a case 10 arrangement we will also need section 4.3.8.

Chapter 5

Birational Topological Equivalence

5.1 Introduction

In sections 3 and 4 we saw that the topology of the link of an arrangement did not necessarily determine the topology of the arrangement itself, although for this implication to be false the arrangements of curves had to be very special indeed. However in the cases where the implication is false, we did see the following.

Theorem 5.1.1 *If A_1, A_2 are two arrangements of nonsingular curves with all crossings pairwise transversal such that the links of A_1 and A_2 are homeomorphic, then A_1 and A_2 are BTE ("Birationally Topologically Equivalent").*

We make this precise. For any birational map β on $\mathbb{CP}^2$, denote by $s(\beta)$, the set of points in $\mathbb{CP}^2$ for which β is not an injection either because it is not one to one or because it is not well defined. One can see that $s(\beta)$ consists precisely of those curves which are blown down.

Definition 5.1.1 (BE) *Two specific arrangements of curves* A_1*,* A_2 *are said to be* BE *(birationally equivalent) if there is a birational map* β *on* $\mathbb{CP}^{\not\leq}$ *such that*

- $s(\beta)$ *are curves in* A_1*,*
- $s(\beta^{-1})$ *are curves in* A_2*,*
- *and* β *induces a bijection*

  $$\beta : A_1 \setminus s(\beta) \to A_2 \setminus s(\beta^{-1})$$

  between the curves of A_1 *and the curves of* A_2 *outside of* $s(\beta)$*.*

Definition 5.1.2 (BTE) *Two arrangements of curves* A_1*,* A_2 *are said to be* BTE *(birationally topologically equivalent) if there is an arrangement* A'_1 *equivalent to* A_1 *such that* A'_1*,* A_2 *are BE.*

For example let A_1 be an arrangement of four lines in general position, and A_2 be the arrangement of a conic passing through the intersection points of a triangle of lines and in particular has equation

$$xyz(xy + yz + xz) = 0$$

in homogeneous coordinates. Now choose the map T

$$T : (x : y : z) \mapsto (yz : xz : xy).$$

T is in fact well known as a Cremona transformation. It is a birational equivalence on $\mathbb{CP}^2$ and is also an involution. Furthermore T is not a well defined map on the three point set $\{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\}$ and is not injective on the set $xyz = 0$, but away from this three line set T is a homeomorphism. Now applying T to A_2 yields

$$X^2Y^2Z^2(XYZ^2 + X^2YZ + XY^2Z) = 0$$

that is

$$XYZ(X + Y + Z) = 0$$

Hence the image of A_2 is four lines in general position. Since T is a homeomorphism away from $xyz = 0$, it is certainly a homeomorphism on the complement provided that the moduli space of A_1 is connected (which it is). (We are using the term: moduli space, in a loose sense to mean any parameter space. See theorem 6.2.1 and the paragraph which follows.)

We now develop a simpler characterisation of BTE.

Definition 5.1.3 (RE) *Two arrangements A_1, A_2 are said to be* RE *(rationally equivalent) if we can obtain $Pl(A_2)$ from $Pl(A_1)$ using only (-1)-blow-ups or (-1)-blow-downs. That is only R2 moves.*

Theorem 5.1.2 *Assume we do a sequence of (-1)-operations starting with $\mathbb{CP}^2$ such that we have done m_1 blow-ups and m_2 blow-downs. Then $m_1 \geq m_2$ and $m_1 = m_2$ if and only if we finish up with $\mathbb{CP}^2$*

This result can be deduced from [1] pp468–469 and using the fact that $\mathbb{CP}^2$ contains no (-1)-curves which can be blown down.

Theorem 5.1.3 *Assume arrangements A_1, A_2 are RE, then the number of (-1)-blow-ups in the passage from A_1 to A_2 via the plumbing diagrams*

$$A_1 \to Pl(A_1) \to Pl(A_2) \to A_2$$

equals the number of (-1)-blow-downs.

Proof: Assume in the passage from A_1 to A_2 there are m_1 blow-ups and m_2 blow-downs. If $m_1 \neq m_2$ we can assume without loss of generality that

$m_1 > m_2$. Now consider the passage from A_2 to A_1. Even though the (-1)-operations between $Pl(A_1)$ and $Pl(A_2)$ have no ambient space in which to do them we can still do these operations on A_2 which carries its ambient space with it. Since we do more blow-downs than blow-ups, there must be some point where we have done as many blow-downs as blow-ups, and the next (-1)-operation is a blow-down. However by theorem 5.1.2 we end up with an arrangement in $\mathbb{CP}^2$ at this point and cannot blow down any curves in $\mathbb{CP}^2$ as we need to.

□

Corollary 5.1.4 *RE is equivalent to BTE.*

Proof: If A_1 and A_2 are RE then by theorem 5.1.3 the number of blow-ups equals the number of blow downs. It is well known that if this is the case then the blowing-up and blowing down can be achieved by a birational map on $\mathbb{CP}^2$. The reverse implication holds because any birational map can be constructed as a combination of blow-ups and blow-downs. (See [1] pp468–469.)

□

It would be tempting to conjecture that the only situations in which the link does not determine the arrangement are the situations in sections 3 and 4. But this can be seen not to be the case. In fact there are infinite families using singular curves where the link does not determine the arrangement. In fact one such family can be seen to be an extension of cases 1 and 4 in section 3. A_1 consists of a degree n curve C with an ordinary order $n-1$ point P and with $2n$ lines $l_1, \dots l_{2n}$ passing through P transversally. There is also one additional line m in the arrangement which we now describe. By Bezout's theorem $l_i \cap C$ consists of two points P and another which we shall name Q_i. We take m to be any line passing through Q_1 and intersecting everything else in general position. We form its plumbing diagram by blowing up points

P and all the Q_i. (Strictly we do not need to blow up $Q_2, \ldots, Q_{2n}$ but this makes the example transparent.) The plumbing diagram is shown in figure 5.1 where the marked vertices are the top $2n+1$ vertices.

A_2 consists of two degree n curves C_1, C_2 with ordinary $n-1$ points both at P but with all intersections being pairwise transversal and $2n$ lines $l_1, \ldots l_{2n-1}, m$ which we now describe. The intersection multiplicity of C_1, C_2 at P is $(n-1)^2$. By Bezout's theorem C_1, C_2 must intersect in $2n-1$ other distinct points $Q_1, \ldots Q_{2n-1}$. We take l_i to be the line joining P, Q_i and we take m to be a line in general position passing through P. We form the plumbing diagram by blowing up points P and all the Q_i and also the point where m intersects C_1. (Again, strictly we do not need to blow up this last point, but it makes the example clearer.) The plumbing diagram is shown in figure 5.1 where the marked vertices are the bottom $2n+1$ vertices.

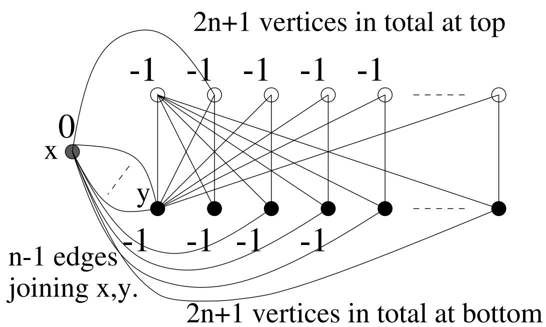


Figure 5.1:

The significance of this family is that the ambiguous family can be arbitrarily large. Indeed for the nth case we have a degree n curve and $2n$ lines, that is $2n+1$ ambiguous curves. However we still see the passage between the two arrangements as being a sequence of (-1)-blow-ups and blow-downs, that is a BTE.

This conjecture (BTE) would be highly desirable to prove in full generality, and amounts to the following:

Conjecture 5.1.5 *Given arbitrary arrangements A_1 and A_2 with homeomorphic links, first blow up relevant points in each arrangement so as to obtain plumbing diagrams for A_1 and A_2. Now we know that $Pl(A_1)$ and $Pl(A_2)$ describe the same 3-manifold and so we can transform $Pl(A_1)$ into $Pl(A_2)$ via plumbing calculus moves. We conjecture that only (-1)-blow-ups and blow-downs are needed for this.*

Although a proof of this has not yet been found, we do prove it for a large number of cases.

Before closing this section, however, we need a few lemmas that justify the infinite family of counterexamples and also the processes used in the next section.

Lemma 5.1.6 *Let C be an algebraic curve in an algebraic surface M. Let $P \in C$ be a singular point of order n, then after blowing up at P the total transform of C counting multiplicities is $\tilde{C} \cup nE$ where $\tilde{C}$ is the strict transform of C and E is the exceptional divisor. Furthermore we have the intersection number $\tilde{C}.E = n$.*

Proof: C can be locally described as follows.

$$\sum_{i=0}^{n} a_i x^i y^{n-i} + O(n+1) = 0$$

Where $a_0 \neq 0$ and $O(n+1)$ denotes those terms of degree at least $n+1$. Since $a_0 \neq 0$ we can see that all the action occurs in the chart $y = uv$, $x = u$. In the preimage we see that $O(n+1)$ is divisible by u^{n+1} so that the equation becomes

$$\sum_{i=0}^{n} a_i u^i u^{n-i} v^{n-i} + u^{n+1} q(u,v) = 0$$

thus

$$u^n(\sum_{i=0}^{n} a_i v^{n-i} + uq(u,v)) = 0.$$

And this establishes the lemma since $u = 0$ is the exceptional divisor E in this chart and $\sum_{i=0}^{n} a_i v^{n-i} + uq(u,v) = 0$ is the strict transform $\tilde{C}$ and of course intersects $u = 0$ n times counting multiplicities because $a_0 \neq 0$.

□

Corollary 5.1.7 *Let C be an algebraic curve in an algebraic surface M. Let $P \in C$ be a singular point of order n, then blowing up at P reduces the self-intersection number of C by n^2 in the strict transform of C.*

Proof: Using the lemma we have the following.

$$\begin{aligned} C.C &= (\tilde{C} + nE).(\tilde{C} + nE) \\ &= \tilde{C}.\tilde{C} + 2n\tilde{C}.E + n^2 E.E. \end{aligned}$$

Now $\tilde{C}.E = n$ by the lemma and also we have $E.E = -1$, thus

$$\begin{aligned} \tilde{C}\tilde{C} &= C.C - 2n\tilde{C}.E - n^2 E.E \\ &= C.C - 2n^2 + n \\ &= C.C - n^2. \end{aligned}$$

□

Comment: This allows us to conclude for example that the curve $y^n = x^n + x^{n+1}$ of genus zero with the origin being an nth order singularity has plumbing diagram as calculated in figure 5.2.

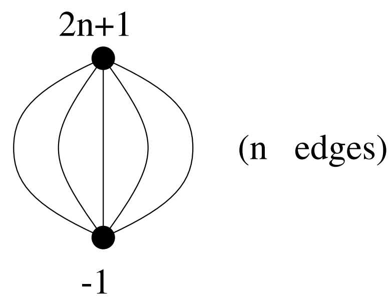


Figure 5.2:

Lemma 5.1.8 *The genus of a nonsingular curve of degree n is $\frac{1}{2}(n-1)(n-2)$. If the curve has only ordinary singularities, then each n_ith order singularity reduces the genus by $\frac{1}{2}n_i(n_i-1)$.*

This result can be found in [1] p614.

Lemma 5.1.9 *There is no genus zero curve with exactly two singularities, both being ordinary.*

Proof: Assume there is a curve of degree n with exactly two singular points of orders $a, b > 1$, both being ordinary singularities. Then by lemma 5.1.8 we have:

$$\frac{1}{2}(n-1)(n-2) - \frac{1}{2}a(a-1) - \frac{1}{2}b(b-1) = 0$$

thus if $a + b \leq n - 1$ we have

$$\begin{aligned} n^2 - 3n + 2 &= a^2 + b^2 - (a+b) \\ &< a^2 + b^2 + 2ab - (a+b) \\ &= (a+b)(a+b-1) \\ &\leq (n-1)(n-2) \end{aligned}$$

Which is a contradiction. Thus we must have $a + b \geq n$. But Bezout's theorem after considering a line passing through the two singular points, forces $a + b = n$. Hence

$$\begin{aligned}(a+b)^2 - 3(a+b) + 2 &= a^2 + b^2 - (a+b)\\ a^2 + 2ab + b^2 + 2 &= a^2 + b^2 + 2a + 2b\\ ab - a - b + 1 &= 0\\ (a-1)(b-1) &= 0\end{aligned}$$

Thus either $a = 1$ or $b = 1$, a contradiction.

5.2 The Pairwise Transversal Crossings Singular Case

In this section we prove the following result, which is about as general as we can get without proving the whole result.

Theorem 5.2.1 *Let $\mathcal{C}$ be the class consisting of arrangements A, where A consists of the distinct irreducible curves $C_1, C_2, \ldots, C_n$, and such that any $A \in \mathcal{C}$ fulfills at least one of the following two conditions.*

1. *$\cup_{i=1}^{n} C_i$ is a (reducible) curve whose singularities are all ordinary.*
2. *The initial plumbing diagram for A contains a cycle and the passage to normal form can be achieved using only (-1)-blow-ups or (-1)-blow-downs.*

Then BTE holds for the class $\mathcal{C}$.

Note that the condition (2) is a very weak condition in that most arrangements satisfy (2). In fact most arrangements which satisfy (1) will also satisfy (2).

Lemma 5.2.2 *If C_1, C_2 are two curves in A such that their intersection consists of at least two distinct points, then the initial plumbing diagram for A will contain a cycle.*

Proof: Let $P, Q \in C_1 \cap C_2$. Then both C_1 and C_2 are represented by vertices v_1, w_1 in the initial plumbing diagram. However since C_1 intersects C_2 at P, then there is a path λ via P (P is either an edge if P is an ordinary double point which is not a singular point of either curve, or an exceptional vertex otherwise.) from C_1 to C_2. Similarly there is a path μ via Q from C_2 to C_1. But since $P \notin \mu$ and $Q \notin \lambda$ we have that the path $\lambda + \mu$ is a cycle as required.

□

Our current task is now to quantify precisely which arrangements of (1) are not in (2).

