Cutting and Pasting of Manifolds; SK-Groups

U. Karras, M. Kreck,

W. D. Neumann, E. Ossa

Bonn University, Germany

and

Institute for Advanced Study, Princeton, N.J.

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CONTENTS

PREFACE

Let M be a closed manifold and $L \subset M$ a closed submanifold of codimension 1 with trivial normal bundle. If one cuts M open along L one obtains a manifold M' with boundary $\partial M' = L + L$ (disjoint union), and by pasting these two copies of L together again in a different way one can obtain a new closed manifold M_1. M_1 is said to have been obtained by cutting and pasting M.

The theory of so-called SK-invariants--invariants under cutting and pasting of manifolds--was born in a series of papers [13], [14], by Klaus Jänich, characterizing signature and euler characteristic by additivity properties. Later Karras and Kreck, in their Diplom theses, extended many of Jänich's results to cutting and pasting of bundles.

The idea of defining SK-groups brought many simplifications and in summer, 1971, a study group was organized in which the authors incorporated these simplifications in a summary of the known results, in particular, of Karras' and Kreck's Diplom theses. The results were also extended somewhat. A survey lecture by Neumann for the Bonn-Heidelberg Colloquium (Dec., 1970) served as a basis for this study group, of which these notes are the proceedings.

Chapter 1 brings the general theory of SK-invariants and SK-groups and proves Jänich's results in this framework. Basic for the theory are Theorems (1.1) and (1.2), which reduce calculations of SK-groups to the solution of problems of the following type: which bordism classes in, say, $\Omega_*(X)$ can be represented by an $M \to X$ where M is a manifold which fibres over S^1? The results of these notes solve this in many cases.

Chapter 2 is mainly the Diplom thesis work of Karras and Kreck on SK of bundles. An important by-product is results on multiplicativity of signature for fibre bundles--this was originally the main motivation for much of this work.

Chapter 3 on unoriented equivariant SK is based on work of Neumann and Ossa at a miniconference in Regensburg in June, 1970. It generalizes a result of Karras from $\mathbb{Z}_2$ to arbitrary groups. Since euler characteristics of fixpoint sets and similar invariant subsets are SK-invariants, a complete calculation of equivariant SK-invariants would give some general Smith-type theorems.

Chapter 4 brings a generalization of the concept of SK-invariant, due to K. Jänich. The complete calculation of the corresponding universal group, denoted by SKK_*, is based on work of K. Jänich, Ossa and Neumann. Ossa has proved that SKK_* can be identified with the vector-field bordism groups of Reinhart [16]. The index of an elliptic operator is an important example for an SKK-invariant which is generally not an SK-invariant; this was originally the main motivation for SKK-invariants.

The cutting and pasting concepts which have previously appeared in the literature differ in some cases from ours, and Chapter 5 fits them into the framework of these notes. Finally in Chapter 6 some recent results of Neumann which result from Elmar Winkelnkemper's "open book theorem" are described. In particular, it is shown that in odd dimensions $\neq 5$ all SK-invariants for bundles over orientable manifolds vanish, and the connection between SK and multiplicativity of signature is reconsidered.

An appendix by Gottfried Barthel on the extension of the theory to categories of manifolds with (B,f)-structure completes the notes.

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After these notes were typed it was noticed that the methods of chapters 2 and 6 easily lead to the result that for a simply connected space X, the oriented SK-groups $SK_n(X)$ are equal to $SK_n(pt)$ for $n \neq 4,6$, and that this still holds up to torsion if X has a non-trivial but finite fundamental group. This lends a small amount of credibility to the probably very wild conjecture that $SK_n(X)$ only depends on the fundamental group $\pi_1(X)$. This conjecture has been confirmed for $n \leq 3$.

Since it was too late to incorporate these latter results into these notes, they are left as exercises for the reader and may possibly appear in a later paper by the third named author.

CHAPTER 1: Introduction.

In these notes <u>manifold</u> always means smooth manifold, usually compact, and an <u>invariant</u> ρ for n-dimensional manifolds is assumed to take values in an abelian group and to be additive with respect to disjoint union $+$. That is, if $M = M_1 + M_2$ then $\rho(M) = \rho(M_1) + \rho(M_2)$.

Let ρ be an invariant in this sense for closed oriented n-manifolds. ρ is called an SK-<u>invariant</u> if whenever N and N' are compact oriented n-manifolds and $\varphi,\psi : \partial N \longrightarrow \partial N'$ orientation preserving diffeomorphisms, then

$$\rho(N\cup_{\varphi} -N') = \rho(N\cup_{\psi} -N').$$

Here $-N'$ means N' with reversed orientation, and $N\cup_{\varphi} - N'$ means N pasted to N' along the boundary by φ and smoothed. In other words ρ is invariant under "cutting and pasting" (= <u>S</u>chneiden and <u>K</u>leben) of the closed manifold $M = N\cup_{\varphi} - N'$ along the submanifold $L = \partial N$.

Note that L is a 1-codimensional two-sided submanifold which separates M. It is no gain in generality to drop the condition that L separate M, since the union of L with a second copy of L, suitably embedded near L, will separate M.

In the non-orientable case "cutting and pasting" and "SK-invariant" are defined analogously.

<u>Examples</u>: 1) Euler characteristic e is an SK-invariant for arbitrary manifolds. This follows from the fact that euler characteristic is zero for closed odd-dimensional manifolds, together with the additivity property

$$e(X \cup Y) = e(X) + e(Y) - e(X \cap Y)$$

which holds for any "nice" spaces X and Y which intersect nicely.

2) Signature τ is an SK-invariant for orientable manifolds. This is due to the Novikov additivity property

$$\tau(N \cup_{\varphi} -N') = \tau(N) - \tau(N'),$$

where N, N', φ are as above. A proof can for instance be found in Atiyah-Singer [3].

If G is a compact Lie group one can also consider equivariant cutting and pasting of G-manifolds. The case that G acts freely is of particular interest, as clearly the problem of calculating SK-invariants for free G-actions with oriented (resp. arbitrary) orbit space is the same as the problem of calculating invariants for cutting and pasting of locally trivial fibre bundles with fixed fibre, structure group G, and oriented (resp. arbitrary) closed base manifold.

If the total space of the fibre bundle is also a closed orientable manifold, then τ(Base manifold) and τ(Total space) are both SK-invariants, so non-multiplicity of signature will show up in the SK-invariants. This will be discussed in more detail in Chapter 2.

We now construct the basic tools for calculating SK-invariants.

Let X be a space. A <u>singular</u> <u>oriented</u> n-<u>manifold</u> in X is an equivalence class (M,f), where M is a closed oriented n-manifold, $f : M \to X$ a continuous map, and (M,f) is equivalent to (M',f') if there is an orientation

preserving diffeomorphism $M \to M'$ such that

$$\begin{tikzcd} M \arrow[rr] \arrow[dr, "f"'] & & M' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

commutes. Let

$$\mathfrak{M}_n^{SO}(X) := \{\text{singular oriented } n\text{-manifolds in } X\}.$$

$\mathfrak{M}_n^{SO}(X)$ is a commutative semigroup with respect to disjoint union $+$ and has a zero given by $M = \emptyset$.

Let $M_1 = N \cup_{\varphi} - N'$ and $M_2 = N \cup_{\psi} - N'$ be closed orientable manifolds obtainable from each other by cutting and pasting along $\partial N \subset M_1$. Let $f_i : M_i \to X$ be continuous maps. We say the singular manifold (M_2, f_2) is obtained from (M_1, f_1) by cutting and pasting in X if there are homotopies

$$f_1|N \simeq f_2|N, \qquad f_1|N' \simeq f_2|N'.$$

Two singular oriented n-manifolds $(M_1,f_1), (M_2,f_2) \in \mathfrak{M}_n^{SO}(X)$ are called SK-<u>equivalent</u> if there is an $(M,f) \in \mathfrak{M}_n^{SO}(X)$ such that the disjoint union $(M_2,f_2) + (M,f)$ can be obtained from $(M_1,f_1) + (M,f)$ by a sequence of cutting and pastings in X (Ed Miller at Harvard has recently observed that for non-empty M_1, M_2, this definition is equivalent to the "unstabilized version"--without adding (M,f). See end of Chapter 5.) The quotient semigroup

$$\mathcal{Y}_n^{SO}(X) := \mathfrak{M}_n^{SO}(X)/\text{SK-equivalence}$$

is a cancellative semigroup. Define

$$SK_n^{SO}(X) := \text{Grothendieck group of } \mathcal{Y}_n^{SO}(X).$$

Since $\mathcal{Y}_n^{SO}(X)$ is cancellative, it injects into $SK_n^{SO}(X)$, so two singular manifolds represent the same element in $SK_n^{SO}(X)$ if and only if they are SK-equivalent. In fact it follows from Theorem (1.1) below that $\mathcal{Y}_n^{SO}(X)$ actually equals $SK_n^{SO}(X)$, but we won't need this.

By construction, any SK-invariant for singular oriented n-manifolds in X factors over the natural map

$$\mathfrak{M}_n^{SO}(X) \longrightarrow SK_n^{SO}(X),$$

and this map is itself an SK-invariant. Thus $SK_n^{SO}(X)$ yields the universal SK-invariant.

Example: $X = BG$ (classifying space for G) where G is a Lie group. Then $SK_n^{SO}(BG)$ gives the universal SK-invariant for fibre bundles with fixed fibre and structure group G, over oriented n-manifolds.

$X = *$ (the one-point space). $SK_n^{SO}(*)$ gives the universal SK-invariant for oriented n-manifolds.

One can make completely analogous definition in the not-necessarily-oriented case, to obtain a universal SK-invariant $\mathfrak{M}_n^{O}(X) \longrightarrow SK_n^{O}(X)$.

Conventions: In the oriented case we omit the superscript SO and write $SK_n(X) := SK_n^{SO}(X)$. Furthermore, we write

$$SK_n := SK_n(*),$$

$$SK_n^O := SK_n^O(*),$$

the SK-groups for oriented resp. arbitrary n-manifolds.

Remarks: SK_n clearly defines a covariant functor from the homotopy category of topological spaces to the category of abelian groups. Product of singular manifolds induces a functorial bilinear map

$$SK_n(X) \times SK_n(Y) \longrightarrow SK_{n+m}(X \times Y).$$

In particular $SK_* = \coprod_n SK_n$ is a graduated ring, and for any X, $SK_*(X)$ is a graduated SK_*-module. There is an augmentation

$$\varepsilon : SK_*(X) \longrightarrow SK_*$$

induced by $X \longrightarrow *$.

Similar remarks hold in the unoriented case.

Statement of Results.

Let $\overline{SK}_n(X)$ be $SK_n(X)$ factored by the bordism relations, that is, $SK_n(X)$ factored by the subgroup generated by all elements which have a representative (M,f) which bounds in X. $\overline{SK}_n^O(X)$ is defined analogously. A basic tool in these notes will be:

THEOREM (1.1): Let X be path-connected. There is a split exact sequence

$$0 \longrightarrow I_n \longrightarrow SK_n(X) \longrightarrow \overline{SK}_n(X) \longrightarrow 0,$$

where I_n is the subgroup of $SK_n(X)$ generated by $[S^n,*]$ (here $*$ denotes the -unique up to homotopy- constant map $S^n \longrightarrow X$) and

$$I_n \cong \mathbb{Z} \quad n \text{ even},$$

$$= 0 \quad n \text{ odd}.$$

In the non-orientable case exactly the same holds except that the sequence does not split for n even.

A useful corollary of Theorem (1.1) is:

THEOREM (1.1b): If $[M,f] = [M',f']$ in $\overline{SK}_n(X)$ and $e(M) = e(M')$, then $[M,f] = [M',f']$ in $SK_n(X)$. The same also in the non-oriented case.