Theorem 5.2.3 *The set of arrangements in (1) which are not in (2) are precisely the arrangements which follow.*

1. *Line and Pencil*
2. *Pencil*
3. *Double line*
4. *Single line*
5. *Single conic*
6. *Conic and Line*

7. *Nodal Cubic*

8. *Degree n genus zero curve with ordinary order $n-1$ point with m lines passing through the singular point with pairwise transversal crossings.*

9. *Single nonsingular curve of degree n.*

Lemma 5.2.4 *Let P, Q be two points of orders x, y respectively on a curve C of degree $n \geq 2$ with only ordinary singularities, then $x^2 + y^2 < n^2$.*

Proof: We certainly have $x(x-1) + y(y-1) \leq (n-1)(n-2)$ and thus

$$\begin{aligned} x^2 + y^2 &\leq n^2 - 3n + 2 + x + y \\ &\leq n^2 - 3n + 2 + 2(n-1) \text{ since } x, y \leq n-1. \\ &= n^2 - n \\ &< n^2 \end{aligned}$$

□

Lemma 5.2.5 *Let C, D be two curves of degree at least 2 such that $C \cup D$ is a curve with ordinary singular points only. Then the number of intersection points on C is at least 3.*

Proof: Let the degrees of C and D be m and n respectively. Assume first that C, D intersect only in one point P. If x is the order of C at P and u is the order of D at P, then using Bezout yields

$$\begin{aligned} mn &= xu \\ &\leq (m-1)(n-1) \\ &< mn \end{aligned}$$

a contradiction. Alternatively if C, D intersect in precisely two distinct points P, Q. Let x, y be the orders of C at P, Q respectively and let u, v be the orders of D at P, Q respectively. Bezout yields

$$\begin{aligned} mn &= xu + yv \\ &\leq \sqrt{x^2+y^2}\sqrt{u^2+v^2} \text{ (Cauchy Schwartz)} \\ &< \sqrt{m^2}\sqrt{n^2} \text{ by the lemma} \\ &= mn \end{aligned}$$

another contradiction.

□

Lemma 5.2.6 *Let C be a curve of degree $n \geq 2$ and l a line such that $C \cup l$ is a reducible curve with only ordinary singularities. Then C intersects l in at least two points.*

Proof: This is immediate from the fact that

$$\sum_{P_{n_i}} n_i(n_i - 1) \leq (n-1)(n-2)$$

where the sum ranges over all singular points P_{n_i} of order n_i. □

Corollary 5.2.7 *The initial plumbing diagram for an arrangement A contains a cycle, provided that A contains at least two curves, one of which is not a line.*

Lemma 5.2.8 *In an arrangement of at least two curves with only pairwise transversal crossings and containing at least two curves which are not lines, the passage to normal form can be achieved using only (-1)-blow-down operations.*

Proof: If there are at least two curves of degree at least two, then they will all have at least three intersection points on them (lemma 5.2.5) so that their valences in the plumbing diagram are all at least three. Thus only lines have the possibility of having valence at most two in the plumbing diagram. Let l be any such line, thus the valence of l is equal to two by lemma 5.2.6. Now let C, D be the two curves of degree at least two. Then l intersects C twice in points P, Q only. Furthermore l intersects D twice in points P, Q only. Thus there are at least three curves passing through both P and Q, namely C, D and l, making it necessary for these points to be blown up. Hence the vertex representing l in the plumbing diagram has weight -1. Thus we can do simultaneous (-1)-blow-downs on all vertices in the plumbing diagram which represent a line with exactly two intersection points on it. This does not change the valences of any other vertices, and since all other vertices have valence exceeding two, we attain normal form.

□

Definition 5.2.1 *Let C be a curve in an arrangement A. A* special *point on C is any point P on C which is either an intersection point of C with some other curve, or a singular point of C.*

Lemma 5.2.9 *A genus zero curve C of degree $n \geq 3$ intersecting a line l always has the property that the number of special points from $l \cup C$ which lie on C is at least three, unless C contains an ordinary order $n-1$ point through which l passes.*

Proof: We already know that l intersects C in at least two points so assume that $l \cap C = \{P, Q\}$, with the order of C at P, Q being $x, y \geq 2$ respectively. Then $x + y = n$ by Bezout. However we also have

$$\begin{aligned} x(x-1) + y(y-1) &= (n-1)(n-2) \\ &= (x+y-1)(x+y-2) \end{aligned}$$

$$\begin{aligned} x^2 - x + y^2 - y &= x^2 + 2xy + y^2 - 3x - 3y + 2 \\ \Rightarrow 2xy - 2x - 2y + 2 &= 0 \\ \Rightarrow (x-1)(y-1) &= 0 \end{aligned}$$

a contradiction. If say $x = 1$ then $y = n - 1$ and we get the situation which we have excluded.

□

Corollary 5.2.10 *In an arrangement of at least three curves, all curves of degree at least two have at least three special points on them.*

Lemma 5.2.11 *Let C be a curve of nonzero genus and of degree at least three. Let l be a line which intersects C in exactly two points P, Q. Then both P, Q are singular points of C.*

Proof: If say P were not a singular points of C then using Bezout's theorem we see that the multiplicity of the intersection at Q is $n - 1$ which violates the nonzero genus premise.

□

Corollary 5.2.12 *Let C be a curve of degree at least three and l a line intersecting C in exactly two points P, Q. Then both P, Q are singular points of C unless C has an ordinary $n - 1$ order point through which l passes.*

Corollary 5.2.13 *Let A be an arrangement consisting of a curve C of degree $n \geq 3$ and some lines $l_1, l_2, \ldots, l_m$. Then the passage to normal form can be achieved using only (-1)-blow-down operations unless C has an ordinary $n - 1$ order point through which all the l_i pass.*

Proof: Assume otherwise, then we quickly see that C must have an ordinary $n-1$ order point P. If $P \notin l_i$ for some i, then the number of special points on l_i is at least three. Also if $P \in l_j$ for some j then either the number of special points on l_j is at at least three if $l_j \cap (l_i \cap C) = \phi$ or else l_j must have have two points blown up on it, namely P and $l_i \cap C$. In either case the passage to normal form only uses (-1)-blow-down operations.

□

Putting everything together proves theorem 5.2.3.

We can now prove BTE for the class $\mathcal{C}$.

Proof: Class $\mathcal{C}$ can be resubdivided into the following mutually exclusive subclasses.

- $(1)' = (1) \setminus (2)$ - The list in theorem 5.2.3
- $(2)' = (2)$ - as before.

We have three comparisons to do.

If $A_1, A_2 \in (1)'$ we see that all the normal forms in $(1)'$ are different so that BTE (in fact topological equivalence) holds for A_1, A_2.

If $A_1 \in (1)'$ and $A_2 \in (2)'$, then the normal form for A_2 will contain a cycle because the (-1)-operations do not change the number of cycles. Thus A_1 could only possible be the seventh or eighth situations in the list from theorem 5.2.3. But this is impossible because (-1)-operations cannot introduce negative signs on edges. Thus we have topological equivalence again and hence BTE.

If $A_1, A_2 \in (2)'$ then of course we have BTE.

This completes the proof of theorem 5.2.1.

□

We note that BTE holds almost vacuously for another large class $\mathcal{C}^\infty$.

Theorem 5.2.14 *Let $\mathcal{C}^\infty$ be the class of arrangements such that the passage to normal form only requires (-1)-operations. Then in $\mathcal{C}^\infty$ we have BTE.*

Comment: I can as yet see no way of obtaining BTE for the class $\mathcal{C}^{\in} = \mathcal{C} \cup \mathcal{C}^{\infty}$. For example it may be possible using only (-1)-operations on some unusual arrangement to achieve the normal form for a pencil of lines.

5.3 BTE Difficulties

It is very tempting to try and generalise BTE to all arrangements, however this turns out to be very difficult and it is the purpose of this section to highlight some of the difficulties.

It can be shown that an arrangement such that each curve contains at least three special points yields a plumbing graph which is already in normal form except for whatever simple double points which are singular points of a curve where the exceptional divisor involved needs to be blown down. However it is possible for each pair of curves to have only one point in common even if we permit nonzero genus.

Consider the two singular curves

$$\begin{aligned} p(x,y) &= y - x^n \\ q(x,y) &= y^n \end{aligned}$$

In the projective completion one computes that they intersect in exactly one point in $\mathbb{CP}^2$, namely $(0:0:1)$ and with multiplicity n^2. Furthermore $(0:0:1)$ is not a singular point of the curve $p = 0$, thus a generic linear combination of p and q will have no singular points. In particular taking $r = p + q$ and $s = p - q$ it is routine to check that r, s are two nonsingular curves of degree n and intersect in a single point $(0:0:1)$ with multiplicity n^2. In fact even in the resolution the vertices representing the curves defined by $r = 0$ and $s = 0$ both have valence 1.

Nonzero genus however is helpful, since reduction to normal form involves genus zero vertices. Thus it is the genus zero curves which really cause the most trouble. There are many examples of arrangements whose normal form cannot be realised unless one uses operations in addition to the (-1)-operations.

We have for example the infinite family of arrangements $A_1, A_2, \ldots, A_n, \ldots$ which all define homeomorphic links. The arrangement A_n is the union of the two curves $y^n = x^{n+1}$ and $y = 0$. See figure 5.3 for resolution.

Forming the plumbing diagram and then performing all the obvious (-1)-blow-downs yields figure 5.4

Doing a zero chain absorption yields its normal form. (Figure 5.5.)

Yet we can see that the nth diagram is equivalent to the $(n-1)$th diagram using only (-1)-operations as follows (figure 5.6) which inductively establishes the claim. Yet we cannot reduce to normal form using only (-1)-operations.

The problem however is that the singularities could get very complicated. In fact in the next section we show how bad it can get.

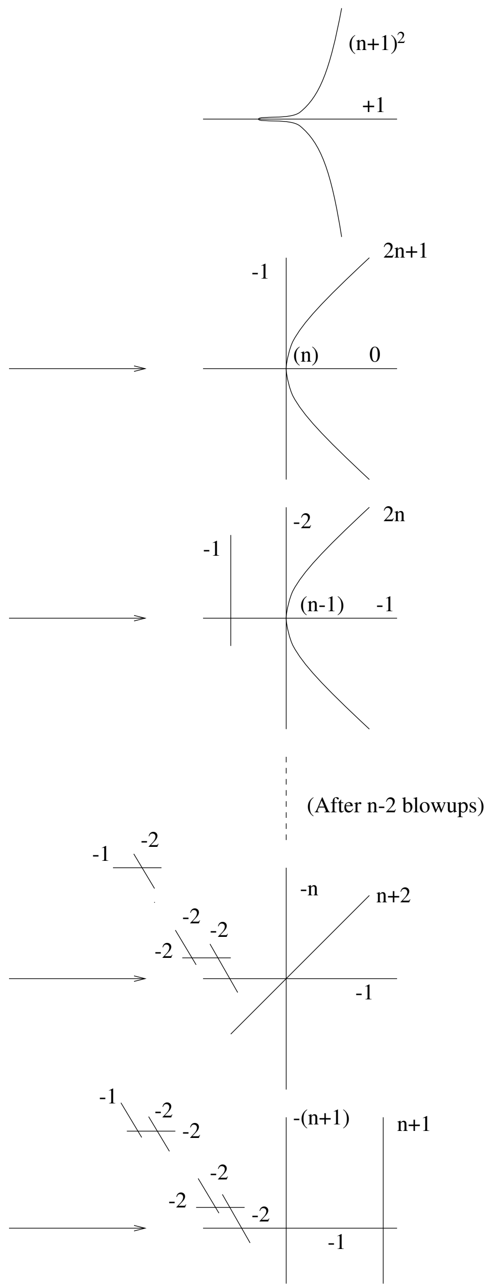

(n+1)^2
+1
-1
2n+1
(n)
0
-2
2n
-1
(n-1)
-1
(After n-2 blowups)
-1
-2
-2
-2
-2
-n
n+2
-1
-1
-2
-2
-(n+1)
n+1
-2
-2
-2
-1

Figure 5.3:

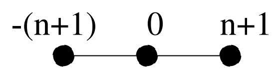


Figure 5.4:

Figure 5.5:

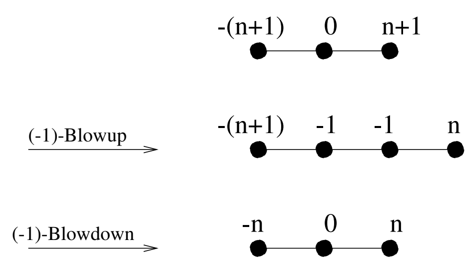


Figure 5.6:

5.4 Cremona Transformations

Recall the basic Cremona transformation of $\mathbb{CP}^2$ up to linear equivalence can be written as

$$T_0 : (x : y : z) \mapsto (yz : xz : xy)$$

It is in fact a bicontinuous bijection outside of the set $xyz = 0$ which is a union of three distinct nonconcurrent lines. We see that the lines $x = 0$, $y = 0$ and $z = 0$ are mapped to the points $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$ respectively. We also loosely say that the points $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$ get mapped to the lines $x = 0$, $y = 0$ and $z = 0$ respectively even though T_0 is not well defined on the three points. Note also that T_0 is an involution.

In terms of plumbing diagrams T_0 looks like figure 5.7

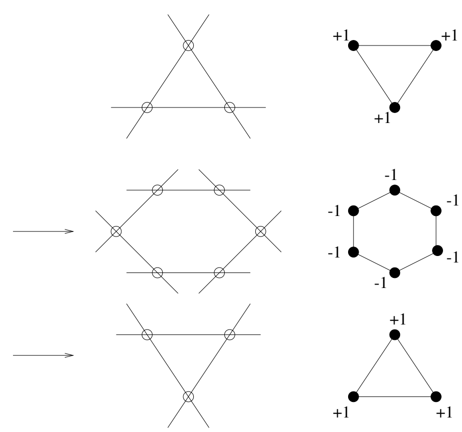


Figure 5.7:

We also have the two singular Cremona transformations T_1 and T_2 (see [1] p469). The transformation T_1 is a limiting case of where two of the lines of the triangle in T_0 are permitted to coalesce. Up to linear equivalence T_1 is given by

$$T_1 : (x : y : z) \mapsto (x^2 : xy : yz)$$

Both lines are collapsed to a point, but one line is more badly collapsed. Also two points are blown up, but a further blow-up is done on the exceptional divisor where the original lines intersected. Figure 5.8 shows what it looks like in terms of plumbing diagrams.