Indeed, the assumptions of (1.1b) imply $[M,f] - [M',f'] \in \mathrm{Ker}(SK_n(X) \longrightarrow \overline{SK}_n(X)) = I_n$ and $e([M,f] - [M',f']) = 0$. Since euler characteristic clearly classifies the elements of I_n by Theorem (1.1), it follows that $[M,f] - [M',f']$ is zero in I_n and hence certainly in $SK_n(X)$.

There are obvious epimorphisms $\Omega_n(X) \longrightarrow \overline{SK}_n(X)$ and $\mathfrak{N}_n(X) \longrightarrow \overline{SK}^0_n(X)$. Let $F_n(X) \subset \Omega_n(X)$ and $F^0_n(X) \subset \mathfrak{N}_n(X)$ be the subgroups of all elements which admit a representative (M,f) such that M fibres over the circle S^1.

THEOREM (1.2): The sequences

$$0 \longrightarrow F_n(X) \longrightarrow \Omega_n(X) \longrightarrow \overline{SK}_n(X) \longrightarrow 0$$

$$0 \longrightarrow F^0_n(X) \longrightarrow \mathfrak{N}_n(X) \longrightarrow \overline{SK}^0_n(X) \longrightarrow 0$$

are exact.

This theorem reduces the calculation of $\overline{SK}_n(X)$ and $\overline{SK}^0_n(X)$ to a bordism problem.

The calculation of the absolute SK-groups is as follows:

THEOREM (1.3a): For n odd both SK_n and SK^0_n are zero. For even n one has:

$$SK_n \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{with basis } [S^n], [P_{n/2}\mathbb{C}], \text{ for } n \equiv 0 \pmod 4; \\ \mathbb{Z} & \text{with basis } [S^n] \quad , \text{ for } n \equiv 2 \pmod 4; \end{cases}$$

$$SK^0_n \cong \quad \mathbb{Z} \quad \text{with basis } [P_n\mathbb{R}] \quad , \text{ for } n \equiv 0 \pmod 2.$$

Recall that for oriented manifolds euler characteristic and signature are congruent modulo 2. The claim as to what one can choose as bases of the above groups is clearly equivalent to: the above three isomorphism can be given by $\frac{e-\tau}{2} \oplus \tau$, $\frac{e}{2}$, e respectively. Thus

COROLLARY (1.4): Any SK-invariant for smooth manifolds is a linear combination of euler characteristic, and signature in the oriented case.

In view of Theorems (1.1) and (1.2) we can give two equivalent formulations of Theorem (1.3a):

THEOREM (1.3b): For n odd both $\overline{SK}_n$ and $\overline{SK}^O_n$ are zero. For even n one has isomorphisms

$$\tau : \overline{SK}_n \overset{\cong}{\to} \begin{cases} \mathbf{Z} & n \equiv 0 \pmod 4; \\ 0 & n \equiv 2 \pmod 4; \end{cases}$$
$$e(\mathrm{mod}\ 2) : \overline{SK}^O_n \overset{\cong}{\to} \mathbf{Z}_2 \quad n \equiv 0 \pmod 2.$$

THEOREM (1.3c):

$$F_n = \{[M] \in \Omega_n \mid \tau(M) = 0\}$$
$$F^O_n = \{[M] \in \mathcal{N}_n \mid e(M) \equiv 0 \pmod 2\}.$$

Theorem (1.3c) has been proved by Conner and Floyd [9] in the non-oriented case, and up to torsion by Conner and Burdick [8] and [5] in the oriented case (that is $F_n + \mathrm{Tors}(\Omega_n) = \{[M] \in \Omega_n \mid \tau(M) = 0\}$). Thus to prove (1.3c), and hence also (1.3b) and (1.3a), it suffices to prove

$$\mathrm{Tors}(\Omega_n) \subset F_n.$$

The proof we shall give is based on Jänich's proof [14] of (1.3b). Actually Jänich works with invariants and uses a different concept of SK-invariant but as we shall show in Chapter 5, his concept is equivalent to "$\overline{SK}$-invariant." Essentially the same proof of (1.3b) has also been found independently by Rowlett [17], who also had independently had the idea of defining SK-groups. He also had

a different SK-concept, which also turns out to give precisely $\overline{SK}$ (see Chapter 5). An independent proof of (1.3c) in the oriented case for $n > 5$ can be found in H. E. Winkelnkemper's dissertation [19] (see also [20]). Theorems (1.1) and (1.2), which show the equivalence of the three formulations of (1.3), are of later vintage, though they are latent already in the work of Jänich, Burdick and others.

The proof of Theorem (1.1):

We first give some lemmas on cutting and pasting which will also be useful later on. If (M,f) is a singular manifold in X we write $[M,f]_{SK}$, $[M,f]_\Omega$, etc., for the class of (M,f) in $SK_*(X)$, $\Omega_*(X)$, etc., but omit the subscript if no confusion can occur. If $X = *$ is the one-point space, we simply write $[M]_{SK}$, $[M]_\Omega$, etc., for classes in the respective groups.

LEMMA (1.5): For any space X we have in $SK_*(X)$ and $SK^O_*(X)$:

i) $[S^1,f] = 0$ for any $f : S^1 \longrightarrow X$.

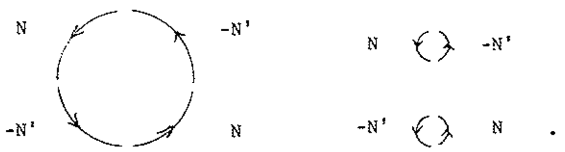
ii) If M fibres over S^n with typical fibre F and $f : M \longrightarrow X$ then $[M,f] = [S^n][F,f|F]$ (recall that $SK_*(X)$ is an SK_*-module).

iii) If M fibres over $P_n\mathbf{C}$ with typical fibre F and $f : M \longrightarrow X$ then $[M,f] = [P_n\mathbf{C}][F,f|F]$.

iv) In the non-oriented case iii) also holds with $P_n\mathbf{C}$ replaced by $P_n\mathbf{R}$.

Proof: We prove the orientable case; in the non-orientable case the proofs are the same.

i) Let $N = -N' = I + I$, where $I = [0,1]$ is the unit interval. We can paste N to $-N'$ in two ways to obtain either S^1 or $S^1 + S^1$:



Hence $[S^1] = 2[S^1]$, so $[S^1] = 0$. This cutting and pasting can clearly also be done in any space X.

ii) We can write $S^n = D^n \cup -D^n$, pasted along the boundary S^{n-1}. Since a fibration over the disc D^n is trivial, we have $M = (D^n \times F) \cup -(D^n \times F)$. If $f : M \longrightarrow X$ is any map, then restricted to each piece $D^n \times F$, f is homotopic to $* \times f|F$. On the other hand $(S^n \times F, * \times f|F)$ is also of the form $((D^n \times F) \cup -(D^n \times F), * \times f|F)$, so $[M,f] = [S^n \times F, * \times f|F] = [S^n][F,f|F]$.

iii) We prove iii) by induction on n; for $n = 0$ it is trivial. Suppose M fibres over $P_n\mathbf{C}$ with fibre F. We can write $P_n\mathbf{C}$ as

$$P_n\mathbf{C} = D^{2n} \cup -N^{2n}$$

where $-N^{2n}$ is diffeomorphic to the normal disc bundle of $P_{n-1}\mathbf{C}$ in $P_n\mathbf{C}$. Let

$$M_0 = M|D^{2n} + N \times F = D^{2n} \times F + N \times F$$

$$M_1 = M|N + D^{2n} \times F .$$

If $f : M \longrightarrow X$ is a map, we define maps of M_0 and M_1 to X by taking the restriction of f on $M|D^{2n}$ and $M|N$ and taking $* \times f|F$ on $N \times F$ and

$D^{2n} \times F$. On the boundaries $S^{n-1} \times F$ of these pieces all these maps are homotopic to $* \times f|F$, so we can paste M_0 to M_1 in two ways in X to obtain

$$(M_0 \cup_\varphi - M_1, f_1) = (M,f) + (-P_n\mathbf{C} \times F, * \times f|F),$$

$$(M_0 \cup_\psi - M_1, f_2) = (E,g) + (S^{2n} \times F, * \times f|F).$$

In the second case we have pasted the first part of M_0 to the second part of M_1 and vice versa. E is a fibration over the double $\mathcal{D}N = N\cup_{id} - N$ of N with fibre F, and g is a map with $g|F = f|F$. However, $\mathcal{D}N$ fibres over $P_{n-1}\mathbf{C}$ with fibre F', where F' fibres over S^2 with fibre F. By part ii) we have $[F',g|F'] = [S^2][F,g|F] = [S^2][F,f|F]$, so by induction hypothesis $[E,g] = [P_{n-1}\mathbf{C}][F',g|F'] = [P_{n-1}\mathbf{C}][S^2][F,f|F]$. The above cutting and pasting thus shows

$$[M,f] + [-P_n\mathbf{C}][F,f|F] = [P_{n-1}\mathbf{C}][S^2][F,f|F] + [S^{2n}][F,f|F].$$

That is,

$$[M,f] = ([P_{n-1}\mathbf{C}][S^2] + [S^{2n}] - [-P_n\mathbf{C}])[F,f|F].$$

It hence only remains to prove that

$$[P_n\mathbf{C}] = [P_{n-1}\mathbf{C}][S^2] + [S^{2n}] - [-P_n\mathbf{C}],$$

but this follows by taking $F = *$ in the above. The proof of iv) is completely analogous to iii). Q.E.D.

LEMMA (1.6): Suppose the singular manifold (M',f') in X results from (M,f) by surgery of type $(k+1,n-k)$ in X. Then in $SK_n(X)$ (resp. $SK_n^0(X)$)

$$[M,f] + [S^n,*] = [M',f'] + [S^k \times S^{n-1},*].$$

<u>Proof:</u> We must look closely at the surgery and its trace. Let

$$i : S^k \times D^{n-1} \hookrightarrow M$$

be the embedding on which surgery was done. Then

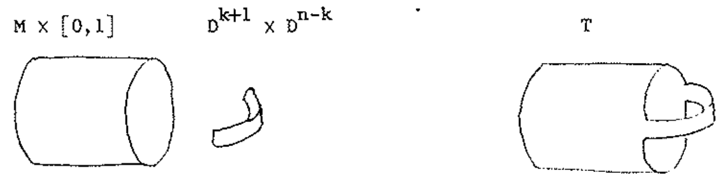
$$M' = (M - (S^k \times D^{n-k}))\cup - (D^{k+1} \times S^{n-k-1})$$

where "$\cup$" is the obvious identification of boundaries $S^k \times S^{n-k-1}$. The trace T of the surgery can be constructed as follows.

Recall that

$$S^n = (S^k \times D^{n-k})\cup - (D^{k+1} \times S^{n-k-1})$$

pasted by the obvious identification of boundaries (think of S^n as $\partial(D^{k+1} \times D^{n-k})$!). T is the manifold obtained by taking the disjoint union of $M \times [0,1]$ and $D^{k+1} \times D^{n-k}$ and then identifying $S^k \times D^{n-k} = (i(S^k \times D^{n-k}),1) \subset M \times [0,1]$ with $S^k \times D^{n-k} \subset S^n = \partial(D^{k+1} \times D^{n-k})$, and then smoothing corners.



The boundary of T is clearly $\partial D = M + (-M')$. The fact that we did surgery in X means by definition that we have a continuous map

$$g : T \longrightarrow X$$

with $g|M = f$ and $g|M' = f'$.

Now

$$\text{A)}\quad \begin{aligned} M + S^n &= (M - (S^k \times D^{n-k}))\cup-(S^k \times D^{n-k}) + (S^k \times D^{n-k})\cup-(D^{k+1} \times S^{n-k-1}) \\ M' + S^k \times D^{n-k} &= (M - S^k \times D^{n-1}))\cup-(D^{k+1} \times S^{n-k-1}) + (S^k \times D^{n-k})\cup-(S^k \times D^{n-k}), \end{aligned}$$

always with the obvious identification of boundaries, so $M' + S^k \times D^{n-k}$ results by cutting and pasting $M + S^n$. But we must cut and paste in X. For this, consider $S^k \times D^{n-k}$ and $D^{k+1} \times S^{n-k-1}$ as subsets of $\partial(D^{k+1} \times D^{n-k}) \subset D^{k+1} \times D^{n-k} \subset T$. Then we have maps into X of all the pieces on the right hand side of A) by restricting the map g. The cutting and pasting is compatible with these maps and the resulting maps of M and M' into X are the ones we want. The resulting maps of S^n and $S^k \times S^{n-k}$ into X factor over $g|D^{k+1} \times D^{n-k} : D^{k+1} \times D^{n-k} \longrightarrow X$, and are hence both homotopic to the constant map. This completes the proof of the lemma. Q.E.D.

As an application of this lemma note that $S^{k+1} \times S^{n-k-1}$ results from S^n by surgery of type $(k+1,n-k)$, since

$$\begin{aligned} S^{k+1} \times S^{n-k-1} &= (D^{k+1} \times S^{n-k-1})\cup-(D^{k+1} \times S^{n-k-1}) \\ &= (S^n - S^k \times D^{n-k})\cup-(D^{k+1} \times S^{n-k-1}). \end{aligned}$$

Thus the lemma gives

$$[S^n,*] + [S^n,*] = [S^{k+1} \times S^{n-k-1},*] + [S^k \times S^{n-k},*].$$

Putting $k = 0$ (alternatively, by Lemma (1.5) i) and ii)) we have

$[S^1 \times S^{n-1},*] = 0$, and a simple induction now shows

COROLLARY (1.7): In $SK_*(X)$:

$$[S^k \times S^{n-k},*] = \begin{cases} 2[S^n,*], & k \text{ even} \\ 0 , & k \text{ odd} . \end{cases}$$

COROLLARY (1.8): Let (Y,g) be a bordism in X between the singular manifolds (M_1,f_1) and (M_2,f_2). Then in $SK_*(X)$:

$$[M_1,f_1] = [M_2,f_2] - (e(Y) - e(M_1))[S^n,*].$$

Proof: First suppose Y is an elementary bordism, that is the trace of a surgery of type $(k+1,n-k)$ say. Then by Lemma (1.6) and Corollary (1.7)

$$[M_1,f_1] = [M_2,f_2] + (-1)^k[S^n,*],$$

so it suffices to prove that $e(Y) - e(M_1) = (-1)^{k+1}$. But Y is obtained by pasting $D^{k+1} \times D^{n-k}$ to $M_1 \times I$ along submanifolds $S^k \times D^{n-k}$ of $\partial(D^{k+1} \times D^{n-k})$ and $\partial(M_1 \times I)$ and then smoothing the result, so

$$\begin{aligned} e(Y) &= e(M_1 \times I) + e(D^{k+1} \times D^{n-k}) - e(S^k \times D^{n-k}) \\ &= e(M_1) + (-1)^{k+1}, \end{aligned}$$

proving this case.

In the general case we can split Y up into a sequence of elementary bordisms and the corollary then follows easily from the case just proved and the additivity property of euler characteristic. Q.E.D.

Theorem (1.1) is now easily proved. Namely, the kernel I_n of $SK_n(X) \longrightarrow \overline{SK}_n(X)$ is clearly generated by all classes $[M,f]$ such that (M,f) bounds in X. By Corollary (1.8) such an $[M,f]$ is a multiple of $[S^n,*]$, so I_n is generated by $[S^n,*]$. If n is odd, say $n = 2k+1$, then S^n fibres over $P_k\mathbf{C}$ with fibre S^1, so by Lemma (1.5), iii) and i), it follows that $[S^n,*] = 0$. If n is even the fact that $e(S^n) = 2$ shows that $[S^n,*]$ has infinite order in $SK_*(X)$, so $I_n \cong \mathbf{Z}$. The same arguments all hold in the non-oriented case, so it only remains to prove the claim on when the sequence of Theorem (1.1) splits.

Assume n is even. In the oriented case the map $(e-\tau)/2 : SK_n(X) \longrightarrow \mathbf{Z} \cong I_n$ is a retraction of $I_n \hookrightarrow SK_n(X)$ which splits the sequence

$$0 \longrightarrow I_n \longrightarrow SK_n(X) \longrightarrow \overline{SK}_n(X) \longrightarrow 0.$$

In the non-oriented case S^n and $2P_n\mathbf{R}$ both bound, so they are in the kernel I_n of $SK^0_n(X) \longrightarrow \overline{SK}^0_n(X)$. But euler characteristic classifies the elements of I_n and $e(S^n) = 2 = e(2P_n\mathbf{R})$, so $[S^n,*] = 2[P_n\mathbf{R},*]$ in $SK^0_n(X)$. Thus the generator of I_n is not indivisible in $SK^0_n(X)$, so the sequence

$$0 \longrightarrow I_n \longrightarrow SK^0_n(X) \longrightarrow \overline{SK}^0_n(X) \longrightarrow 0$$

does not split. The proof of Theorem (1.1) is complete.

Fibrations over S^1.

Let N be a closed manifold and $\varphi : N \longrightarrow N$ a diffeomorphism.

<u>Definition</u>: N_φ is the manifold obtained from $N \times I$ by identifying the ends $N \times \{0\}$ and $N \times \{1\}$ via φ ; that is $(x,1)$ is identified with $(\varphi(x),0)$ for each $x \in N$. N_φ is called the <u>mapping torus</u> of φ.

The projection $N \times I \longrightarrow I$ induces a fibration

$$N_\varphi \longrightarrow S^1$$

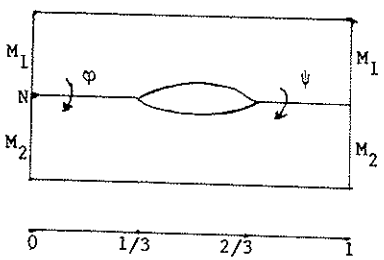
with fibre N. Conversely any fibration over S^1 with fibre N is clearly of this form for suitable φ. N_φ is orientable if and only if N is orientable and φ orientation preserving. The following lemma holds in the orientable and in the non-orientable category. We formulate the orientable case.

<u>LEMMA</u> (1.9): If the singular manifold (M',f') results from (M,f) by cutting and pasting along N in X, say $M = M_1 \cup_\varphi - M_2$, $M' = M_1 \cup_\psi - M_2$, where $\varphi,\psi : \partial M_1 = N \longrightarrow \partial M_2$ are diffeomorphisms, then

$$[M,f]_\Omega = [M',f']_\Omega + [N_{\psi\varphi^{-1}},g]_\Omega$$

in $\Omega_*(X)$ for suitable $g : N_{\psi\varphi^{-1}} \longrightarrow X$.

<u>Proof</u>: A bordism is constructed as follows. Let Y be the union of $M_1 \times [0,1]$ and $M_2 \times [0,1]$ with the following identifications: for $x \in N$ identify $(x,t) \in \partial M_1 \times [0,\frac{1}{3}]$ with $(\varphi(x),t) \in \partial M_2 \times [0,\frac{1}{3}]$ and $(x,t) \in \partial M_1 \times [\frac{2}{3},1]$ with $(\varphi(x),t) \in \partial M_2 \times [\frac{2}{3},1]$.



After smoothing it is easily seen that $\partial Y = M - N_{\varphi\psi^{-1}} - M'$, so Y is the required bordism. Since we are doing cutting and pasting in X we have homotopies $f|M_1 \simeq f'|M_1$ and $f|M_2 \simeq f'|M_2$ which can clearly be used to construct a map $h : Y \longrightarrow X$ with $h|M = f$ and $h|M' = f'$. Putting $g = h|N_{\varphi\psi^{-1}}$, the lemma is proved.

Q.E.D.

To prove Theorem (1.2) note that $\mathrm{Ker}(\Omega_*(X) \longrightarrow \overline{SK}_*(X))$ is generated by classes of the form

$$[M,f]_\Omega - [M',f']_\Omega,$$

where $[M',f']$ results from $[M,f]$ by cutting and pasting in X, so by the above lemma

$$\mathrm{Ker}(\Omega_*(X) \longrightarrow \overline{SK}_*(X)) \subset F_*(X).$$

The reverse inclusion is an immediate consequence of Lemma (1.5) i) and ii), so Theorem (1.2) is proved in the orientable case. The non-orientable case is the same proof.

Q.E.D.

Before we prove Theorem (1.3) we need a lemma:

<u>LEMMA</u> (1.10): Suppose $M_i = (N_i)_{\varphi_i}$ for $i = 1,\ldots,k$, with each N_i orientable and each φ_i orientation reversing. Then there exists orientable N and orientation reversing $\varphi : N \longrightarrow N$ with

$$M_1 \times \ldots \times M_k = N_\varphi.$$

Furthermore if $k \geq 2$ then N itself fibres over S^1.

Proof: The general case follows from $k = 2$ by a trivial induction, so assume $k = 2$. Let $p_i : M_i \longrightarrow S^1$ $(i = 1,2)$ be the projections. Then the fibration

$$p : M_1 \times M_2 \longrightarrow S^1$$
$$(x,y) \longmapsto p_1(x)p_2(y)^{-1}$$

has typical fibre

$$N = p^{-1}(1) = \{(x,y) \in M_1 \times M_2 | p_1(x) = p_2(y)\}.$$

There is a fibration

$$q : \quad N \longrightarrow S^1$$
$$(x,y) \longmapsto p_1(x)$$

with typical fibre

$$q^{-1}(1) = \{(x,y) \in M_1 \times M_2 | p_1(x) = p_2(y) = 1\} = N_1 \times N_2,$$

and one easily checks that this fibration is given by

$$N \cong (N_1 \times N_2)_{\varphi_1 \times \varphi_2}.$$

Since φ_1 and φ_2 both reverse orientations, $\varphi_1 \times \varphi_2$ preserves it, so N is orientable. But $M_1 \times M_2$ is non-orientable, so $M_1 \times M_2$ must be of the form N_φ with φ orientation reversing. Q.E.D.

Recall that to prove the three versions a), b), c) of Theorem (1.3) it only remains to prove

$$T_n := \mathrm{Tor}(\Omega_n) \subset F_n,$$

so the first thing to do is describe T_*. We recall C. T. C. Wall's description in [18].

Let M be a closed manifold. Then one can always find a closed 1-codimensional submanifold $W \subset M$ such that

1) $M - W$ is orientable, and

2) no submanifold of W satisfies 1).

C. T. C. Wall proves that if W can be chosen orientable with trivial normal bundle in M then the class $[W]_\Omega \in \Omega_*$ is a torsion element which only depends on $[M]_{\mathfrak{N}} \in \mathfrak{N}_*$. Under these conditions he defines $\partial_3[M]_{\mathfrak{N}} = [W]_\Omega$, so ∂_3 is a homomorphism from a subgroup of $\mathfrak{N}_*$ to $T_* = \mathrm{Tors}(\Omega_*)$.

Example (1.11): Let $M = N_\varphi$ with N orientable and φ orientation reversing. Then clearly $\partial_3[M] = [N]_\Omega$.

Now let $P(m,n)$ be the quotient manifold of the free involution $(x,z) \longmapsto (-x,\bar{z})$ on $S^m \times P_n\mathbf{C}$ (the "Dold manifold") and $\alpha : P(m,n) \longrightarrow P(m,n)$ the involution induced by the map $(x,z) \longmapsto (x',z)$ on $S^m \times P_n\mathbf{C}$, where $x \longmapsto x'$ is reflection in an equator of S^m. Let

$$Q(m,n) = P(m,n)_\alpha.$$

Remark: $P(m,n)$ is orientable $\Longleftrightarrow$ $m+n$ is odd.

α is orientation reversing $\Longleftrightarrow$ m is odd.