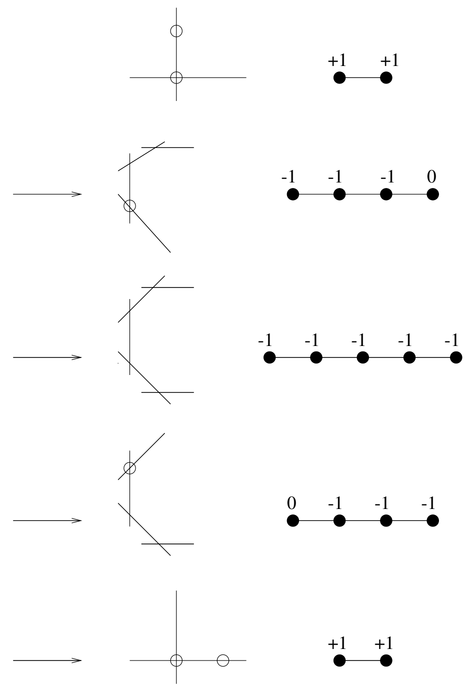


Figure 5.8:

The equation for T_2 up to linear equivalence is given by

$$T_2 : (x : y : z) \mapsto (xy + cz^2 : y^2 : yz) \ (c \neq 0)$$

Here the single line $y = 0$ is badly collapsed to a point. Branches of curves passing through $(1 : 0 : 0)$, however, may be highly separated. Other branches of curves intersecting $y = 0$ at different points are all made to intersect with tangency so that they all become tangent to the line $y = 0$ at the point $(1 : 0 : 0)$. Figure 5.9 shows what is happening in terms of the plumbing diagram.

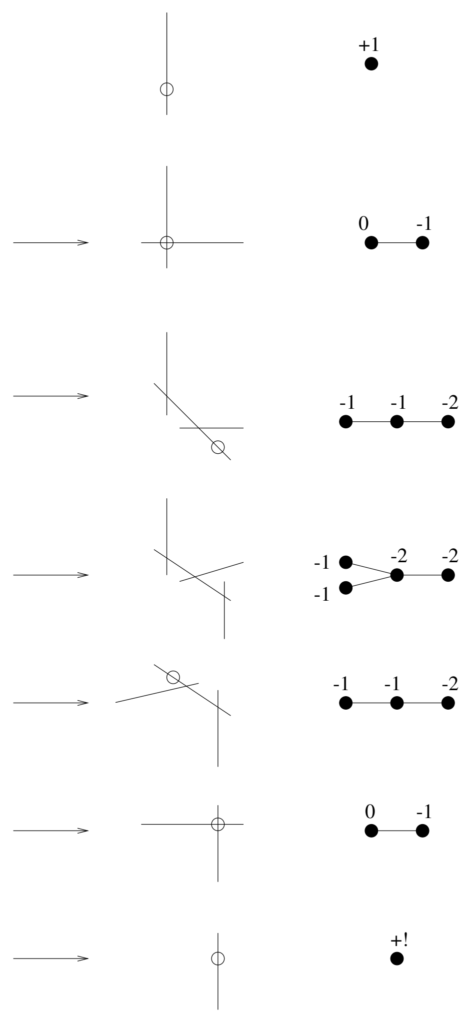


Figure 5.9:

Now if an arrangement contains some lines, we can use a Cremona transformation to obtain a different arrangement, but with same complement. In particular T_2 causes the worst complications. Indeed if B_1, B_2 are branches of curves which intersect a line l transversally in distinct points P_1, P_2 and P is chosen to be different from P_1, P_2, then applying (after a linear map) T_2 causes B_1', B_2' to both be tangential to each other and to both be tangential to the line P' with third order contact.

For a particular example, let the $n+1$ lines $l, l_1, l_2, \ldots, l_n$ all concur at P_0. After applying T_2 along l and at P_0 we end up with n conics all tangent to a line at the same point and tangent to each other with order four contact.

Note that generally T_0, T_1, T_2 all double the degree of every (other) curve unless they are chosen very carefully. Thus if we have an arrangement A_0 containing a line l_0, choose a point P_0 to be a generic point (i.e. not an intersection point) on l. Apply T_2 to get an arrangement A_1. Note that A_1 will contain a line l_1. Choose P_1 to be a generic point on l_1. Apply T_2 to get an arrangement A_2. As we keep repeating this process, the curves and their singularities get quite bad (in fact even A_1 will have only one intersection point) and can be of arbitrarily high degree. Yet they all have the same normal form plumbing diagram. Of course in terms of plumbing diagrams all we are doing is finding the vertex representing the line in question and doing three blow ups as shown in figure 5.10.

This suggests a possible strategy to obtain a BTE result for very complicated arrangements which contain only one line. If a genus zero C curve is present and in the plumbing diagram has valence one. It certainly intersects the line which we shall call l somewhere say at P. Now do a T_0 operation along l at P which separates l and C as much as possible. Hopefully the new situation is simpler and hopefully we can eventually transform the situation into one where we have two or more lines and then use T_1 and T_0 in appropriate ways. This is trying to model the reverse of the previous example where

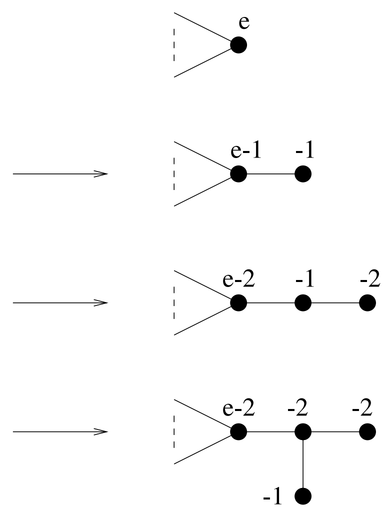


Figure 5.10:

the situation was purposely complicated.

Another bad problem that can arise in reduction to normal form is that a vertex of valence one or two and of weight zero may appear. A really bad example is the family of arrangements $A_2, A_3, \ldots, A_n, \ldots$. Where A_n consists of the singular curve C which has an order $n-1$ point at the origin and the line l which passes through the origin at general position. We see in the immediate resolution plumbing diagram that l is represented by a vertex of degree one and weight zero and thus allows splitting.

Chapter 6

The Complement

6.1 Introduction

Given an arrangement $A \subseteq \mathbb{CP}^2$, we would like to know when the arrangement A itself determines the topology of $M(A)$, the complement of A in $\mathbb{CP}^2$. This is a very difficult question in general. Indeed even in the case where the arrangement consists of a union of lines difficulties arise and we content ourselves with exploring only this case.

In terms of arrangements, a union of lines is very easy to deal with because any reasonable definition of what constitutes equivalent line arrangements will coincide with the definition of an arrangement used throughout this thesis. The definition used basically codes the lattice structure of the arrangement. Here we will describe a line arrangement in terms of its moduli space, $\text{Mod}(A)$. The following result is well known.

Theorem 6.1.1 *Let A be an arrangement. If Mod(A) is connected then the arrangement determines its complement.*

Proof (Sketch): Let A_0, A_1 be two points in $\text{Mod}(A)$, and $\lambda(t)$ a path joining A_0, A_1. Then λ defines a continuously varying set of arrangements

A_t with the same topology as A_0 and A_1, so that A_0, A_1 are isotopic, which in turn implies that they are ambient isotopic and thus have homeomorphic complements. (See [18].)

□

This indicates that we have three possibilities worthy of discussion for an arrangement A.

1. The moduli space of A is connected.
2. Two arrangements $A_1, A_2 \in \mathrm{Mod}(A)$ are isotopic.
3. Two arrangements $A_1, A_2 \in \mathrm{Mod}(A)$ have homeomorphic complements.

Now we certainly know that (1) implies (2) and that (2) implies (3). Hence we investigate (1), (2) and (3) in turn.

6.2 Moduli Spaces

Since we are restricting ourselves to line arrangements, we can see that two line arrangements are equivalent if and only if they generate equivalent lattices which in turn coincides with the moduli space criterion.

Definition 6.2.1 *Given an arrangement A of lines, we say a line l is* removable *from the arrangement if the set of triple or higher order points on l are always collinear for the deleted arrangement $A \setminus l$.*

Definition 6.2.2 (Brittle) *An arrangement A of lines is said to be* brittle *if it is not possible to delete in succession removable lines and end up with the empty set.*

Definition 6.2.3 *We say a brittle arrangement is* reduced *if the arrangement contains no removable lines.*

Theorem 6.2.1 *All non-brittle arrangements have irreducible moduli spaces*

Here we describe the moduli space for line arrangements a little differently. Say that there are n lines and that the line l_i has equation $a_i x + b_i y + c_i z = 0$. Then the line l_i can be described by a single point $(a_i : b_i : c_i) \in \mathbb{CP}^2$. The three lines l_i, l_j and l_k concur if and only if the determinant

$$\begin{vmatrix} a_i & b_i & c_i \\ a_j & b_j & c_j \\ a_k & b_k & c_k \end{vmatrix} = 0$$

We abbreviate this to $R(i, j, k) = 0$. Thus the moduli space can be written as a subspace of $(\mathbb{CP}^2)^n$ determined by a family of relations $R(i, j, k) = 0$ with the subspace determined by some further family of relations $R(i', j', k') = 0$ deleted. Note that we want to delete also the subspace which codes lines being coincident. This will usually have been taken care of solely by the above procedure.

Lemma 6.2.2 *Let A be an arrangement of n lines such that the line l_α is removable. If Mod($A \setminus l_\alpha$) is irreducible, then so is Mod(A).*

Proof: $\mathrm{Mod}(A \setminus l_\alpha)$ can be described as an open subset S of some irreducible closed variety V. Say that l_α has coefficients $(a_\alpha : b_\alpha : c_\alpha)$.

If l_α has no triple points, then no extra relations are added so that $\mathrm{Mod}(A)$ is an open subset of $S \times \mathbb{CP}^2$, which is irreducible.

If l_α has exactly one triple point then one relation is added but is linear in $a_\alpha, b_\alpha, c_\alpha$ and so we see that $\mathrm{Mod}(A)$ is isomorphic to an open subset of a $\mathbb{CP}^1$-bundle over S, and so is irreducible.

If l_α has at least two triple points on it, we only need two corresponding relations to describe l_α. (The other relations will be consequential since l_α is removable.) This completely determines l_α yielding that $\mathrm{Mod}(A)$ is isomorphic to an open subset of S and hence irreducible.

□

In view of the fact that an open subset of an irreducible complex variety is always connected, we have the following corollary.

Corollary 6.2.3 *If A is a non-brittle arrangement, then Mod(A) is connected and thus A determines its complement.*

Non-brittleness is usually very easy to detect and it is a much simpler concept than Jiang and Yau's "nice" criterion found in [8] p139. We now investigate all small arrangements of lines and determine which ones are reduced brittle arrangements.

Theorem 6.2.4 *All arrangements of at most six lines are non-brittle.*

Proof: Let $n \leq 6$ be the smallest positive integer such that there exists a brittle arrangement of n lines. Thus we can further assume that the arrangement is a reduced brittle arrangement and so contains no removable lines. Thus every line contains at least three triple (or higher order) points and this forces the number of lines to be at least seven.

□

Theorem 6.2.5 *All arrangements of at most seven lines are non-brittle.*

If there were a brittle arrangement of seven lines, then again every line would have to contain at least three triple (or higher order) points. Thus every line contains exactly three triple points and no other sorts of intersection points. Choose the l_1 and assume we have $R(1,2,3)$, $R(1,4,5)$ and $R(1,6,7)$. Now the four lines l_2, l_3, l_4, l_5 intersect in four distinct points which themselves must be triple points, so that l_6 and l_7 must pass through them. This very quickly yields figure 6.1 after relabeling.

The incidences are $R(1,2,4)$, $R(1,3,5)$, $R(2,3,6)$, $R(1,6,7)$, $R(2,5,7)$, $R(3,4,7)$ and $R(4,5,6)$. Under a T_0 type Cremona transformation on the

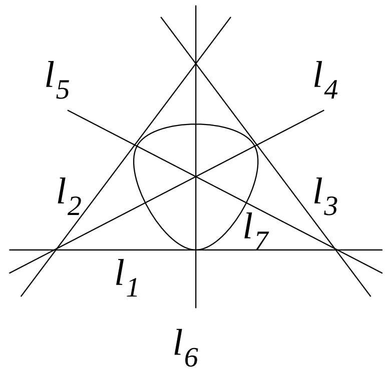


Figure 6.1:

lines l_1, l_2, l_3, we obtain that l_7' becomes a conic which is tangent to each of the three lines l_4', l_5', l_6'. Yet these three lines are concurrent at a point P say. This is impossible because there are at most two distinct lines though a given point which are tangent to a given conic.

□

Theorem 6.2.6 *There is exactly one eight-line arrangement which is brittle. Furthermore its moduli space is disconnected.*

Proof: Again all lines of a reduced brittle arrangement must contain at least three triple or higher order points. If there were a quintuple or higher order point then we immediately require at least nine lines. If four lines l_1, l_2, l_3, l_4 concurred at a quadruple point, then each of those four lines must have in addition two other triple points on them and no other sorts of intersection points. Thus any of the other four lines say l_5 must intersect l_1, l_2, l_3, l_4 in triple points. Thus l_5 has four triple points yielding at least nine lines.

Thus all lines contain exactly three triple points and one double point. Say l_1, l_2, l_3 are concurrent. Now l_1 intersects some line say l_8 at a double point. Say also that l_2, l_3 intersect l_7, l_6 respectively at double points. Now

l_8 intersects l_2, l_3 in triple points and without loss of generality assume that we have $R(3,4,8)$ and $R(2,5,8)$. We also see that l_4, l_5 intersect at a double point. This forces the rest of the incidences. The list of incidences are $R(1,2,3)$, $R(1,4,7)$, $R(1,5,6)$, $R(2,4,6)$, $R(2,5,8)$, $R(3,4,8)$, $R(3,5,7)$ and $R(6,7,8)$. This can be rearranged using a projective linear transformation which sends the line (this line is not part of the arrangement) which joins the two triple points P_{123}, P_{678} to the line at infinity. Restricting our attention now to the affine part, we now further transform so that $P_{147} = (0,0)$, $P_{18} = (1,0)$, $P_{348} = (1,1)$ and $P_{357} = (0,1)$. See figure 6.2.