If a is a natural number write $a = 2^{r-1}(2s+1)$ and define

$$X_{2a} = Q(m,n),\ m = 2^r - 1,\ n = 2^r s.$$

According to Wall (loc. cit.), the torsion $T_* \subset \Omega_*$ is generated as a ring by classes of the form

$$\partial_3[X_{2a_1} \times \dots \times X_{2a_k}].$$

If $k \geq 2$ then by the above remarks, Lemma (1.10) and Example (1.11), $\partial_3[X_{2a_1} \times \dots \times X_{2a_k}]$ is represented by a manifold which fibres over S^1, so $\partial_3[X_{2a_1} \times \dots \times X_{2a_k}] \in F_*$, as was to be shown. If $k = 1$ then by Example (1.11) we have $\partial_3(X_{2a}) = [P(m,n)]_\Omega$, so we must show $[P(m,n)]_\Omega \in F_*$, or equivalently (by Theorem (1.2)), $[P(m,n)] = 0$ in $\overline{SK}_*$. The map $S^m \times P_n\mathbb{C} \longrightarrow S^m$ induces a fibration $P(m,n) \longrightarrow P_m\mathbb{R}$ with fibre $P_n\mathbb{C}$, and $P_m\mathbb{R}$ fibres over $P_q\mathbb{C}$ with fibre S^1, where $q = (m-1)/2 = 2^{r-1} - 1$. Thus $P(m,n)$ fibres over $P_q\mathbb{C}$ with fibre F which fibres over S^1, so by Lemma (1.5) $[P(m,n)] = [P_q\mathbb{C}][F] = 0$ in SK_*, and hence certainly in $\overline{SK}_*$. This completes the proof. Q.E.D.

CHAPTER 2: SK of Fibre Bundles.

Let G be a Lie group. In this chapter we investigate SK for fibre bundles over closed differentiable manifolds with fixed fibre F and structure group G. As in Chapter 1, SK-equivalence for fibre bundles is defined by saying that the fibre bundle $E \cup_\varphi E'$ is obtained from $E \cup_\psi E'$ by cutting and pasting if E and E' are fibre bundles with fibre F and structure group G over compact manifolds M and M' respectively and $\varphi, \psi : E|\partial M \longrightarrow E'|\partial M'$ are bundle isomorphisms which induce diffeomorphisms $\partial M \longrightarrow \partial M'$ in the bases. SK-groups for fibre bundles can then be defined in the obvious way. By the homotopy classification of fibre bundles it is clear that these groups are $SK_*(BG)$ in the oriented case and $SK_*^O(BG)$ in the non-oriented case.

Remark: If the fibre F is a smooth manifold one can consider SK of smooth fibre bundles. This makes no difference for (as is well known) any continuous fibre bundle admits a smooth structure, unique up to bundle isomorphism.

Interpreting $SK_*(BG)$ as the SK-group for fibre bundles with structure group G, the augmentation

$$\varepsilon^{BG} : SK_*(BG) \longrightarrow SK_* = SK_*(pt)$$

is just the map which sends the SK-class of a bundle (E,π,B) to the SK-class $[B]$ of its base manifold. We have the trivial lemma:

LEMMA (2.1): There are natural isomorphisms

$$SK_*(BG) \cong SK_* \oplus \operatorname{Ker} \varepsilon^{BG},$$
$$\overline{SK}_*(BG) \cong \overline{SK}_* \oplus \operatorname{Ker} \varepsilon^{BG}.$$

Proof: The map $pt \longrightarrow BG$, which is unique up to homotopy, induces a retraction $SK_* \longrightarrow SK_*(BG)$ of ε^{BG}, proving the first isomorphism. Similarly, one has that

$$\overline{SK}_*(BG) \cong \overline{SK}_* \oplus \text{Ker } \bar{\varepsilon}^{BG},$$

where $\bar{\varepsilon}^{BG} : \overline{SK}_*(BG) \longrightarrow \overline{SK}_*$ is the augmentation. But Theorem (1.1) implies that $\text{Ker } \bar{\varepsilon}^{BG} = \text{Ker } \varepsilon^{BG}$, so the second isomorphism is also proved.

This lemma can be interpreted as saying that the SK-invariants for bundles split in a natural way into the SK-invariants of the base space, which we already know are euler characteristic and signature, together with certain bordism invariants of the whole bundle, given by $\text{Ker } \varepsilon^{BG}$. As we are about to state precisely, these latter additional invariants are in most cases torsion, and often actually zero.

THEOREM (2.2): i) If G is a Lie group with finitely many components then $\text{Ker } \varepsilon^{BG}$ is a torsion group.

ii) If G is compact and $H^*(BG)$ torsion free, for instance, $G = (S^1)^n$, $U(n)$, $SU(n)$, $Sp(n)$, then $\text{Ker } \varepsilon^{BG} = 0$.

Remark: The conclusion of part i) above can be formulated: given any bundle (E,π,B) with structure group G, some multiple mE of E is SK-equivalent to the trivial bundle with base manifold mB. If now the fibre F is also a compact manifold, so that the signatures $\tau(F)$ and $\tau(E)$ are defined, then it clearly follows from this that $m\tau(E) = m\tau(B \times F) = m\tau(B)\tau(F)$, so $\tau(E) = \tau(B)\tau(F)$. That is, signature is multiplicative for E. Atiyah [2] has given an example of non-multiplicativity of signature, so Theorem (2.2) does not generalize to arbitrary G. We will see another example of this later, but first to the proof of (2.2).

By Lemma (2.1) it is sufficient to prove that

$$\bar{\varepsilon}^{BG} : \overline{SK}_*(BG) \longrightarrow \overline{SK}_*$$

is a mod-torsion isomorphism (kernel and cokernel are torsion groups) and an isomorphism if $H^*(BG)$ is torsion-free. We shall prove this first for G a torus, then for G compact, and then finally in the generality of the theorem. Because of the epimorphism $\Omega_*(X) \longrightarrow \overline{SK}_*(X)$, to calculate $\overline{SK}_*(X)$ one need only do cutting and pasting on a generating set of $\Omega_*(X)$. The basic idea of the proof is that in our case such a generating set can be represented by products of projective spaces up to torsion, so Lemma (1.5) iii) gives the result.

Let

$$\mu : \Omega_*(X) \longrightarrow H_*(X)$$

be the canonical map given by $\mu[M,f] = f_*\sigma$, where σ is the fundamental homology class of M.

THEOREM (2.3): Let X be a CW-complex such that $H_*(X)$ has no torsion. Let singular manifolds (M_i, f_i) in X be given such that $\{\mu[M_i,f_i]\}$ is a generating set of $H_*(X)$. Then $\{[M_i,f_i]\}$ is a generating set of $\Omega_*(X)$ as an Ω_*-module.

Proof: See Conner and Floyd [10], §18, p. 49. In fact, Conner and Floyd prove more, namely that if X is a finite CW-complex then the above holds with "generating set" replaced by "base" each time. The finiteness of X is only used in proving the independence of the base $\{[M_i, f_i]\}$, so it is not needed for our formulation.

An easy application of this theorem is the following lemma, whose proof we leave to the reader. Let $\eta_* : P_k\mathbf{C} \longrightarrow BS^1$ be the classifying map for the canonical line bundle over $P_k\mathbf{C}$.

<u>LEMMA</u> (2.4): The set $\{[P_{i_1}\mathbf{C} \times \ldots \times P_{i_n}\mathbf{C}, \eta_{i_1} \times \ldots \times \eta_{i_n}]\}$ generates $\Omega_*(B(S^1)^n)$ as an Ω_*-module (recall that $B(S^1)^n = (BS^1)^n$). In fact it is an Ω_*-base, but we do not need this.

It follows that $\overline{SK}_*(B(S^1)^n)$ is generated as an $\overline{SK}_*$-module by the elements $[P_{i_1}\mathbf{C} \times \ldots \times P_{i_n}\mathbf{C}, \eta_{i_1} \times \ldots \times \eta_{i_n}]$, so if G is a torus, Theorem (2.2) now follows by Lemma (1.5) iii).

Now let G be any compact Lie group and $T \subset G$ its maximal torus. The projection $BT \longrightarrow BG$ induces a map

$$\rho : \overline{SK}_*(BT) \longrightarrow \overline{SK}_*(BG)$$

and the composition

$$\overline{SK}_*(BT) \xrightarrow{\rho} \overline{SK}_*(BG) \xrightarrow{\varepsilon^{BG}} \overline{SK}_*$$

is just ε^{BT}, which we already know to be an isomorphism. Hence to show that ε^{BG} is an isomorphism or mod-torsion isomorphism it suffices to show that ρ is surjective or mod-torsion surjective respectively. By a result of Borel [4] the map

$$H^*(BG) \longrightarrow H^*(BT)$$

is mod-torsion injective, and even injective for $H^*(BG)$ torsion-free. Hence

$$H_*(BT) \longrightarrow H_*(BG)$$

is mod-torsion surjective, and surjective if $H^*(BG)$ is torsion-free, so all we need is the following lemma:

<u>LEMMA</u> (2.5): Let $f : X \longrightarrow Y$ be a map of CW-complexes. If the induced map $H_*(X) \longrightarrow H_*(Y)$ is mod-torsion surjective, then so is $\Omega_*(X) \longrightarrow \Omega_*(Y)$ and hence also $\overline{SK}_*(X) \longrightarrow \overline{SK}_*(Y)$. If $H_*(X)$ has no odd torsion and $H_*(X) \longrightarrow H_*(Y)$ is surjective, then so is $\Omega_*(X) \longrightarrow \Omega_*(Y)$, and hence also $\overline{SK}_*(X) \longrightarrow \overline{SK}_*(Y)$.

<u>Proof</u>: We need the bordism spectral sequence (see for instance Conner and Floyd [10] for details) so we recall the essentials. For a CW-complex X the E^2-term is

$$E^2_{p,q}(X) = H_p(X;\Omega_q)$$

and the E^∞-term is

$$E^\infty_{p,q}(X) = J_{p,q}/J_{p-1,q+1}$$

where

$$0 \subset J_{0,n} \subset \ldots \subset J_{n,0} = \Omega_n(X)$$

is the skeleton filtration of $\Omega_n(X)$, that is

$$J_{p,q} = \operatorname{Im}(\Omega_{p+q}(X^p) \longrightarrow \Omega_{p+q}(X)).$$

Furthermore, the bordism spectral sequence is trivial modulo odd torsion.

It follows that a map $f : X \longrightarrow Y$ which is mod-torsion surjective in homology, and hence for the E^2-term, stays mod-torsion surjective up to E^∞, and hence also for Ω_*, proving the first statement of the lemma.

Now suppose $H_*(X)$ has no odd torsion and $H_*(X) \longrightarrow H_*(Y)$ is surjective. Then $E^2(X)$ has no odd torsion, so by triviality modulo odd torsion of the spectral sequence, the differential $d^2(X) : E^2_{p,q}(X) \longrightarrow E^2_{p+2,q-1}(X)$ is trivial. Also $E^2(X) \longrightarrow E^2(Y)$ is surjective, so $d^2(Y)$ is also trivial. Hence $E^2(X) = E^3(X)$, $E^2(Y) = E^3(Y)$, and repeating the argument we eventually get that both spectral sequences are trivial and $E^\infty(X) \longrightarrow E^\infty(Y)$ is surjective. Hence $\Omega_*(X) \longrightarrow \Omega_*(Y)$ is surjective, as was to be proved. Q.E.D.

Theorem (2.2) is thus proved for compact G. If G is connected but not necessarily compact, choose a maximal connected compact subgroup $H \subset G$. Since the structure group of any bundle with structure group G can be reduced to H,

$$\overline{SK}_*(BH) \longrightarrow \overline{SK}_*(BG)$$

is surjective. Since the composition with $\varepsilon^{BG} : \overline{SK}_*(BG) \longrightarrow \overline{SK}_*$ is ε^{BH}, which we know to be a mod-torsion isomorphism, ε^{BG} is itself a mod-torsion isomorphism.

Finally if G has finitely many, say n, connected components and G_0 is the component of unity, then $BG_0 \longrightarrow BG$ is an n-fold covering. Hence

$$H_*(BG_0) \longrightarrow H_*(BG)$$

is mod-torsion surjective (the n-fold of any homology class in BG clearly comes from BG_0). By Lemma (2.5)

$$\overline{SK}_*(BG_0) \longrightarrow \overline{SK}_*(BG)$$

is mod-torsion surjective, so again, since the composition with ε^{BG} is ε^{BG_0}, which we know is a mod-torsion isomorphism, ε^{BG} is a mod-torsion isomorphism.

Q.E.D.

We now shall calculate $SK_*(BG)$ in some of the cases not covered by the previous theorem.

THEOREM (2.6): For $G = \mathbb{Z}_{p^t}$, p an odd prime, and for $G = \mathbb{Z}_2$, $\operatorname{Ker} \varepsilon^{BG} = 0$.

Proof: For any X we have the short exact sequence of Theorem (1.2):

$$0 \longrightarrow F_*(X) \longrightarrow \Omega_*(X) \longrightarrow \overline{SK}_*(X) \longrightarrow 0.$$

Denote $\operatorname{Ker}(F_*(X) \longrightarrow F_*(pt))$ by $\tilde{F}_*(X)$. The above sequence surjects at all three places onto the short exact sequence

$$0 \longrightarrow F_* \longrightarrow \Omega_* \longrightarrow \overline{SK}_* \longrightarrow 0,$$

so the kernel sequence

(2.7) $$0 \longrightarrow \tilde{F}_*(X) \longrightarrow \tilde{\Omega}_*(X) \longrightarrow \operatorname{Ker} \varepsilon^{BG} \longrightarrow 0$$

is also exact. In particular $\operatorname{Ker} \varepsilon^{BG}$ is the image of $\tilde{\Omega}_*(X) \subset \Omega_*(X)$.

We first consider the case $X = BG$ with $G = \mathbb{Z}_{p^t}$ (p an odd prime), where we consider G as a subgroup of the circle group S^1. S^1 acts freely on the unit sphere S^{2n-1} in $\mathbb{C}^n$ by

$$t(z_1,\ldots,z_n) = (tz_1,\ldots,tz_n), \quad (|t| = 1).$$

This gives a free action of G on S^{2n-1}, inducing a singular manifold $(S^{2n-1}/G, f)$

in BG. By Conner and Floyd [10], p. 99, the elements $[S^{2n-1}/G, f]$ generate $\widetilde{\Omega}_*(BG)$ as an Ω_*-module. They hence also generate $\operatorname{Ker} \varepsilon^{BG}$ as an $\overline{SK}_*$-module. But S^{2n-1}/G fibres over $S^{2n-1}/S^1 = P_{n-1}\mathbb{C}$ with fibre $S^1/G \cong S^1$, so by Lemma (1.5), $[S^{2n-1}/G, f]$ is zero in $SK_*(BG)$, and hence certainly in $\operatorname{Ker} \varepsilon^{BG}$.

The case $G = \mathbb{Z}_2$ is rather more difficult, and we must first recall some facts on free involutions and bordism of $B\mathbb{Z}_2$.

Let $\pi : \widetilde{M} \longrightarrow M$ be a principal $\mathbb{Z}_2$-bundle, $T : \widetilde{M} \longrightarrow \widetilde{M}$ the covering transformation. Recall that a 1-codimensional submanifold $W \subset M$ is called a <u>characteristic submanifold</u> if $\widetilde{W} = \pi^{-1}(W)$ is the boundary ∂A of a compact submanifold A of $\widetilde{M}$ satisfying: $A \cup TA = \widetilde{M}$ and $A \cap TA = \widetilde{W}$. It is easy to see that such a W exists and is unique up to non-oriented bordism (for instance, by showing that W is a transversal self-intersection of the zero-section of the real line bundle $E \longrightarrow M$ associated with $\widetilde{M} \longrightarrow M$). The characteristic submanifold in fact defines a map

$$w : \Omega_n(B\mathbb{Z}_2) \longrightarrow \mathfrak{N}_{n-1}.$$

By Burdick [6] (see also Hirzebruch and Jänich [11]) the restriction

$$\widetilde{\Omega}_n(B\mathbb{Z}_2) \longrightarrow \mathfrak{N}_{n-1}$$

is an isomorphism whose inverse

$$i : \mathfrak{N}_{n-1} \longrightarrow \widetilde{\Omega}_n(B\mathbb{Z}_2)$$

is given as follows. For $[N] \in \mathfrak{N}_{n-1}$ let $E \longrightarrow N$ be the line bundle associated with the orientation covering $\widetilde{N} \longrightarrow N$, and S the sphere bundle of the Whitney sum $E \oplus 1$ of E with a trivial line bundle. S is oriented and has a free orientation preserving involution given by the antipodal map in the fibres S^1. The induced singular manifold $[S/\mathbb{Z}_2, f]$ in $B\mathbb{Z}_2$ represents $i[N]$.

Observe that this geometric description of i is compatible with cutting and pasting, so we have an induced map

$$i' : \overline{SK}{}^0_{n-1} \longrightarrow \operatorname{Ker} \varepsilon_n^{B\mathbb{Z}_2}.$$

By (2.7) above, $\widetilde{\Omega}_n(B\mathbb{Z}_2) \longrightarrow \operatorname{Ker} \varepsilon_n^{B\mathbb{Z}_2}$ is surjective, so the commutative square

$$\begin{tikzcd}
\mathfrak{N}_{n-1} \arrow[r] \arrow[d, "i", "\cong"'] & \overline{SK}{}^0_{n-1} \arrow[d, "i'"] \\
\widetilde{\Omega}_n(B\mathbb{Z}_2) \arrow[r] & \operatorname{Ker} \varepsilon_n^{B\mathbb{Z}_2}
\end{tikzcd}$$

shows that i' is surjective. Thus for n even it follows that $\operatorname{Ker} \varepsilon_n^{B\mathbb{Z}_2} = 0$, since $\overline{SK}{}^0_{n-1} = 0$ (Theorem (1.3b)).

We can hence assume n is odd. Then by (1.3b) the diagram becomes

$$\begin{tikzcd}
\mathfrak{N}_{n-1} \arrow[r, "\bar{e}"] \arrow[d, "i", "\cong"'] & \mathbb{Z}_2 \arrow[d, "i'"] \\
\widetilde{\Omega}_n(B\mathbb{Z}_2) \arrow[r] & \operatorname{Ker} \varepsilon_n^{B\mathbb{Z}_2}
\end{tikzcd} \qquad (n \text{ odd})$$

where $\bar{e}$ is euler characteristic modulo 2. Since i' is surjective, we must only show that it maps $1 \in \mathbb{Z}_2$ onto zero.

Let $a : P_j\mathbb{R} \longrightarrow B\mathbb{Z}_2$ $(j = 1,3)$ be the classifying map for the double covering $S^j \longrightarrow P_j\mathbb{R}$. Then for $k = [n/4]$ and $j = n-4k = 1$ or 3, we have that $[P_{2k}\mathbb{C} \times P_j\mathbb{R}, * \times a]$ represents an element of $\widetilde{\Omega}_n(B\mathbb{Z}_2)$. Since $i^{-1} = w$ is given by taking a characteristic submanifold, $\bar{e}i^{-1}[P_{2k}\mathbb{C} \times P_j\mathbb{R}, * \times a] = \bar{e}[P_{2k}\mathbb{C} \times P_{j-1}\mathbb{R}] = 1 \in \mathbb{Z}_2$. On the other hand, $P_j\mathbb{R}$, and hence also $P_{2k}\mathbb{C} \times P_j\mathbb{R}$ fibres over S^2 for $j = 3$ and over S^1 for $j = 1$, so by Lemma (1.5) ii) we have that

$[P_{2k}\mathbb{C} \times P_j\mathbb{R}, * \times a] = 0$ in $\overline{SK}_*(B\mathbb{Z}_2)$, and hence certainly also in Ker ε^{BG}. By the commutativity of the diagram it follows that $i'(1) = 0$, as was to be proved.

Q.E.D.

To close the discussion of SK of bundles in the oriented case we mention some isolated results in low dimensions. In dimensions 0 and 1 everything is trivial.

THEOREM (2.8): i) If G is a Lie group with G/G_0 abelian then Ker $\varepsilon_2^{BG} = 0$, i.e., $SK_2(BG) = SK_2 = \mathbb{Z}$.

ii) If G is connected then also Ker $\varepsilon_3^{BG} = 0$, so $SK_3(BG) = SK_3 = 0$.

Proof: i) BG has fundamental group $\pi_1(BG) = \pi_0(G) = G/G_0$, which is hence abelian. We shall in fact show more than required, namely

$$\overline{SK}_2(X) = 0$$

for any space X with abelian fundamental group.

Let (F_n, f) be a singular 2-manifold in X, where F_n is the oriented surface of genus n. We can write F_n as $F_n = F_{n-1} \# (S' \times S')$. Let $S^1 \subset F_n$ be the circle along which the connected sum operation $\#$ was carried out. S^1 represents the zero homology class in $H_1(F_n)$, so $f(S^1)$ represents zero in $H_1(X) = \pi_1(X)$. Thus $f(S^1)$ is null-homotopic in X and we can do surgery of type $(2,1)$ in X on this circle, reducing (F_n, f) to $(F_{n-1} + (S^1 \times S^1), g)$ for some g.

In this way one sees that any oriented singular 2-manifold in X is cobordant to a sum of singular tori in X, and hence equal to zero in $\overline{SK}_2(X)$ by Theorem (1.2). Thus i) is proved.

ii) We again prove more than required, namely

$$\overline{SK}_3(X) = 0$$

for any simply connected X.

Recall that any connected oriented 3-manifold M is bordant to S^3, and can be reduced to S^3 by surgeries of type $(2,2)$. If (M, f) is a connected singular 3-manifold in X, then restricted to each solid torus, f is null-homotopic, so we can do surgery in X to reduce (M, f) to (S^3, g) for some g. By Lemma (1.5) we deduce that $[M, f] = 0$ in $\overline{SK}_3(X)$.

Q.E.D.

Finally we give an example where ε^{BG} is not an isomorphism, not even modulo torsion. Let F be an orientable surface of genus ≥ 2. The universal cover of F is contractible so $F = B\pi_1(F)$.

THEOREM (2.9): If F is an orientable surface of genus ≥ 2 then Ker $\varepsilon_2^{B\pi_1(F)} = \overline{SK}_2(B\pi_1(F)) \cong \mathbb{Z}$.

Proof: The bordism spectral sequence shows for any CW-complex X that $\Omega_2(X) = H_2(X;\mathbb{Z})$. Since $B\pi_1(F) = F$ and $H_2(F,\mathbb{Z}) = \mathbb{Z}$, we must show that $F_2(F) = 0$ and the theorem then follows by Theorem (1.2). That is, we must show that any singular torus in F bounds.

Since $S^1 \times S^1$ and F are $K(\pi,1)$-spaces, the homotopy classes of maps $S^1 \times S^1 \longrightarrow F$ are classified by the set $\mathrm{Hom}(\mathbb{Z} \oplus \mathbb{Z}, \pi_1(F))$ (see for instance Mosher and Tangora [15], p. 3). But it is well known that any abelian subgroup of $\pi_1(F)$ is trivial or infinite cyclic, so any $f \in \mathrm{Hom}(\mathbb{Z} \oplus \mathbb{Z}, \pi_1(F))$ factors as

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\bar{f}} \mathbb{Z} \longrightarrow \pi_1(F),$$

where $\bar{f}$ is, without loss of generality, surjective. By a change of splitting of $S^1 \times S^1$ as a product if necessary, and hence a change of base in $\mathbb{Z} \oplus \mathbb{Z}$, we can assume that $\bar{f}$ is the projection p_1. The corresponding map $S^1 \times S^1 \longrightarrow F$ thus splits as

$$S^1 \times S^1 \xrightarrow{p_1} S^1 \longrightarrow F$$

and hence extends to the solid torus $S^1 \times D^2$. Q.E.D.