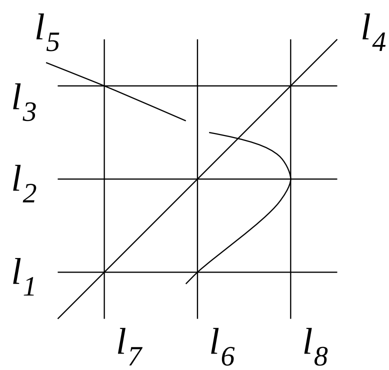


Figure 6.2:

Thus l_4 has equation $y = x$ and so P_{246} has coordinates (α, α). We compute the coordinates of the triple points of l_5. $P_{156} = (\alpha, 0)$, $P_{258} = (1, \alpha)$ and $P_{357} = (0,1)$. We want these to be collinear. Thus

$$\begin{vmatrix} \alpha & 0 & 1 \\ 1 & \alpha & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

which simplifies to

$$\alpha^2 - \alpha + 1 = 0$$

thus $\alpha = \frac{1 \pm i\sqrt{3}}{2}$, either of the two complex cube roots of -1.

Both values yield brittle arrangements of eight lines which are in fact conjugate and neither can be realised as a real arrangement. Furthermore if we consider the moduli space for this arrangement under projective equivalence we see that it is a two point space. The points corresponding to α and $\overline{\alpha}$. Furthermore, applying an automorphism to the arrangement does not flip it. $\square$

This eight line lattice is known as the MacLane arrangement (M^8). See [3] section 7.7 pp312–314.

Theorem 6.2.7 *There are exactly four reduced brittle arrangements of nine lines. Two of them have connected moduli spaces and two have disconnected moduli spaces.*

Proof: Firstly no reduced brittle arrangement of nine lines can contain a quintuple or higher order point P since any line not passing through P would have to intersect at least five other lines in triple points which needs at least eleven lines. We now deal with the case if there is a quadruple point.

If there are two quadruple points on the one line say l_9, then l_9 must contain an additional triple point and no other sorts of intersection points. Assume without loss of generality that we have $R(1,2,3,9)$, $R(4,5,6,9)$ and $R(7,8,9)$. We note that there can be no quadruple point P not on l_9 otherwise this would cause l_9 to have at least four intersection points on it. Thus $l_1, l_2, \ldots, l_6$ must all have precisely one quadruple point, two triple points and one double point on them. Also l_7 and l_8 must either have four triple points or three triple points and two double points.

Now consider all the intersection points on l_1, l_2, l_3 and l_9. They account for $\begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} + 3\left(\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) = 36$ intersections. But the total number of intersections is $\begin{pmatrix} 9 \\ 2 \end{pmatrix} = 36$.

Thus no intersections occur off l_1, l_2, l_3, l_9. Thus there are exactly three double points, one on each of l_1, l_2, l_3. Therefore there are at most six lines with at least one double point, however the lines $l_1, l_2, \ldots, l_6$ account for this. Hence l_7 and l_8 contain no double points and so must contain exactly four triple points.

Thus deleting the line l_9 leaves a brittle eight-line MacLane arrangement. The rigidity of this arrangement does not permit the line l_9 to exist in that the purported intersection points on l_9 cannot be collinear.

If there is a quadruple point P through which the four lines l_1, l_2, l_3, l_4 pass and have no other quadruple points on them, we see that l_1, l_2, l_3, l_4 must each have two further triple points, one double point on them and no other sorts of intersection points. Since there are only five remaining lines but four double points, one of the remaining lines say l_5 must intersect l_1, l_2, l_3, l_4 in triple points only. The number of intersection points,, counting multiplicities on $l_1, \ldots, l_5$ is $\binom{4}{2} + 8\binom{3}{2} + 4\binom{2}{2} = 34$, which is two short of the $\binom{9}{2} = 36$ required. Thus there are two double points amongst $l_6, \ldots l_9$ not on $l_1, \ldots, l_5$. If they both occur on one line, say l_6, then l_6 will have an additional four intersections with $l_1, \ldots, l_4$ that is at least six intersection points, which is not possible for a reduced brittle arrangement of nine lines. Thus the two double points occur on different lines. Without loss of generality say l_6, l_9 intersect in a double points and say l_7, l_8 intersect in a double point. Note that this forces all other intersection points between $l_6, \ldots, l_9$ to be triple points. It also guarantees that the lines $l_6, \ldots, l_9$ all have precisely three triple points and two double points. Thus all the intersections (l_6, l_7); (l_6, l_8); (l_7, l_9) and (l_8, l_9) are triple points. Now relabel $l_1, \ldots, l_4$ so that we have $R(4, 6, 7)$; $R(2, 6, 8)$; $R(3, 7, 9)$ and $R(1, 8, 9)$ This yields figure 6.3.

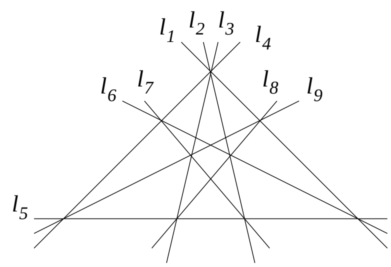


Figure 6.3:

Note that l_5 is the only line in the configuration with exactly four triple points on it and no other sorts of points. Using a projective transformation, we send this line to infinity and concentrate on the affine part only. See figure 6.4.

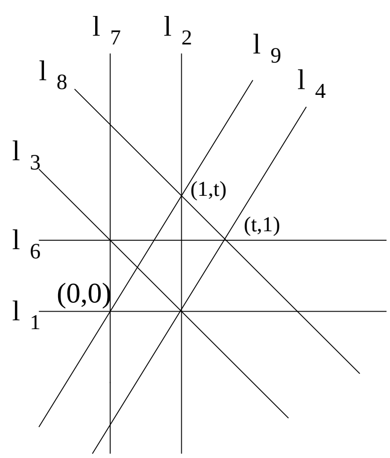


Figure 6.4:

This turns out to be a complete pentagram with two lines deleted (P^9). We want l_9 to be parallel to l_4. The calculation yields $\frac{t}{1} = \frac{1}{t-1}$, that is $t^2 - t - 1 = 0$. Thus $t = \frac{1 \pm \sqrt{5}}{2}$ and the moduli space of the arrangement up to projective equivalence is once again a two point space. The two points corresponding to which sign of $\pm\sqrt{5}$ is taken. This arrangement also has the advantage of having a purely real realisation but the disadvantage of being able to get from one arrangement to the other via an automorphism of the lattice. See [3] section 7.5 pp311-312.

It there are only triple points, deleting one line yields the MacLane arrangement. The rigidity of the MacLane arrangement forces the right four points to be collinear so that this nine-line MacLane arrangement, (M^9) exists. (As in the eight line MacLane arrangement it has disconnected moduli space). See figure 6.5.

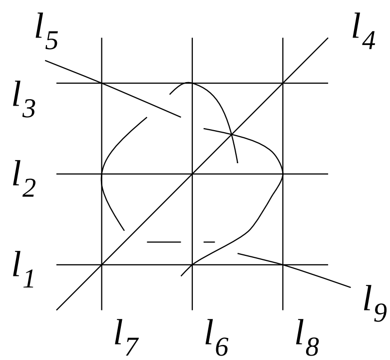


Figure 6.5:

If there is a double point, combinatorics yields that the number of double points must be divisible by three.

If there are exactly three double points, and hence eleven triple points, then removing one of the lines with a double point on it yields the MacLane arrangement whose rigidity prevent restoring the line correctly.

If there are exactly six double points, and hence ten triple points, then we quickly see that there must exist three lines say l_7, l_8, l_9 containing no double points and these must concur at a point P say. Furthermore, since l_7, l_8, l_9 account for ten triple points, there can be no triple points occurring off l_7, l_8, l_9. Thus if we remove these lines, we see that $l_1, \ldots, l_6$ only intersect each other in double points.

Now consider the six triple points on l_7, l_8, l_9 excluding P. One quickly sees that these are joined by $l_1, \ldots, l_6$ in a hexagonal sequence. Figure 6.6 shows that situation without losing generality. The figure also shows nine double points and l_9 must pass through three of these double points (as well as P).

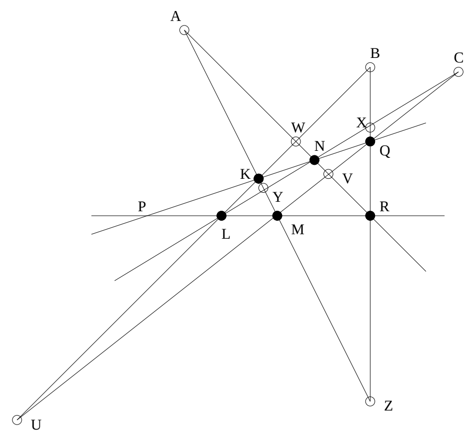


Figure 6.6:

We already know that A, B, C are collinear and so we quickly see that one possibility is that P, A, B, C are collinear, yielding a nongeneric Pascal arrangement as in figure 6.7. This is non-brittle.

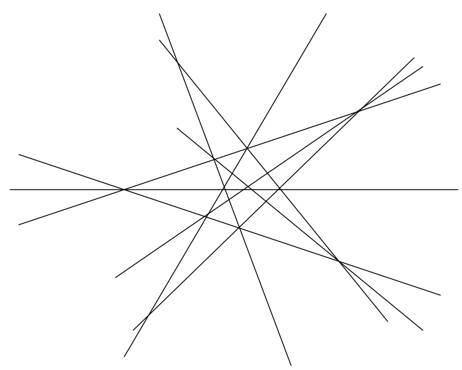


Figure 6.7:

The only remaining possibility is that one of A, B, C; one of U, V, W and one of X, Y, Z must be collinear. Without loss of generality assume that B is involved. This immediately excludes W, X, U, Z. Thus B, V, Y are collinear. It is easy to check if this situation is possible by use of a projective transformation, which sends the three collinear points A, B, C to infinity and the points K, L, M to the coordinates $(0,1)$, $(0,0)$ and $(1,0)$ respectively. Let the coordinates of N be (r,s). See figure 6.8.

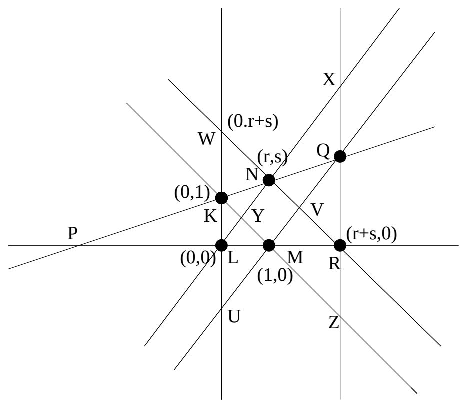


Figure 6.8:

We can now calculate the equation of the line joining L and N to be $y = \frac{s}{r}x$. Intersecting this with the line $x + y = 1$ yields the coordinates of Y to be $(\frac{r}{r+s}, \frac{s}{r+s})$. The line joining M and Q has equation $y = \frac{s}{r}(x-1)$. Intersecting this with the line $x + y = r + s$ yields the x-coordinate of V to be $\frac{r^2+rs+s}{r+s}$. Now for the points B, Y, V to be collinear, we must have the x-coordinates of Y and V being equal. Thus $\frac{r}{r+s} = \frac{r^2+rs+s}{r+s}$. This can be rearranged to give $s = \frac{r-r^2}{r+1}$. However we must remember that P is also collinear with B, V, Y and so must have same x-coordinate. The equation of the line joining K and N is $y = \frac{s-1}{r}x+1$. Intersecting this with the line $y = 0$ yields the x-coordinate of P to be $\frac{r}{1-s}$. Using that the x-coordinates of P and Y are equal, we quickly have $\frac{r}{1-s} = \frac{r}{r+s}$. Thus $s = \frac{1}{2}(1-r)$. Putting this together with the previous expression yields $\frac{1-r}{2} = \frac{r-r^2}{r+1}$, which rearranges to $(r-1)^2 = 0$. Thus the only possibility is $r = 1$ and this forces $s = 0$ so that N coincides with M - a contradiction. Thus B, V, Y may be collinear, but not with P.

If there are exactly nine double points, and hence nine triple points, form a graph G whose vertices are the double points such that two points are joined by an edge if and only if the double points lie on a line of the arrangement. We see that all vertices in G have valence two and there are no double edges or loops. Thus G is a union of disjoint polygons. One quickly sees that there is either a triangle, a quadrilateral or a nonagon in G. If there is a triangle, say l_7, l_8, l_9 form a triangle of double points, then we are back in the six double point case analysis, except that l_7, l_8, l_9 do not concur. In one case we get a generic Pascal arrangement as in figure 6.9. In the other case we saw that the moduli space corresponds to an open subset of the curve $s = \frac{r-r^2}{r+1}$. See figure 6.10 which shows the case $r = -2$

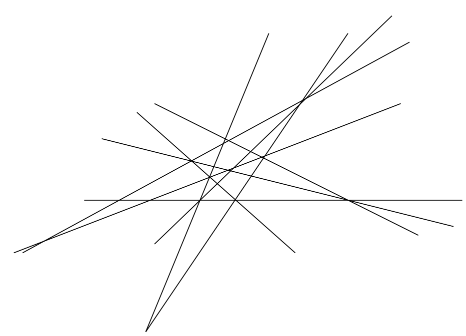


Figure 6.9:

If there is a quadrilateral in G say l_1, l_2, l_3, l_4 form a quadrilateral with l_i, l_{i+1} (indices taken modulo 4) intersect in double points, then l_1, l_3 intersect in a triple point and likewise for l_2, l_4. However each line contains two double points and three triple points, thus each line contains a further two triple points. Thus there are ten triple points involved in the four lines $l_1, \ldots l_4$

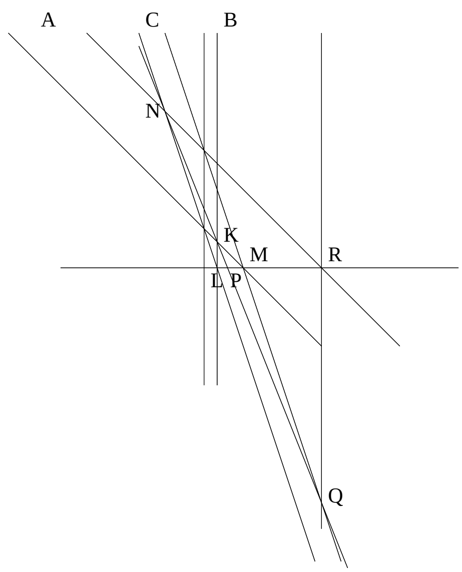


Figure 6.10:

which exceeds the nine calculated. Thus G cannot contain a quadrilateral.

If there is a nonagon, assume without loss of generality that l_i intersects l_{i+1} in a double point for all i and where the indices are taken modulo 9. Say that the triple points on l_1 are A_1, B_1, C_1 and that the triple points on l_2 are A_2, B_2, C_2. Now l_3 intersects l_1 in what must be a triple point say A_1. l_3 has two further triple points say A_3, B_3. l_4 intersects l_2 and l_1 in triple points say A_2 for l_2 and B_1 for l_1, note that it cannot be A_1 because l_3 intersects l_4 in a double point. l_4 has one further triple point say A_4. Thus far we have now accounted for all the nine triple points. Now l_5 intersects l_1, l_2, l_3 in triple points. If l_5 passes through A_1 then we have accounted for l_5 intersecting both l_3 and l_1 in triple points via the one point A_1. Furthermore l_5 would gain a triple point from its intersection with l_2 but would still need one more triple point and none of these can come from the nine triple points A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3 and A_4. This would force us to exceed nine triple points. Note also that l_5 cannot pass through B_1 or A_2 because l_4 passes through those points. Thus l_5 is forced to intersectl_1 at C_1 and without loss of generality intersects l_2 and l_3 in B_2 and A_3 respectively. Note that in particular we have shown that l_i, l_{i+2}, l_{i+4} are not concurrent for any i. Figure 6.11 shows what we have so far.

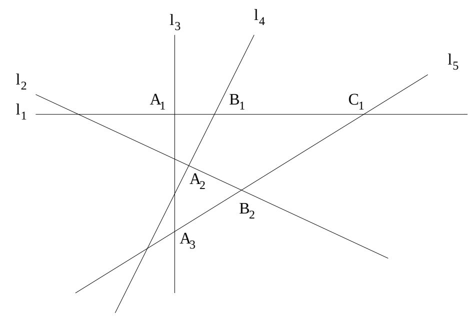


Figure 6.11:

Considering l_6 now, we see that it intersects l_1, l_2, l_3, l_4 in triple points. Since we know l_i, l_{i+2}, l_{i+4} cannot be concurrent for any i, l_6 cannot pass through A_2. It must pass through one of A_1, B_1 so that either $R(1,3,6)$ or $R(1,4,6)$. In fact we have shown that for every i modulo 9, we have either $R(i, i{+}2, i{+}5)$ or $R(i, i{+}3, i{+}5)$. Now colour i white if we have $R(i, i{+}2, i{+}5)$ and black if we have $R(i, i+3, i+5)$. Now consider where l_9 intersects l_4 and assume $i = 1$ is white. It must be at either B_1, A_2 or A_4. B_1 is discounted because l_1 passes through it and A_2 is discounted because this would cause l_i, l_{i+2}, l_{i+4} to be concurrent for $i = 9$. Thus l_9 passes though A_4 so that that we have $R(4,6,9)$. Hence i being white implies that $i + 3$ is white for any i.

Now we have nine colours around a circle and nine is odd, thus there must exist an i such that both i and $i + 1$ have the same colour. Assume without loss of generality that that $i = 1$ and $i = 2$ have the same colour. Furthermore by the automorphism of indices $i \to 7 - i \pmod 9$ if necessary, we can assume that we have $R(1,3,6)$ and $R(2,4,7)$. Now consider where l_7 and l_8 intersect l_3. l_7 does not pass through A_1 because l_6 does and l_7

does not pass through A_3 because then we would have l_i, l_{i+2}, l_{i+4} for $i = 3$. Thus l_7 passes through B_3. Thus l_8 does not pass through B_3 and neither does it pass through A_1 since A_1 already has three lines through it. Hence l_8 passes through A_3 and so we have $R(3, 5, 8)$. But now we have $i = 1, 2, 3$ all white and when we consider that i is white implies that $i+3$ is white, we see that all $i \pmod 9$ are white. Thus $R(i, i+2, i+4)$ for all $i \pmod 9$ are the complete set of triple points and this in conjunction with l_i, l_{i+1} intersecting in a double point make up the complete set of incidences.

We now projectively transform as follows. The intersection of l_1, l_2 is sent to the origin (in the affine part). The intersection of l_1, l_4, l_8 is sent to the point $(1, 0)$. The intersection of l_4, l_6, l_9 is sent to $(1, 1)$ and the intersection of l_2, l_6, l_8 is sent to $(0, 1)$. Note the this forces l_1, l_3, l_6 to be all parallel to the X-axis l_1 and l_2, l_4, l_7 to be all parallel to the Y-axis l_2. Let l_3 have equation $y = s$ and l_7 have equation $x = r$. Figure 6.12 shows the situation so far.

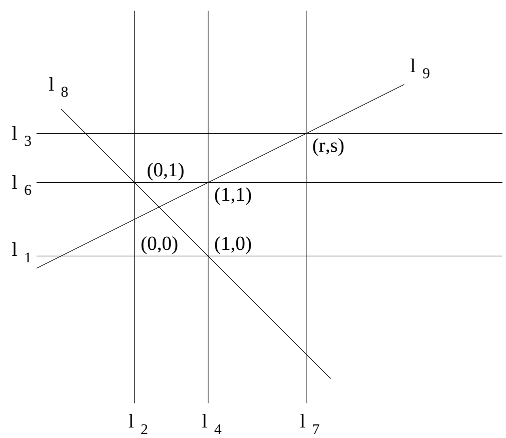


Figure 6.12:

The equation of the line l_9 is $y - 1 = \frac{s-1}{r-1}(x - 1)$ and this intersects the Y-axis l_2 at $(0, \frac{r-s}{r-1})$. The equation of l_8 is $x + y = 1$ and this intersects the line l_3 at $(1 - s, s)$. Finally l_5 must pass through these two points as well as $(r, 0)$ the intersection of l_1 and l_7. Thus $\frac{\frac{r-s}{r-1} - 0}{0 - r} = \frac{s - 0}{1 - s - r}$. This rearranges to give

$(s+r^2-r)(s-1) = 0$. Thus the moduli space of this arrangement corresponds to an open subset of $s = r - r^2$. Figure 6.13 shows the arrangement for $r = 2$.

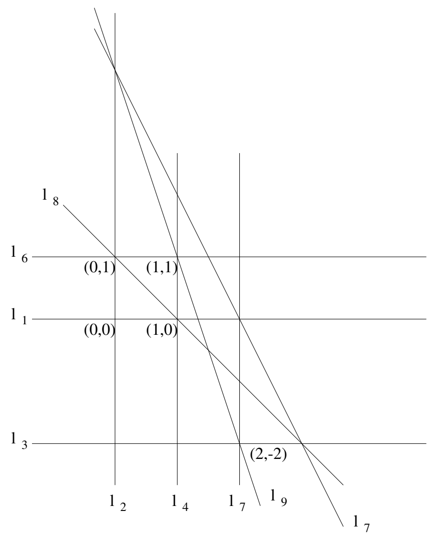


Figure 6.13:

□

Comment: It would seem to me that the classification of small brittle arrangements should be known. However I have been unable to locate such results nor find the concept of a brittle arrangement in the literature.

Comment: There will be nonreducible ten-line arrangements. One such is the ten-lined deleted pentagram (P^{10}) in figure 6.14.

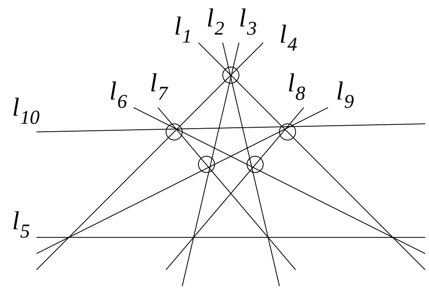


Figure 6.14:

Again the moduli space is disconnected but this time there is no automorphism of the lattice which takes an arrangement in one component of the moduli space to the other.

There are nonbrittle ten line arrangements such that each line contains at least three triple or higher order points. Unfortunately this complicates a search for nonreducible ten-line arrangements. One such is the Desargue arrangement and its degenerations. See figure 6.15.

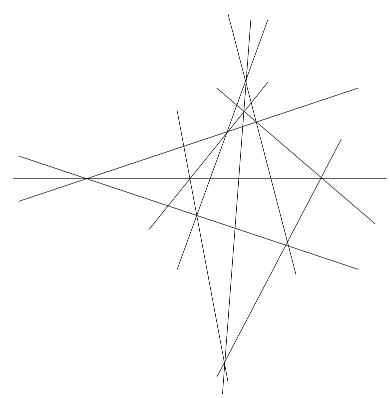


Figure 6.15:

6.3 Isotopy and the Complement

There is not much to say here on isotopy except that it makes sense to deal with labeled arrangements. Say for example we had two arrangements of lines $A_1 = \{l_1, l_2, \ldots, l_n\}$ and $A_2 = \{m_1, m_2, \ldots, m_n\}$ where l_i corresponds to m_i under a given lattice isomorphism. But suppose that as arrangements A_1 and A_2 were isotopic and that the induced map on lattices did not correspond to the given isomorphism of lattices but to a different isomorphism of lattices. Then we could mark the lines in A as an abstract arrangement as follows. For the line l_i, create an $(i+1)$-tuple point P_i in general position on l_i by adding i lines through P_i in general position with respect to the rest of the arrangement. Thus if the new arrangement has a unique isotopy class, then so does the old. However the new arrangement forces us to deal with labeled arrangements.

Now as labeled arrangements the two eight-line conjugate MacLane configurations are isotopic. They are related by complex conjugation which is ambient isotopic to the identity map on $\mathbb{CP}^2$.

Here is a nonrigorous sketch of two arrangements which are lattice equivalent but not ambient isotopic (and hence not isotopic).

Assume the following proposition.

Proposition: Any ambient isotopy of a labeled MacLane arrangement to itself is isotopic to the identity map.

Corollary: Any ambient isotopy of the MacLane arrangement to itself acts trivially on the generators for the homology of the complement.

Let M^8 denote one of the eight-line MacLane arrangements and $\overline{M^8}$ a member of the class of its conjugate. We can define M^{13} to be two copies of M^8 pasted together along l_1, l_2, l_3 and $\overline{M^{13}}$ to be a copy of M^8 pasted to a copy of $\overline{M^8}$ along l_1, l_2, l_3. These define lattice equivalent thirteen-line arrangements.

Corollary: M^{13} and $\overline{M^{13}}$ are not ambient isotopic and thus not isotopic as labeled arrangements.

Proof: To match M^8s we consider the induced ambient isotopies $M^8 \to M^8$ and $M^8 \to \overline{M^8}$. One wants to preserve the generators of H_1 while the other wants to reverse them. This is incompatible.

□

We also have the ten-line deleted pentagram P^{10} whose moduli space is a two point set and no lattice automorphism transfers between these points. These points are related algebraically by the surd conjugation $\sqrt{5} \to -\sqrt{5}$. This does not induce a nice continuous automorphism of $\mathbb{CP}^2$ like complex conjugation does. Hence it is unclear if P^{10} determines its (ambient) isotopy class.

A comment finally on the complement proper. This is very difficult. However Rybnikov [19] claims that M^{13} and $\overline{M^{13}}$ have different complements because they have different fundamental groups. Conjugation is sort of detected in the fundamental group.

Note the contrast to real arrangements of non-intersecting lines in $\mathbb{R}^3$ where the complements are all homeomorphic.

6.4 Questions

1. Can a single irreducible curve have disconnected moduli space?

2. Can an arrangement consisting of a single irreducible curve not determine its complement?

3. The definition of an (abstract) arrangement is quite satisfactory. What other definitions for an arrangement could be reasonably considered?

4. Apart from M^9 are there any other non-trivial line arrangements with no double points?

5. What rigidity properties are causing unusual incidences such as the Pascal, Desargues and M^9 arrangements?
6. Can the Rybnikov arrangement M^{13} be more easily seen to not determine its complement?
7. It is easier to compute the fundamental group of the complement of an arrangement which has a real realisation. Is it possible to find an arrangement with disconnected moduli space and with two real realisations which have different fundamental groups for their complements? The ten-lined deleted pentagram arrangement P^{10} is such a candidate.
8. The sphere can be realised topologically by a curve of any degree. In general what two dimensional complexes can be realised by projective complex plane curves, and in which degrees?
9. Is it true that two arrangements are BTE if and only if they have homeomorphic links?
10. Is there a way to detect when a plumbing diagram represents the link of some arrangement?
11. Is there a way to detect when a plumbing diagram represents an immediate plumbing diagram for the link of some arrangement?
12. Is there an effective normal form for a plumbing diagram if we are restricted to using only R2 type moves?
13. To what extent can we carry the investigation of arrangements and links to curves of higher dimension/codimension?

Appendix A

In section 3.3, we studied how a curve may intersect the ambiguous set. However this was only done in case 1. The remaining cases 2, 3 and 4 are computed here. These computations justify the summary diagrams found in figures 3.47, 3.48 and 3.49. Refer to figure 3.45 on page 74 for the notation used in this section.

Case 2:

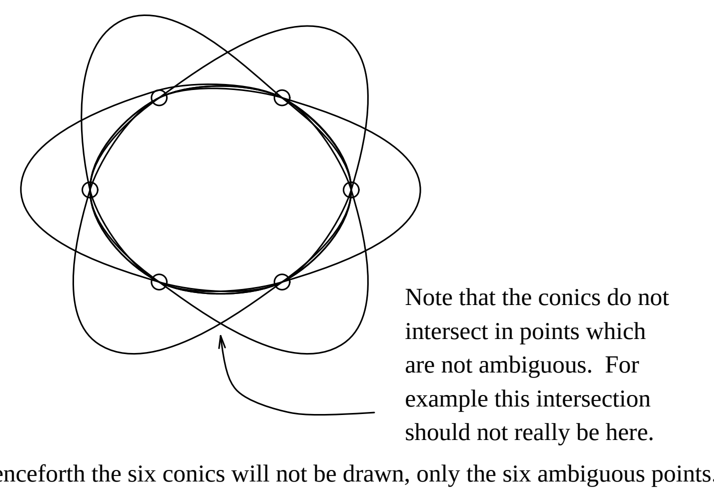


Figure A.1: Case 2

Case 2(i): A line l passes through 0 or 1 intersection points. See figure A.2.

Then consider the conic C passing through five ambiguous points none of which l passes through. Then l intersects C in two non-ambiguous points contradicting lemma 3.3.3.
Hence this case does not occur.