The above proof in fact shows that for any discrete group G, all of whose abelian subgroups are cyclic, $\overline{SK}_2(BG) = \Omega_2(BG) = H_2(BG;\mathbb{Z})$. The finite groups of this type are just the groups with periodic cohomology (see Cartan-Eilenberg [7]), which all have zero second homology and hence do not yield anything interesting here.

The Non-orientable Case.

In the non-orientable case, the analog of Lemma (2.1) of course still holds. The analog of Theorem (2.2) i) is trivial: Ker ε^X, being a subgroup of $\overline{SK}^0_*(X)$, is always a torsion group for any connected space X. We hence have:

Triviality (2.10): For any connected space X

$$SK^0_n(X) = SK^0_n \oplus 2\text{-Torsion},$$

where SK^0_n is $\mathbb{Z}$, given by euler characteristic, in even dimensions and zero otherwise.

THEOREM (2.11): The argumentation $\varepsilon^{BG} : SK^0_*(BG) \longrightarrow SK^0_*$ is an isomorphism for $G = (\mathbb{Z}_2)^k$, $O(k)$, $SO(k)$, $(S')^k$, $U(k)$, $SU(k)$, $Sp(k)$, and products of these groups.

The proof is by showing that one can generate $\mathfrak{N}_*(BG)$ as a $\mathfrak{N}_*$-module, and hence $\overline{SK}^0_*(BG)$ as an $\overline{SK}^0_*$-module by singular manifolds (M,f), where M is a product of real and complex projective spaces. Lemma (1.5) iii) and iv) then shows $\varepsilon : \overline{SK}^0_*(BG) \longrightarrow \overline{SK}^0_*$ is an isomorphism, so the theorem follows by (1.1).

It is convenient to work with vector bundles having G as structure group rather than with singular manifolds in BG. If $\omega = (n_1, \ldots, n_k)$ is a tuple of positive integers. Let ξ_ω be the bundle $\xi_{n_1} \times \ldots \times \xi_{n_k}$ over $P_\omega = P_{n_1}\mathbb{R} \times \ldots \times P_{n_k}\mathbb{R}$, where ξ_{n_i} is the canonical line bundle over $P_{n_i}\mathbb{R}$.

LEMMA (2.12): The following bundles represent a generating set of $\mathfrak{N}_*$-module $\mathfrak{N}_*(BG)$:

i) the bundles ξ_ω for $G = (\mathbb{Z}_2)^k$,

ii) the bundles ξ_ω with $n_1 \geq \ldots \geq n_k$ for $G = O(k)$,

iii) the bundles $\xi_\omega \oplus \det \xi_\omega$ with $n_1 \geq \ldots \geq n_k$ for $G = SO(k+1)$.

In cases i) and ii) the generating set is even a base.

Proof: The analogon of Theorem (2.3) holds in the non-oriented case (see for instance Conner and Floyd [10], Theorem 8.3). Hence we need only show that under the canonical map

$$\mu : \mathfrak{N}_*(BG) \longrightarrow H_*(BG;\mathbb{Z}_2)$$

the set in question goes over to a generating set or base of $H_*(BG;\mathbb{Z}_2)$.

The proof of i) is completely analogous to Lemma (2.4) and therefore also left as an exercise.

For ii) recall that $H^*(BO(k);\mathbb{Z}_2)$ is the polynomial ring $\mathbb{Z}_2[w_1,\dots,w_k]$ in the Stiefel-Whitney classes. In fact the inclusion $(\mathbb{Z}_2)^k \subset O(k)$ induces an inclusion $H^*(BO(k);\mathbb{Z}_2) \subset H^*(B(\mathbb{Z}_2)^k;\mathbb{Z}_2) = \mathbb{Z}_2[t_1,\dots,t_k]$ and w_i is the i-th elementary symmetric polynomial in $t_1,\dots,t_k$. For $\omega = (n_1,\dots,n_k)$ with $n_1 \geq \dots \geq n_k$ let s_ω be the smallest symmetric polynomial in the t_i containing the monomial $t_1^{n_1} \dots t_k^{n_k}$. The s_ω clearly form a base of $H^*(BO(k);\mathbb{Z}_2)$. On the other hand the homology class represented by the bundle ξ_ω is $(f_\omega)_*\sigma_\omega$, where $f_\omega : P_\omega \longrightarrow BO(k)$ is the classifying map for ξ_ω and σ_ω the fundamental $\mathbb{Z}_2$-homology class of P_ω. A trivial computation shows

$$<s_{\omega'},(f_\omega)_*\sigma_\omega> = <f_\omega^* s_{\omega'},\sigma_\omega> = \begin{cases} 1 , & \omega = \omega' \\ 0 , & \omega \neq \omega' ,\end{cases}$$

so the set $\{(f_\omega)_*\sigma_\omega\}$ is the basis of $H_*(BO(k);\mathbb{Z}_2)$ dual to $\{s_\omega\}$.

iii) Let γ^k be the universal $\mathbb{R}^k$-bundle over $BO(k)$. Then $\gamma^k \oplus \det \gamma^k$ is orientable, hence has a classifying map $g : BO(k) \longrightarrow BSO(k+1)$. Now $H^*(BSO(k+1);\mathbb{Z}_2) = \mathbb{Z}_2[w_2,\dots,w_{k+1}]$ and

$$g^* : H^*(BSO(k+1);\mathbb{Z}_2) \longrightarrow H^*(BO(k);\mathbb{Z}_2)$$

is given by $g^*(w_i) = w_i + w_i w_{i-1}$ for $i \leq k$ and $g^*(w_{k+1}) = w_1 w_k$. Since these elements are algebraically independent, g^* is injective. Thus g_* is surjective and case iii) follows from ii). Q.E.D.

Theorem (2.11) is hence proved for $G = (\mathbb{Z}_2)^k$, $O(k)$, $SO(k)$. If η_ω is the complex analogon of the bundle ξ_ω then Lemma (2.12) and its proof carry over to $G = (S^1_*)^k$, $U(k)$ and $SU(k)$ if one replaces ξ_ω by η_ω everywhere. Also a proof similar to the proof of iii) above shows that $\mathcal{N}_*(BSp(k))$ has a generating set represented by the bundles $\eta_\omega \oplus \eta_\omega$. This proves (2.11) for $(S^1)^k$, $U(k)$, $SU(k)$ and $Sp(k)$.

Finally if $\mathcal{N}_*(BG_1)$ is generated by singular manifolds (M_i,f_i) and $\mathcal{N}_*(BG_2)$ by singular manifolds (N_j,g_j), then $\mathcal{N}_*(B(G_1 \times G_2))$ is generated by the singular manifolds $(M_i \times N_j, f_i \times g_j)$. If the M_i and N_j are products of projective spaces, then so are the $M_i \times N_j$. Hence (2.11) also holds for products of the groups listed. Q.E.D.

CHAPTER 3: Equivariant SK

In this chapter G always denotes a compact Lie group and G-manifolds are manifolds with smooth G-actions. We are interested in invariants for equivariant cutting and pasting of closed G-manifolds. As usual, the Grothendieck group of n-dimensional G-manifolds modulo the relations given by cutting and pasting gives a universal such invariant. We denote this group by $SK^O_{G,n}$ (respectively $SK^{SO}_{G,n}$ in the oriented case).

The calculation of equivariant SK-groups is made difficult by the fact that we no longer have Theorem (1.1). In this chapter we calculate $SK^O_{G,n}$ up to 2-torsion. To state and prove the result it is convenient to have the language of "slice types" which we therefore recall briefly. For details see Jänich [12], §4.

If H is a closed subgroup of G and V a smooth H-manifold, then $G \times_H V$ denotes the fibre bundle over G/H with fibre V, associated to the principal H-bundle $G \longrightarrow G/H$. Recall that $G \times_H V$ is $G \times V$ factored by the equivalence relation: $(g,x) \sim (gh, h^{-1}x)$ for $h \in H$. With the G-action induced by left multiplication $G \times_H V$ is a G-manifold.

If V is a vector space and the H-action is given by a representation $\sigma : H \longrightarrow GL(V)$ then we also write $G \times_H \sigma$ for $G \times_H V$.

A <u>slice type</u> for G is a conjugacy class in G of pairs $(H,(\sigma))$, where H is a closed subgroup of G and (σ) an equivalence class of real representations of H. The slice type represented by (H,σ) is denoted by $[H,\sigma]$. One checks that $[H,\sigma] = [H',\sigma']$ if and only if $G \times_H \sigma$ and $G \times_{H'} \sigma'$ are isomorphic G-manifolds.

If M is a G-manifold and $x \in M$, then the <u>slice type at the point</u> x

is $[G_x,\sigma_x]$, where σ_x is the representation of the isotropy subgroup G_x normal to the orbit through x (the "slice representation"). Slice type determines the local structure of M completely, for the "slice theorem" states (see for instance Jänich [12], p. 3).

<u>THEOREM</u> (slice theorem): There is a G-invariant open neighborhood of x in M which is G-diffeomorphic to $G \times_{G_x} \sigma_x$.

There is a partial order on the set of all slice types for G given by: $[H,\sigma] \leq [U,\tau]$ means $[U,\tau]$ is a slice type of the G-manifold $G \times_H \sigma$. A family $\mathcal{F}$ of slice types for G will be called <u>permissible</u> if it contains with each $[H,\sigma]$ also each $[U,\tau]$ greater than $[H,\sigma]$. By the slice theorem, the family $\mathcal{F}(M)$ of all slice types of a G-manifold M is a permissible family.

If $\mathcal{F}$ is a permissible family of slice types, a G-manifold <u>of type</u> $\mathcal{F}$ is a G-manifold M all of whose slice types are in $\mathcal{F}$. That is $\mathcal{F}(M) \subset \mathcal{F}$. Denote by $SK^O(G,\mathcal{F})$ the SK-group resulting from cutting and pasting G-manifolds of type $\mathcal{F}$.

<u>Examples</u>. If $\mathcal{F} = \{[\{e\},\theta_n]\}$ where θ_n is the n-dimensional trivial representation, then $SK^O(G,\mathcal{F}) = SK^O_n(BG)$.
If $\mathcal{F}$ is the family of all n-dimensional slice types for G (by $\dim[H,\sigma]$ we mean $\dim(G \times_H \sigma)$), then $SK^O(G,\mathcal{F}) = SK^O_{G,n}$.

If M is a G-manifold and $[H,\sigma]$ a slice type, define

$$M_{[H,\sigma]} := \{x \in M \mid [G_x,\sigma_x] = [H,\sigma]\}.$$

Via the slice theorem $M_{[H,\sigma]} \subset M$ is given locally by $G \times_H \sigma_0 \subset G \times_H \sigma$, where

σ_0 is the trivial component of σ, so $M_{[H,\sigma]}$ is a smooth submanifold of M. Clearly $M_{[H,\sigma]}$ is a closed submanifold if $[H,\sigma]$ is a minimal element of $\mathcal{F}(M)$. Note that it also follows that any 1-codimensional G-invariant submanifold $N \subset M$ along which one can cut and paste M intersects each $M_{[H,\sigma]}$ transversally, as G, and hence certainly also H, acts trivially normal to N.

$M_{[H,\sigma]}$ fibres over $M_{[H,\sigma]}/G$ with fibre G/H. By the above comments it follows that $e_{[H,\sigma]}$, defined by

$$e_{[H,\sigma]}(M) := e(M_{[H,\sigma]}/G),$$

is an SK-invariant. It will turn out that the $e_{[H,\sigma]}$ give all equivariant SK-invariants up to 2-torsion. We first need a further definition.