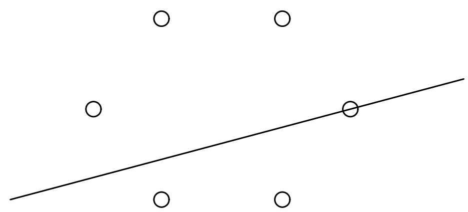


Figure A.2: Case 2(i)

Case 2(ii): A line l passes through two ambiguous intersection points say P_1 and P_2. See figure A.3.

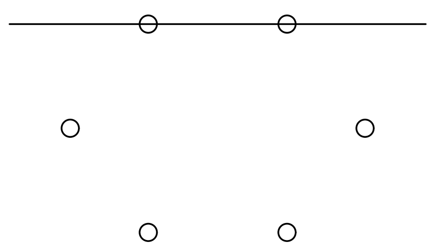


Figure A.3: Case 2(ii)

Then l intersects four conics C_3, C_4, C_5, C_6 (C_i is the conic not passing through P_i) in the two points P_1 and P_2 and the other two conics C_2, C_1 in P_1, P_2 respectively and in another point each of which is non-ambiguous. Hence blowing up reduces the self intersection number of l from $+1$ to -1, then blowing down increases it back to $+1$. The result is once again essentially unique and self-inverse as we can easily understand this in terms of the symmetry in the plumbing diagram (figure A.4).

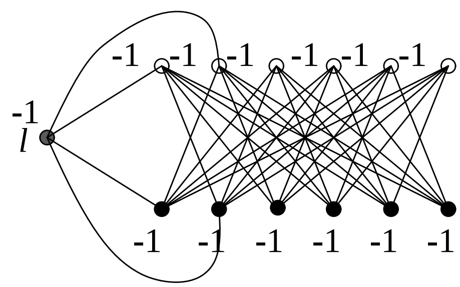


Figure A.4: Case 2(ii)

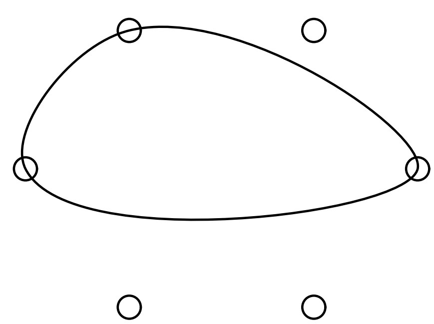


Figure A.5: Case 2(iii)

Case 2(iii): A conic γ passes through at most three ambiguous points. See figure A.5.

Let P_1, P_2, P_3 be ambiguous points which γ does not pass through. Consider C_6, the ambiguous conic which passes through P_1, P_2, P_3, P_4, P_5. Now by Bezout's theorem C_6 intersects γ in four points. However the only possible ambiguous points they could have in common are P_4 and P_5, hence C_6 and γ intersect in at least two other non-ambiguous points contradicting lemma 3.3.3.
Hence this case does not occur.

Case 2(iv): A conic γ passes through exactly four of the ambiguous points say P_1, P_2, P_3, P_4. See figure A.6.

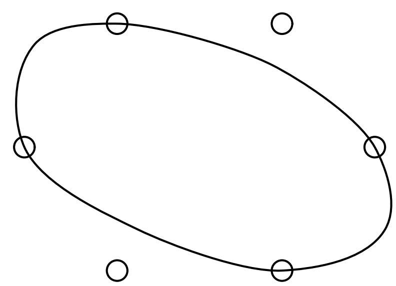


Figure A.6: Case 2(iv)

Hence γ intersects the two ambiguous conics C_5, C_6 in the four points P_1, P_2, P_3, P_4 and intersects the remaining four conics in three of the four ambiguous points P_1, P_2, P_3, P_4 and in one other non-ambiguous point. Hence blowing up decreases the self intersection number of γ from $+4$ to 0, and blowing down increases it back to $+4$. The result is once again essentially unique and self-inverse, and we can easily understand this in terms of the symmetry of the plumbing diagram (figure A.7).

Note that a conic cannot pass through five of the ambiguous points without being one of the ambiguous conics since five points define a conic.

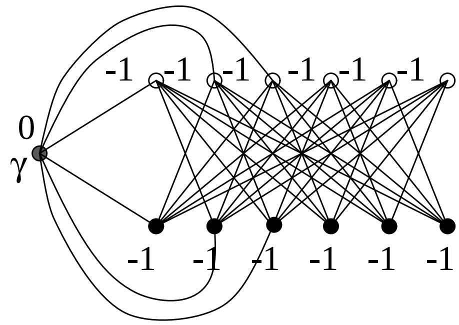


Figure A.7: Case 2(iv)

Case 2(v): A cubic δ passes through at most five ambiguous points. See figure A.8.

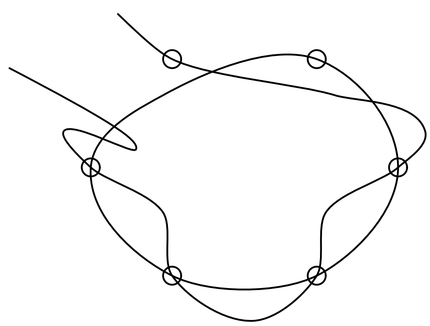


Figure A.8: Case 2(v)

Say δ does not pass through P_1. Consider C_6. Now by Bezout's theorem C_6 and δ intersect in six points, however they have only at most four ambiguous points in common, namely some subset of $\{P_2, P_3, P_4, P_5\}$, hence they must intersect in at least two non-ambiguous points contradicting lemma 3.3.3.
Hence this case does not occur.

Case 2(vi): A cubic δ passes through all six ambiguous intersection points (which is o.k. since the six are not conconic). See figure A.9.

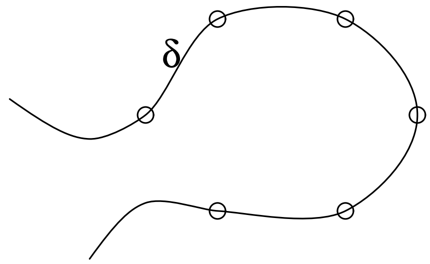


Figure A.9: Case 2(vi)

By Bezout's theorem δ and C_i intersect in six points, but they only have five of the ambiguous points P_j in common. Hence they intersect at one non-ambiguous point. Hence blowing up decreases the self intersection number of δ from +9 to +3 and blowing down restores it to +9. The result is once again essentially unique and the scenario is self-inverse. Again we understand this in terms of the symmetry of the plumbing diagram (figure A.10).

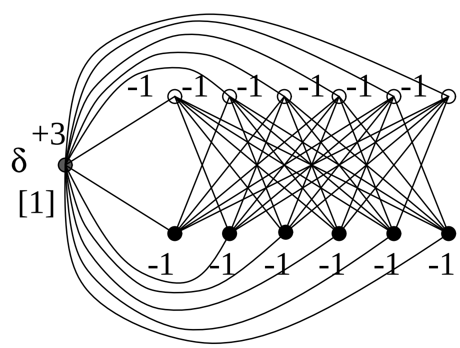


Figure A.10: Case 2(vi)

Case 2(vii): A degree n curve $n \geq 4$ δ is in the configuration.
Again we can appeal to Bezout's theorem. The curve δ must intersect one of the conics say C_1 in $2n \geq 8$ places since $n \geq 4$. But at most six of these are ambiguous points, hence at least two intersections are non-ambiguous points

contradicting lemma 3.3.3.
Hence this case does not occur.

Case 3 For notation first label the ambiguous points and curves as in figure A.11.

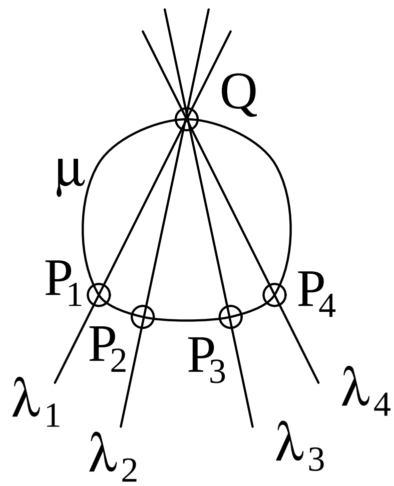


Figure A.11: Case 3

Case 3(i): A line l passes through no ambiguous intersection points. See figure A.12.

Then l intersects μ in two non-ambiguous intersection points contradicting lemma 3.3.3.
Hence this case does not occur.

Case 3(ii): A line l passes through only the ambiguous point Q. See figure A.13.

This can easily be seen to be self inverse from the symmetry in the plumbing diagram (figure A.14).

Case 3(iii): A line l passes through exactly one of the P_i (without loss of generality P_1). See figure A.15.

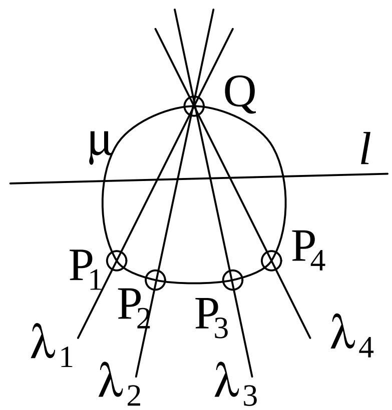


Figure A.12: Case 3(i)

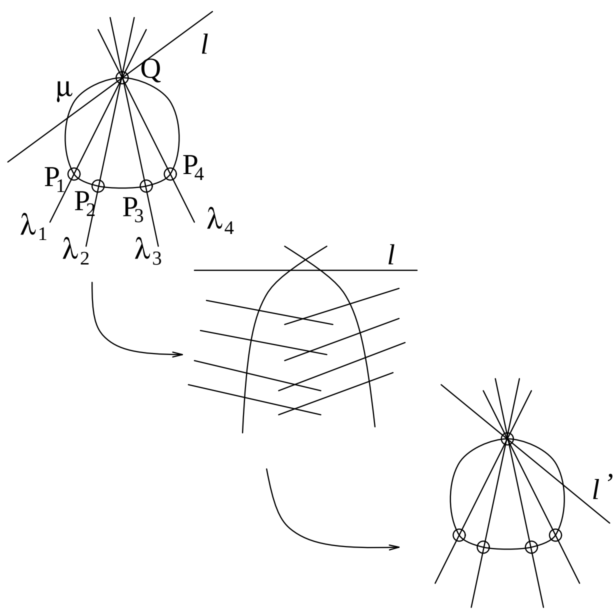


Figure A.13: Case 3(ii)

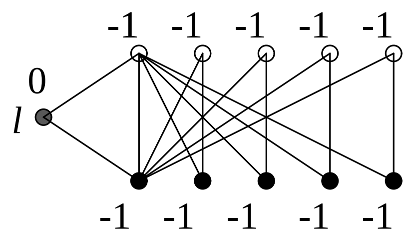


Figure A.14: Case 3(ii)

Blowing up decreases the self intersection number of l from $+1$ to 0 and blowing down increases it from 0 to $+4$, so that we should end up with a conic passing through the four points $\lambda_2', \lambda_3', \lambda_4', \mu'$ and crossing the curve P_1' at a non-ambiguous point.

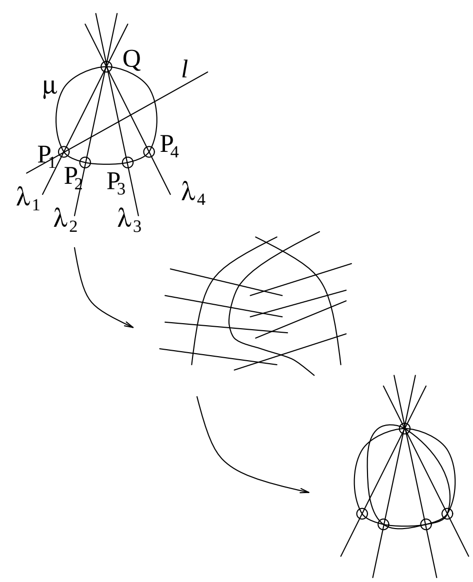


Figure A.15: Case 3(iii)

The result is essentially unique but not self inverse as can be seen in the asymmetry of the plumbing diagram (figure A.16).

Case 3(iv): A line l passes through two of the P_i (without loss of generality P_1 and P_2). See figure A.17.

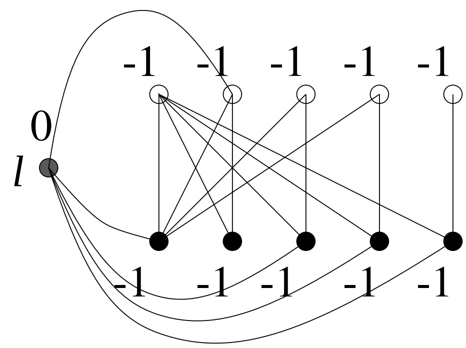


Figure A.16: Case 3(iii)

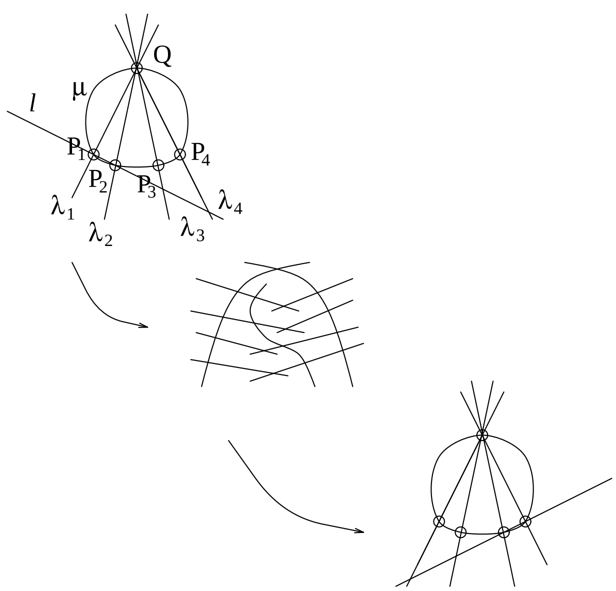


Figure A.17: Case 3(iv)

This is easily seen to be self-inverse from the plumbing diagram (figure A.18) after reshuffling.