Let $\pi : E \longrightarrow B$ be a differentiable G-vector-bundle over a differentiable manifold B. Let $[H,\sigma]$ be a slice type for G. We say $\pi : E \longrightarrow B$ <u>has type</u> $[H,\sigma]$ if just the points of the zero-section of E have slice type $[H,\sigma]$; that is, $E_{[H,\sigma]}$ is the zero-section $B \subset E$. The typical example of this is the normal bundle $\nu(M_{[H,\sigma]})$ of $M_{[H,\sigma]}$ in a G-manifold M.

Equivariant cutting and pasting of G-vector-bundles of type $[H,\sigma]$ whose bases are closed manifolds leads to an SK-group $SK^0[H,\sigma]$.

Now let $\mathcal{F}$ be an admissible family of slice types for G and $[H,\sigma] \in \mathcal{F}$ a minimal element in the partial ordering of $\mathcal{F}$. Then $\mathcal{F}' = \mathcal{F} - \{[H,\sigma]\}$ is also an admissible family and we have an obvious homomorphism

$$i : SK^0(G,\mathcal{F}') \longrightarrow SK^0(G,\mathcal{F}).$$

Furthermore, if M is a G-manifold of type $\mathcal{F}$ then the minimality of $[H,\sigma]$ implies that $M_{[H,\sigma]}$ is closed, so $M \longmapsto \nu(M_{[H,\sigma]})$ defines a homomorphism

$$n : SK^0(G,\mathcal{F}) \longrightarrow SK^0[H,\sigma].$$

<u>THEOREM</u> (3.1): If $\mathcal{F}$ is an admissible family of slice types, $[H,\sigma] \in \mathcal{F}$ a minimal element, and $\mathcal{F}' = \mathcal{F} - \{[H,\sigma]\}$, then the following sequence is split exact.

$$0 \longrightarrow SK^0(G,\mathcal{F}') \otimes \mathbb{Z}[\tfrac{1}{2}] \xrightarrow{i} SK^0(G,\mathcal{F}) \otimes \mathbb{Z}[\tfrac{1}{2}] \underset{d}{\overset{n}{\rightleftarrows}} SK^0[H,\sigma] \otimes \mathbb{Z}[\tfrac{1}{2}] \longrightarrow 0.$$

<u>Proof:</u> We first describe the splitting homomorphism d. Recall that for any manifold X the "double" $\mathcal{D}X$ is defined as $X \cup X$ pasted along the common boundary by the map $id : \partial X \longrightarrow \partial X$. If E is a vector bundle of type $[H,\sigma]$, define

$$d([E] \otimes 1) = [\mathcal{D}DE] \otimes \tfrac{1}{2},$$

where DE is the disc bundle of E. Clearly $n \circ d = id$.

It follows that n is surjective. Since it is clear that i is injective and $n \circ i = 0$, it only remains to show $Ker(n) \subset Im(i)$.

Suppose $n([M]) = 0$. Let N be a small tubular neighborhood of $M_{[H,\sigma]}$ in M, isomorphic to the normal bundle $\nu(M_{[H,\sigma]})$ as a G-manifold. Since $n([M]) = 0$, certainly $d \circ n([M]) = 0$, that is $[\mathcal{D}\bar{N}] = 0$. But by cutting and pasting one has

$$\begin{aligned} 2[M] &= [\mathcal{D}(M-N)] + [\mathcal{D}\bar{N}] \\ &= [\mathcal{D}(M-N)] \end{aligned}$$

in $SK^0(G,\mathcal{F})$, and the right hand side is clearly in $Im(i)$. Q.E.D.

LEMMA (3.2): Assigning to a G-vector-bundle $E \longrightarrow B$ of type $[H,\sigma]$ the SK-class $[B/G]$ defines an isomorphism

$$SK^0[H,\sigma] \otimes \mathbb{Z}[\tfrac{1}{2}] \longrightarrow SK^0_p \otimes \mathbb{Z}[\tfrac{1}{2}],$$

where p is the dimension of the trivial component of σ.

Proof: Write $\sigma = \sigma_0 \oplus \sigma$, where σ_0 is the trivial component of σ. The composite map $E \longrightarrow B \longrightarrow B/G$ identifies E as a fibre bundle over B/G with fibre $G \times_H \sigma_1$ and structure group $\Gamma(\sigma_1) = \mathrm{Aut}_G(G \times_H \sigma_1)$. Since $\dim(B/G) = \dim(\sigma_0) = p$, we hence have

$$SK^0[H,\sigma] = SK^0_p(B\Gamma(\sigma_1))$$

so the lemma follows from (2.10).

Remark: It is not hard to calculate the structure group $\Gamma(\sigma_1)$ explicitly. Since H is compact we can assume $\sigma_1 : H \longrightarrow O(k)$ is an orthogonal representation, and then

$$\Gamma(\sigma_1) = N_{GXO(k)}(\overline{H})/\overline{H},$$

where $\overline{H} = \{(h,\sigma_1(h)) \in G \times O(k) | h \in H\}$.

Now by Theorem (1.3) it follows that $SK^0[H,\sigma] \otimes \mathbb{Z}[\frac{1}{2}]$ is zero if $p = \dim(\sigma_0)$ is odd and is $\mathbb{Z}[\frac{1}{2}]$, generated by the bundle $E_\sigma = P_p \mathbb{R} \times (G \times_H \sigma_1)$, if p is even. Thus by Theorem (3.1) and a trivial induction, $SK^0(G,\mathcal{F}) \otimes \mathbb{Z}[\frac{1}{2}]$ is the free $\mathbb{Z}[\frac{1}{2}]$-module with basis $\{[\mathcal{D}DE_\sigma] | [H,\sigma] \in \mathcal{F},\ \dim(\sigma_0) \text{ even}\}$.

COROLLARY (3.3): The SK-invariants $e_{[H,\sigma]}$ with $[H,\sigma] \in \mathcal{F}$ and $\dim(\sigma_0)$ even define an isomorphism

$$(e_{[H,\sigma]} \otimes \mathrm{id}) : SK^0(G,\mathcal{F}) \otimes \mathbb{Z}[\tfrac{1}{2}] \longrightarrow \coprod_{[H,\sigma]} \mathbb{Z}[\tfrac{1}{2}]$$

where the sum is over all $[H,\sigma] \in \mathcal{F}$ with $\dim(\sigma_0)$ even.

Proof: Let $[H^1,\sigma^1],[H^2,\sigma^2],\ldots$ be those $[H,\sigma]$ in $\mathcal{F}$ with even dimensional trivial component, with indexing so chosen that $[H^i,\sigma^i] \leq [H^j,\sigma^j]$ implies $i \leq j$. Order the basis of $SK^0(G,\mathcal{F}) \otimes \mathbb{Z}[\frac{1}{2}]$ mentioned above correspondingly. Now

$$e_{[H^i,\sigma^i]}(\mathcal{D}DE_{\sigma^i}) = \begin{cases} 2 & \text{if } i = j \\ \\ 0 & \text{if } i < j\,. \end{cases}$$

That is, the matrix of the map $(e_{[H,\sigma]} \otimes \mathrm{id})$ with respect to the above basis is triangular with invertible diagonal entries, so the map is an isomorphism. Q.E.D.

The above corollary can also be formulated that the map

$$E = (e_{[H,\sigma]}) : SK^0(G,\mathcal{F}) \longrightarrow \coprod_{[H,\sigma]} \mathbb{Z}$$

(as usual $[H,\sigma] \in \mathcal{F}$ with $\dim(\sigma_0)$ even) is a modulo 2-torsion isomorphism. That is $\mathrm{Ker}(E)$ and $\mathrm{CoKer}(E)$ are 2-groups. Thus $\mathrm{Ker}(E)$ is the torsion subgroup of $SK^0(G,\mathcal{F})$ and its calculation would complete the calculation of $SK^0(G,\mathcal{F})$. The calculation of $\mathrm{CoKer}(E)$ is equivalent to finding the relations between the $e_{[H,\sigma]}$ and would be in a sense a general Smith type theorem. Note that the $e_{[H,\sigma]}$ with $\dim(\sigma_0)$ odd are not necessarily zero. However, they are linear combinations of the $e_{[U,\tau]}$ with $[U,\tau] \geq [H,\sigma]$ and $\dim(\tau_0)$ even.

Jänich [14] and Rowlett [17] have some further results on equivariant SK for $G = \mathbb{Z}_2$. They both use different SK-relations and it turns out that what they are actually calculating is respectively $SK^{SO}_{\mathbb{Z}_2}/J$ and $SK^{O}_{\mathbb{Z}_2}/J$, where J is the ideal generated by manifolds of the form $\mathcal{D}X$, with X an oriented resp. arbitrary compact $\mathbb{Z}_2$-manifold. Rowlett obtains complete results, however Jänich's result is not quite complete and is only modulo torsion.

Using these results, it is probably not too hard to obtain a complete calculation of $SK_{\mathbb{Z}_2}$ in both the oriented and unoriented case, using the following two remarks:

Remark (3.4): $\overline{SK_G}$ is a quotient of SK_G/J.

Remark (3.5): Since for finite G, bordism of G-manifolds is given by G-equivariant surgery, the analog of Theorem (1.1) holds with I_n replaced by the subgroup of $SK_{n,G}$ generated by all effective linear G-actions on S^n.

CHAPTER 4: Controllable Invariants

In this chapter we discuss a generalization of the concept of SK-invariant, due to K. Jänich (unpublished).

Let $M_1 = N \cup_\varphi - N'$ and $M_2 = N \cup_\psi - N'$ be two closed oriented manifolds obtained from each other by cutting and pasting via the diffeomorphisms $\varphi, \psi : \partial N \longrightarrow \partial N'$. An invariant λ for closed oriented manifolds (as usual additive with respect to disjoint union) is called SK-controllable if $\lambda(N \cup_\varphi - N') - \lambda(N \cup_\psi - N')$ only depends on the diffeomorphisms $\varphi, \psi : \partial N \longrightarrow \partial N'$ and not on the choice of the manifolds N and N'. We then speak briefly of an SKK-invariant (SK-Kontrollierbar).

Clearly any SK-invariant is an SKK-invariant, and the SKK-invariant λ is an SK-invariant if and only if the "correction term"

$$\lambda(\varphi,\psi) := \lambda(N\cup_\varphi -N') - \lambda(N\cup_\psi -N')$$

is always zero.

The above definition is obviously equivalent to the following: for any oriented manifolds N_1, N_1', N_2, N_2' with $\partial N_1 = \partial N_2$ and $\partial N_1' = \partial N_2'$ and any orientation preserving diffeomorphisms $\varphi, \psi : \partial N_1 \longrightarrow \partial N_1'$ one has

$$\lambda(N_1\cup_\varphi -N_1') - \lambda(N_1\cup_\psi -N_1') = \lambda(N_2\cup_\varphi -N_2') - \lambda(N_2\cup_\psi -N_2').$$

This makes it clear how one can define a "universal" SKK-group SKK_n^{SO}, which gives the universal SKK-invariant for closed oriented n-manifolds: factor the semigroup $\mathfrak{M}_n^{SO}$ of diffeomorphism classes of closed oriented n-manifolds by all relations of the form

$$N_1 U_\varphi -N_1' + N_2 U_\psi -N_2' = N_2 U_\varphi -N_2' + N_1 U_\psi -N_1'$$

and then take the Grothendieck group of the result. One can make precisely the same definitions in the non-oriented case to obtain a graded group SKK_*^O. As usual, we drop the superscript in the oriented case and just write SKK_* for SKK_*^{SO}.

THEOREM (4.1): a) Assigning to an oriented manifold M its bordism class in Ω_* is an SKK-invariant and hence defines a surjective homomorphism $SKK_* \longrightarrow \Omega_*$.

b) The analogous statement holds in the non-oriented case.

Proof: This is just Lemma (1.9) carried over to the (un)-oriented category, with $X = pt$. Q.E.D.

K. Jänich (unpublished) had shown that for oriented manifolds bordism class and euler characteristic give all SKK-invariants up to torsion. It turns out that there can be further torsion invariants; the following theorem gives a complete description of SKK-invariants.