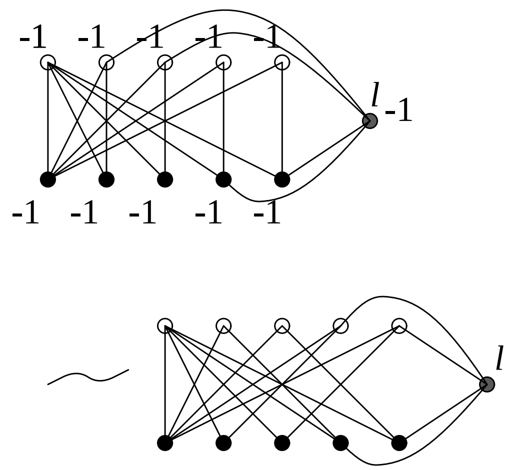


Figure A.18: Case 3(iv)

Case 3(v): A conic γ does not pass through Q, but passes through all of P_1, P_1, P_3, P_4. See figure A.19.

This is easily seen to be self-inverse from the plumbing diagram (figure A.20.

Case 3(vi): A conic γ does not pass through Q and does not pass through at least one of P_1, P_2, P_3, P_4 (without loss of generality γ does not pass through P_1. See figure A.21.

However γ must intersect λ_1 in two places by Bezout's theorem and neither of these places are Q or P_1, hence γ intersects λ_1 in two non-ambiguous places contradicting lemma 3.3.3.
Hence this case does not occur.

Case 3(vii): A conic γ passes through Q and does not pass through at least three of P_1, P_2, P_3, P_4 (without loss of generality γ does not pass through P_2, P_3, P_4). See figure A.22.

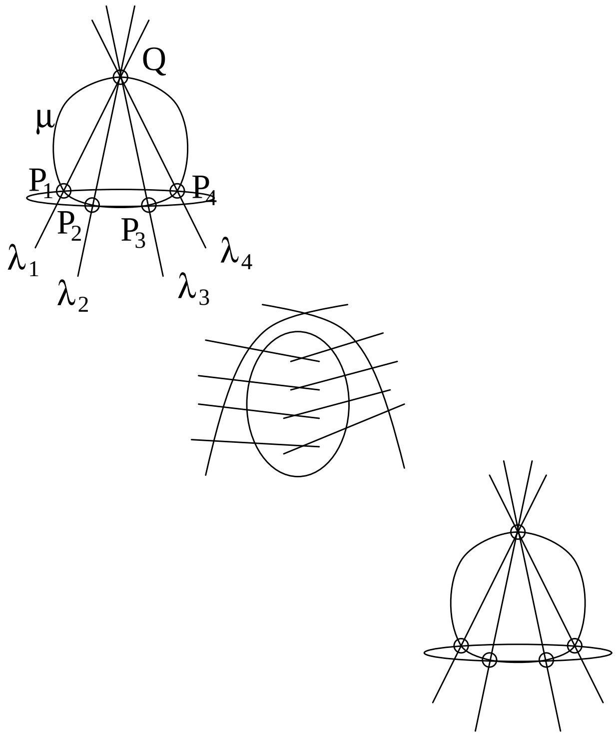


Figure A.19: Case 3(v)

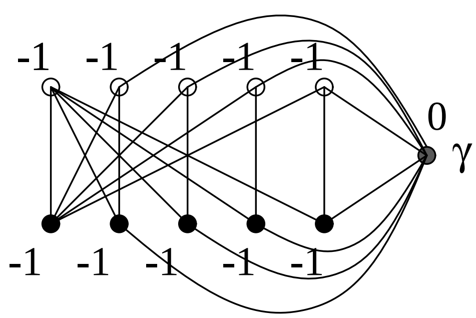


Figure A.20: Case 3(v)

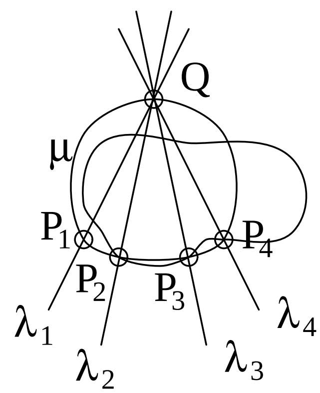


Figure A.21: Case 3(vi)

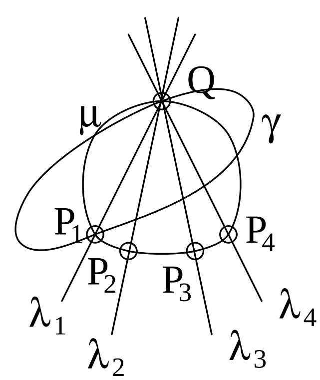


Figure A.22: Case 3(vii)

Now then γ and μ have only at most two ambiguous points (Q and possibly P_1) in common, but by Bezout's theorem they must have four points in common, hence they intersect in at least two non-ambiguous points contradicting lemma 3.3.3.
Hence this case does not occur.

Case 3(viii): A conic γ passes through Q and exactly two of P_1, P_2, P_3, P_4 (without loss of generality P_1 and P_2). See figure A.23.

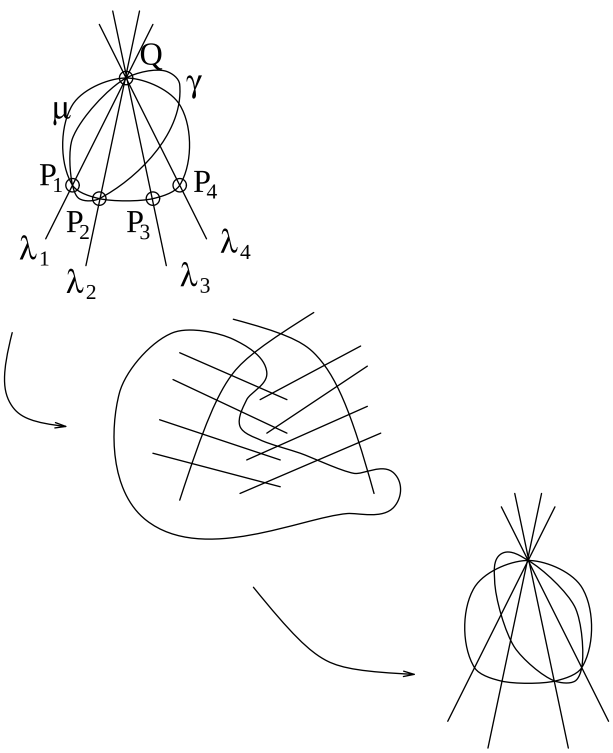


Figure A.23: Case 3(viii)

This is easily seen to be self-inverse from the symmetry in the plumbing diagram (figure A.24) after reshuffling.

Case 3(ix): A conic γ passes through Q and exactly three of P_1, P_2, P_3, P_4

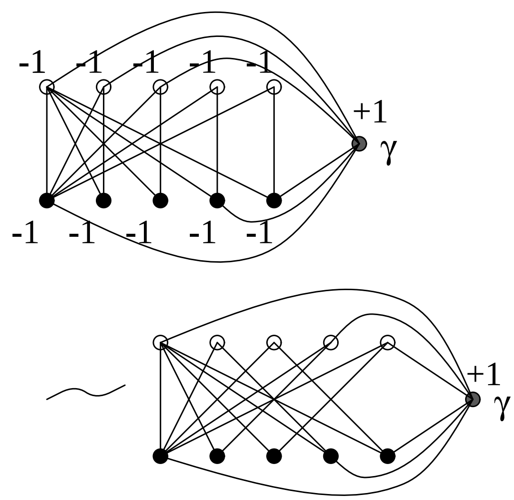


Figure A.24: Case 3(viii)

(without loss of generality P_2, P_3, P_4). See figure A.25.

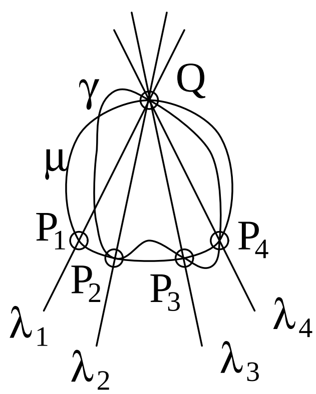


Figure A.25: Case 3(ix)

This case is inverse to case 3(iii), so the reverse argument applies.

Case 3(x): A conic γ passes through all of Q, P_1, P_2, P_3, P_4.
This is impossible since then it has five points in common with μ and hence $\gamma = \mu$.

Case 3(xi): A cubic δ does not pass through all of Q, P_1, P_2, P_3, P_4. In this case it is easy to see that δ cannot have two ambiguous points in common with some line say λ_1. Hence δ and λ_1 have at most one ambiguous point in common. By Bezout's theorem they have three points in common, hence δ and λ_1 intersect in at least two non-ambiguous points contradicting lemma 3.3.3.
Hence this case does not occur.

Case 3(xii): A cubic δ passes through all of Q, P_1, P_2, P_3, P_4. See figure A.26.

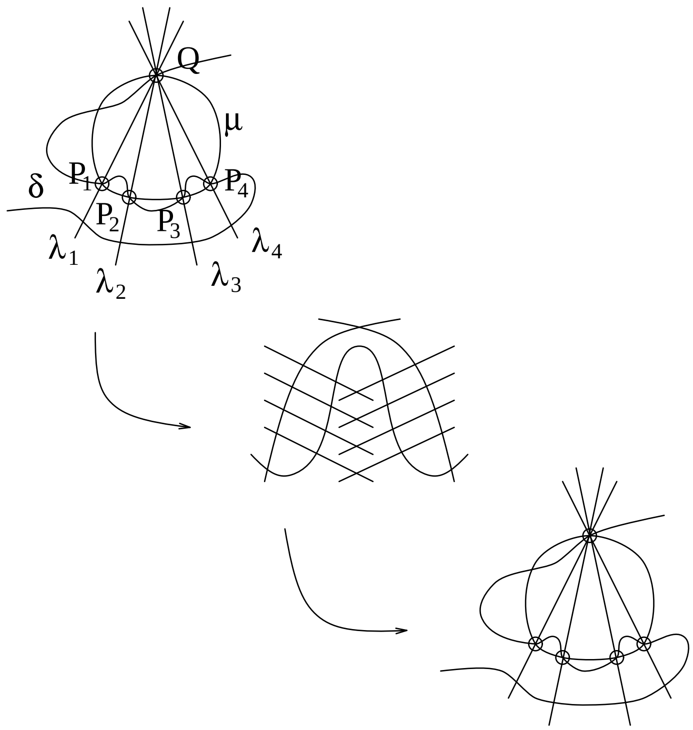


Figure A.26: Case 3(xii)

By Bezout's theorem we quickly see that δ intersects each of the curves

$\mu, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ in one non-ambiguous point each. Hence blowing up decreases the self intersection number of δ from $+9$ to $+4$ and blowing down restores it back to $+9$. And we can see this case to be self-inverse from the symmetry in the plumbing diagram (figure A.27).

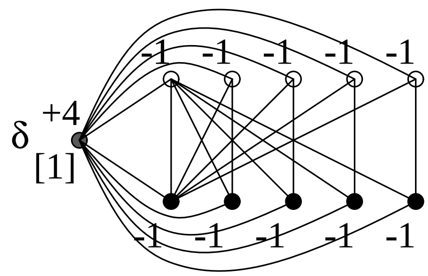


Figure A.27: Case 3(xii)

Case 3(xiii): A degree n curve ($n \geq 4$) δ is in the configuration. Then this must (by Bezout's theorem) intersect λ_1 in $n \geq 4$ places and at most two of these places can be ambiguous points (namely Q and P_1). Hence δ intersects P_1 in at least two non-ambiguous places contradicting lemma 3.3.3.
Hence this case does not occur.

Case 4: For notation we label the ambiguous curves and points as in figure A.28.

Case 4(i): A line l does not pass through any P_i or Q_i, $i = 1, 2, 3$.
Then l intersects μ_1 say in two non-ambiguous points contradicting lemma 3.3.3.
Hence this case does not occur.

Case 4(ii): A line l does not pass through any of P_1, P_2, P_3 but passes through exactly one of Q_1, Q_2, Q_3 (without loss of generality l passes through Q_1).
However $Q_1 \notin \mu_1$ thus we see that l and μ_1 don't have any ambiguous points

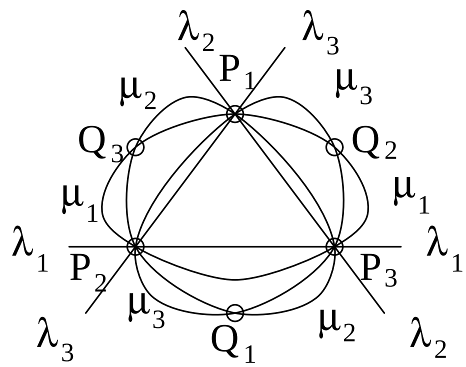


Figure A.28: Case 4

in common, hence they must intersect in two non-ambiguous points contradicting lemma 3.3.3.
Hence this case does not occur.

Case 4(iii): A line l does not pass through any of P_1, P_2, P_3, but passes through exactly two of Q_1, Q_2, Q_3 (without loss of generality l passes through Q_2 and Q_3). See figure A.29.

Note that the plumbing diagram (figure A.30) is not symmetric and so this case is not self-inverse.

We verify that this case can occur as follows. Firstly l passes through $\lambda_1, \lambda_2, \lambda_3$ at non-ambiguous points and also μ_2 and μ_3 at Q_2, Q_3 and a non-ambiguous point each. Thus after blowing up and blowing down we see that l now must pass through the five ambiguous points $\lambda_1', \lambda_2', \lambda_3', \mu_2', \mu_3'$ and also intersect the lines Q_2', Q_3' in one non-ambiguous point each. No other intersections occur. Furthermore the self intersection number of l decreases from +1 to -1 after blowing up, then increases to +4 after blowing down. All of this points to the image of l being a conic which passes through the five points $\lambda_1', \lambda_2', \lambda_3', \mu_2', \mu_3'$ Furthermore the conic l' has four (ambiguous) points in common with each of the conics P_1', P_2', P_3'. Thus all incidences match up

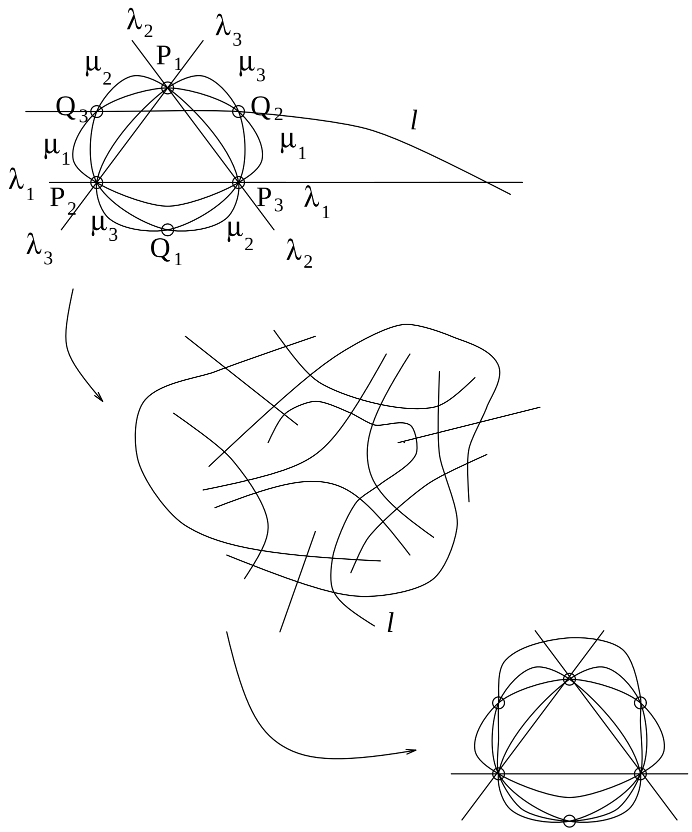


Figure A.29: Case 4(iii)

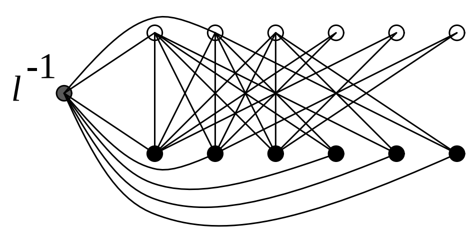


Figure A.30: Case 4(iii)

and we have found that case 4(iii) occurs in precisely the way described in the figure A.29.

Case 4(iv): A line l passes through exactly one of P_1, P_2, P_3 (say P_1), but not through any Q_i. See figure A.31.

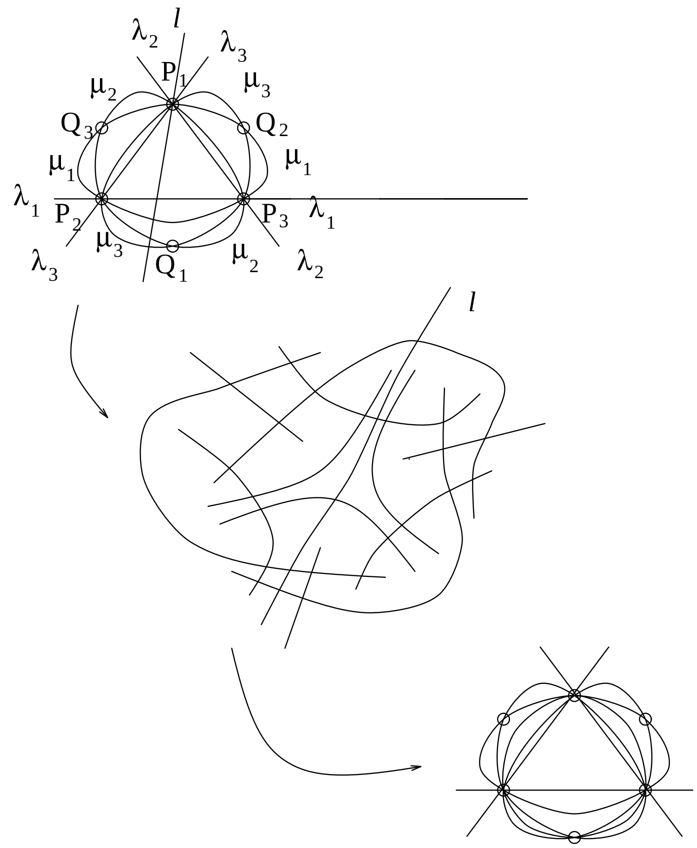


Figure A.31: Case 4(iv)

Note that the plumbing diagram (figure A.32) is not symmetric and so this case is not self-inverse.

Now we can see that l passes through P_1 hence l' intersects the conic P_1' in one non-ambiguous point. Also l intersects all the curves $\lambda_1, \mu_1, \mu_2, \mu_3$ in one non-ambiguous point each and hence l' passes through the vertices

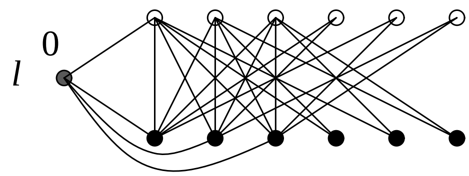


Figure A.32: Case 4(iv)

$\lambda_1', \mu_1', \mu_2', \mu_3'$. Also the self intersection number of l decreases from +1 to 0 after blowing up, then increases to +4 after blowing down. All this shows that case(iv) occurs in precisely the way described in figure A.31 - a conic l' passing through $\lambda_1', \mu_1', \mu_2', \mu_3'$, but not λ_2', λ_3'.

Case 4(v): A line l passes through exactly one P_i and exactly one Q_j. A little thought shows that without loss of generality we can assume that l passes through P_1 and Q_1. See figure A.33.

This is easily seen to be a self-inverse case from the plumbing diagram (figure A.34) after reshuffling.

Case 4(vi): The points Q_1, Q_2, Q_3 are collinear and a line l passes through them. See figure A.35.

This case is easily seen to be self-inverse from the plumbing diagram, and hence occurs precisely in the way described in the diagram above provided the three points $\lambda_1', \lambda_2', \lambda_3'$ are collinear in A_2.

Case 4(vii): A conic γ passes through at most one of the P_i (without loss of generality say γ does not pass through P_2 or P_3).
Then γ and λ_1 have no ambiguous points in common and hence must have two non-ambiguous points in common contradicting lemma 3.3.3.
Hence this case does not occur.

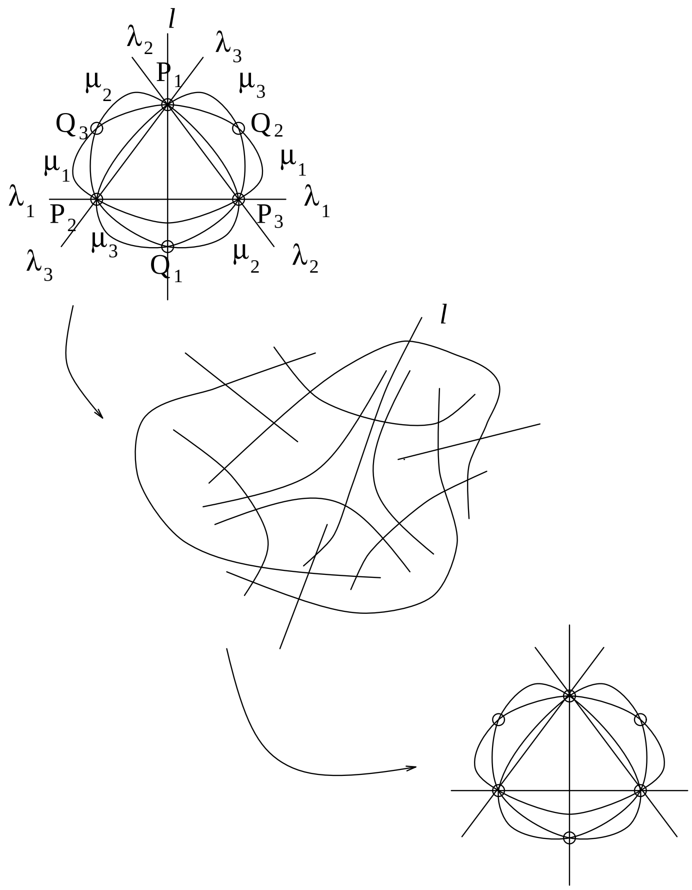


Figure A.33: Case 4(v)

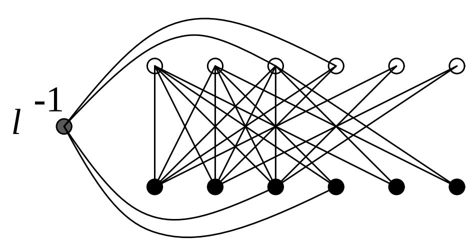


Figure A.34: Case 4(v)

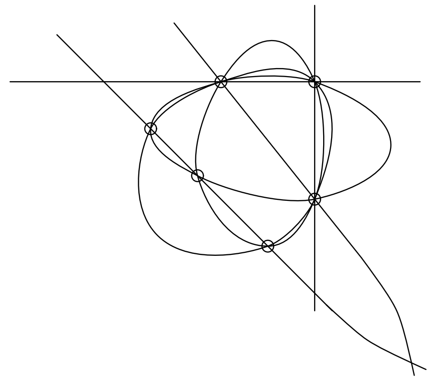


Figure A.35: Case 4(vi)

Case 4(viii): A conic γ passes through exactly two of the P_i (without loss of generality P_2 and P_3) and passes through at most one Q_j (without loss of generality say γ does not pass through Q_2 and Q_3).
Then γ and μ_1 have only P_2 and P_3 as common ambiguous points. Hence they must intersect in two non-ambiguous points contradicting lemma 3.3.3. Hence this case does not occur.

Case 4(ix): A conic γ passes through exactly two of the P_i (without loss of generality P_2 and P_3) and passes through exactly two of the Q_j (without loss of generality Q_2 and Q_3). See figure A.36.

This is easily seen to be self-inverse from the plumbing diagram (figure A.37) after reshuffling.

Case 4(x): A conic γ passes through exactly two of the P_i and all three Q_j.
This case is inverse to case 4(iii) and hence the reverse argument applies.

Case 4(xi): A conic γ passes through all three P_i but through no Q_j. See figure A.38.

This is easily seen to be self-inverse from the plumbing diagram (fig-

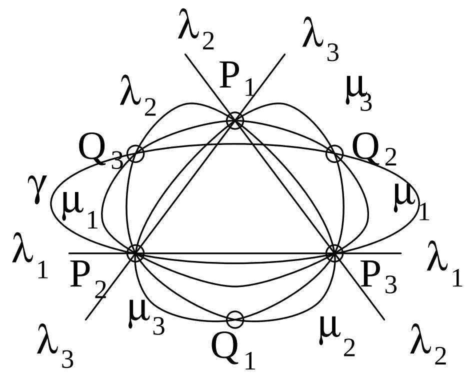


Figure A.36: Case 4(ix)

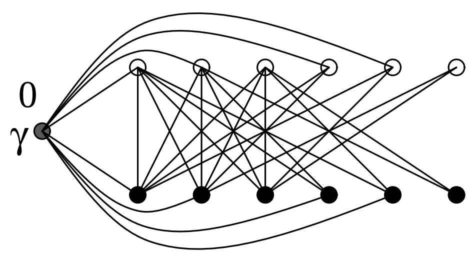


Figure A.37: Case 4(ix)

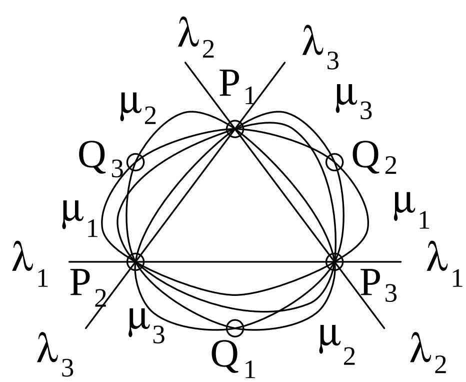


Figure A.38: Case 4(xi)

ure A.39).

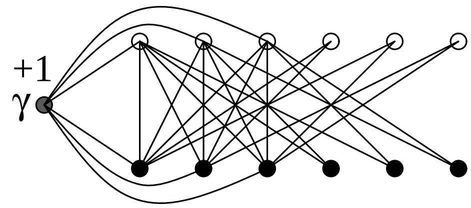


Figure A.39: Case 4(xi)

Case 4(xii): A conic γ passes through P_1, P_2, P_3 and exactly one Q_j. This case is inverse to case 4(iv) and so the reverse argument applies.

Case 4(xiii): A conic γ passes through P_1, P_2, P_3 and at least two of the Q_i (without loss of generality Q_1 and Q_2).
Then γ has five points in common with μ_3 contradicting Bezout's theorem.

Case 4(xiv): A cubic δ does not pass through all P_i (without loss of generality δ does not pass through P_1).
Then δ has only one ambiguous point in common with λ_1. Applying Bezout's theorem to these two curves yields a contradiction to lemma 3.3.3.

Case 4(xv): A cubic δ does not pass through all Q_j without loss of generality δ does not pass through Q_1).
Thus δ has at most four ambiguous points in common with μ_2. Applying Bezout's theorem to these two curves yields a contradiction to lemma 3.3.3.

Case 4(xvi): A cubic δ passes through all six points $P_1, P_2, P_3, Q_1, Q_2, Q_3$. See figure A.40. This case is easily seen to be self-inverse from the plumbing diagram (figure A.41) and hence occurs.

Case 4(xvii): A degree n ($n \geq 4$) curve δ is in the configuration.

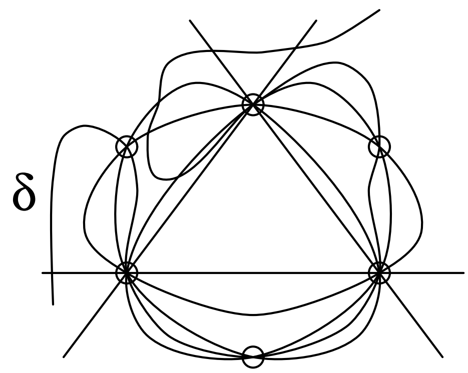


Figure A.40: Case 4(xvi)

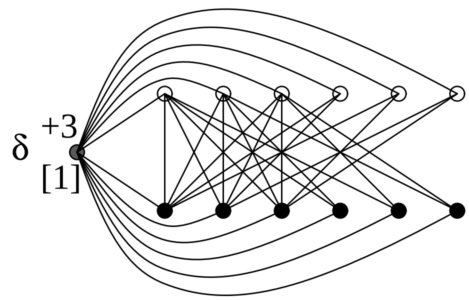


Figure A.41: Case 4(xvi)

Then δ intersects λ_1 in $n \geq 4$ places by Bezout's theorem, of which at most two can be ambiguous contradicting lemma 3.3.3.
Hence this case does not occur.

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