THEOREM (4.2): Let $I_n \subset SKK_n$ (resp. $I_n^O \subset SKK_n^O$) be the cyclic subgroup generated by $[S^n]$. Then the sequences

$$0 \longrightarrow I_n \longrightarrow SKK_n \longrightarrow \Omega_n \longrightarrow 0$$

$$0 \longrightarrow I_n^O \longrightarrow SKK_n^O \longrightarrow \mathfrak{N}_n \longrightarrow 0$$

are exact. Furthermore I_n (I_n^O) is the quotient of $\mathbb{Z}$ by the subgroup generated by euler characteristics of closed (n+1)-dimensional (un)-oriented manifolds, that is:

$$I_n \cong \begin{cases} \mathbb{Z} & n \equiv 0 \pmod 2 \\ \mathbb{Z}_2 & n \equiv 1 \pmod 4 \\ 0 & n \equiv 3 \pmod 4 \end{cases}$$

$$I_n^O \cong \begin{cases} \mathbb{Z} & n \equiv 0 \pmod 2 \\ 0 & n \equiv 1 \pmod 2). \end{cases}$$

Proof: We shall first prove the exactness of the above sequences. Suppose we have two oriented manifolds M_1^n and M_2^n which are cobordant. We must show that in SKK_n they differ by a multiple of $[S^n]$. We shall in fact prove more, namely

LEMMA (4.3): Let Y be an (un)-oriented bordism between M_1^n and M_2^n. Then in SKK_n (resp. SKK_n^O)

$$[M_1] = [M_2] - (e(Y) - e(M_1))[S^n].$$

We have proved this lemma for SK_n as Corollary (1.8), so we need only show that wherever equality in SK_n occurred in the proof of (1.8) it can be replaced by equality in SKK_n.

Let N and N' be oriented manifolds with $\partial N = \partial N' = 2P$, the disjoint union of two copies of a manifold P, and let $t : 2P \longrightarrow 2P$ be the involution exchanging these two copies. Suppose further that P bounds an oriented manifold Q. Then by definition of SKK_n

$$[NU_{id}-N'] + [2QU_t-2Q] = [2QU_{id}-2Q] + [NU_t-N'],$$

so since $2QU_t-2Q = 2QU_{id}-2Q$, we have

$$[NU_{id}-N'] = [NU_t-N'] \quad \text{in} \quad SKK_n.$$

But in the proof of (1.8) only cutting and pasting of the above type occurred (namely the cutting and pasting (A) involved in surgery in the proof of Lemma (1.6)), so the proof can be carried over to the SKK-case, as desired. The same arguments hold in the unoriented case. Q.E.D.

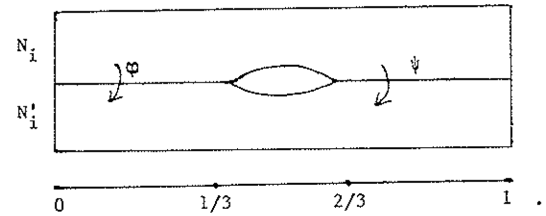
To complete the proof of Theorem (4.2) we must calculate the order of $[S^n]$ in SKK_n (resp. SKK_n^0). For n even, euler characteristic is an SKK-invariant which is non-zero on the generator $[S^n]$ of I_n (resp. I_n^0), showing that $I_n \cong I_n^0 \cong \mathbb{Z}$. We may hence assume n is odd, say $n = 2m-1$.

Observe first that Lemma (4.3) with $M_1 = M_2 = \emptyset$ and $Y = S^{2m}$ shows that $[S^{2m-1}]$ has order at most 2 in SKK_{2m-1} and SKK^0_{2m-1}. Furthermore, if M^{2n} is a closed manifold of odd euler characteristic, then Lemma (4.3) with $M_1 = M_2 = \emptyset$ and $Y = M^{2m}$ now shows that $[S^{2m-1}] = 0$; we can take $M = P_{2m}\mathbb{R}$ in the unoriented case, and for m even we can take $M = P_n\mathbb{C}$ in the oriented case. It hence only remains to show in the oriented case that $[S^{2m-1}] \neq 0$ in SKK_{2m-1} for m odd. We shall prove this by showing that $[S^{2m-1}] = 0$ implies the existence of a closed manifold M^{2m} of odd euler characteristic, which is impossible in the orientable category if m is odd.

Suppose therefore that $[S^{2m-1}] = 0$. By definition of SKK_{2m-1} this means that there exist orientable manifolds N_i and N_i' $(i = 1,2)$ with $\partial N_1 = \partial N_2$ and $\partial N_1' = \partial N_2'$, and diffeomorphisms $\varphi,\psi : \partial N_1 \longrightarrow \partial N_1'$, such that

$$S^{2m-1} + (N_1U_\varphi-N_1') + (N_2U_\psi-N_2') = (N_2U_\varphi-N_2') + (N_1U_\psi-N_1').$$

For $i = 1,2$, let Y_i be the union of $N_i \times [0,1]$ and $N_i' \times [0,1]$ with the following identifications: for $x \in \partial N_i$ identify $(x,t) \in \partial N_i \times [0,1/3]$ with $(\varphi(x),t) \in \partial N_i' \times [0,1/3]$ and $(x,t) \in \partial N_i \times [2/3,1]$ with $(\psi(x),t) \in \partial N_i' \times [2/3,1]$.



As in the proof of (1.9), after smoothing, $\partial Y_i = (N_iU_\varphi-N_i') + -(\partial N_i)_{\varphi\psi^{-1}} + -(N_iU_\psi-N_i')$, so by using the above equation it follows that the disjoint union $Y_2 + -Y_1$ has boundary

$$\partial(Y_2 + -Y_1) = S^{2m-1} + (N_1U_\varphi-N_1') + (N_2U_\psi-N_2') + (\partial N_1)_{\varphi\psi^{-1}} +$$
$$-((N_1U_\varphi-N_1') + (N_2U_\psi-N_2') + (\partial N_1)_{\varphi\psi^{-1}}).$$

Thus by pasting boundary components of $Y_2 + -Y_1 + D^{2m}$ pairwise together we get a closed manifold M^{2m}, whose euler characteristic is easily calculated to be $1 - 2e(\partial N_1)$. Since this is odd, the proof of Theorem (4.2) is completed. Q.E.D.

<u>Remark:</u> For unoriented manifolds, Theorem (4.2) shows that bordism class and euler characteristic give all SKK-invariants.

For orientable manifolds one can show that Kervaire semi-characteristic, defined by

$$k(M^{4k+1}) = \sum_{i=0}^{2k} b_i(M) \pmod 2,$$

where the $b_i(M)$ are the betti numbers, is an SKK-invariant $SKK_{4k+1} \longrightarrow \mathbb{Z}_2$, which splits the sequence (4.2). So bordism class, euler characteristic, and Kervaire semi-characteristic in dimensions $4k+1$ give all SKK-invariants for orientable manifolds.

We sketch a proof of the SKK-invariance of the Kervaire semi-characteristic k. For any oriented manifold Y^{2m} an elementary homological argument using Poincaré duality shows that

$$k(\partial Y) = e(Y) - \tau(Y) \pmod 2.$$

Assume m odd, say $m = 2k+1$, and apply this equation to the manifold Y used in the proof of (1.9). This gives

$$k(M_1 \cup_\varphi -M_2) - k(M_1 \cup_\psi -M_2) - k((\partial M_1)_{\psi\varphi^{-1}}) \equiv -e(\partial M_1) \pmod 2$$

which shows that k is an SKK-invariant with correction term $k(\varphi,\psi) = k(N_{\psi\varphi^{-1}}) - e(N) \pmod 2$. A simple homological calculation puts this in the neater form

$$k(\varphi,\psi) = \operatorname{rank}((\psi\varphi^{-1})_* - \mathrm{id}) \pmod 2,$$

where, since other dimensions pair off, we need only consider the middle dimension $(\psi\varphi^{-1})_* : H_{2k}(N) \longrightarrow H_{2k}(N)$.

Bordism with Vector Fields.

Reinhart [16] introduced bordism with vector fields in order to make euler characteristic into a bordism invariant.

Let M_1 and M_2 be closed (oriented) manifolds. A vector-field bordism between M_1 and M_2 is a usual (oriented) bordism N between M_1 and M_2 together with a non-singular vector field on N which is the inward normal on M_1 and the outward normal on M_2.

It is well known (Reinhart, loc. cit.) that if N is connected, such a vector field exists on N if and only if $e(M_1) = e(M_2) = e(N)$.

THEOREM (4.4): Two (oriented) manifolds M_1 and M_2 are vector field cobordant if and only if they are equivalent in SKK_*^0 (resp. SKK_*). Thus one can identify SKK_* with Reinhart's vector field bordism groups.

We prove only the oriented version, because the same arguments hold in the unoriented case.

We must show that two oriented manifolds M_1^n and M_2^n represent the same class in SKK_n if and only if there exists an oriented bordism N between them with

$$e(M_1) = e(M_2) = e(N).$$

The sufficiency of this condition is immediate from (4.3), so it remains to prove the necessity. Suppose therefore that $[M_1] = [M_2]$ in SKK_n. Since euler characteristic is an SKK-invariant, $e(M_1) = e(M_2)$. Also the bordism classes are equal, so we can find a bordism Y between M_1 and M_2. Lemma (4.3) implies that $(e(Y) - e(M_1))[S^n] = 0$, so for n even Theorem (4.2) shows that $e(Y) = e(M_1)$, and we can take $N = Y$ and are finished. For n of the form $4k+1$ Theorem (4.2) shows that $e(M_1) - e(Y)$ is even, so for arbitrary odd n we can certainly find a closed manifold M^{n+1} with $e(M^{n+1}) = e(M_1) - e(Y)$. In this case, the connected sum of Y and M^{n+1} gives a bordism N of M_1 and M_2 with $e(N) = e(Y) + e(M^{n+1}) = e(M_1)$, completing the proof. Q.E.D.

Tangential Characteristic Numbers.

Jänich (unpublished) has shown for oriented manifolds that the index of an ellipic operator is an SKK-invariant. Here, a version of this theorem will be proved in a more general setting.

Let $\bar{\gamma}_n$ be the universal bundle over $BSO(n)$ and γ_n the universal bundle over $BO(n)$. By $D\bar{\gamma}_n$ and $S\bar{\gamma}_n$ we denote the corresponding disc bundle and its boundary sphere bundle.

Let M be a closed oriented n-manifold. The classifying map for the tangent bundle of M induces a map

$$(tM,\partial tM) \longrightarrow (D\bar{\gamma}_n, S\bar{\gamma}_n),$$

where tM is the tangent disc bundle of M. Since tM has a natural stable almost complex structure, we obtain an element

$$\chi(M) \in \Omega^U_{2n}(D\bar{\gamma}_n, S\bar{\gamma}_n).$$

In the unoriented case we obtain an element

$$\chi(M) \in \Omega^U_{2n}(D\gamma_n, S\gamma_n).$$

LEMMA (4.5): χ defines a homomorphism

$$\chi : SKK_n \longrightarrow \Omega^U_{2n}(D\bar{\gamma}_n, S\bar{\gamma}_n)$$

respectively

$$\chi : SKK^O_n \longrightarrow \Omega^U_{2n}(D\gamma_n, S\gamma_n).$$

Proof: Suppose $[M^n] = 0$ in SKK_n; we must show that $\chi(M) = 0$. By Theorem (4.4) we can find an oriented manifold Y with $\partial Y = M$ and a non-singular vector field ξ on Y which is the inward normal on M. Let $t'Y$ be the disc bundle of the bundle obtained by splitting the line bundle corresponding to ξ off from the tangent bundle of Y, and $f : (t'Y,\partial t'Y) \longrightarrow (D\bar{\gamma}_n, S\bar{\gamma}_n)$ its classifying map. f is clearly a zero bordism of $\chi(M)$. The argument also holds in the unoriented case. Q.E.D.

Now let h_* and h^* be corresponding homology and cohomology theories for which stably almost complex manifolds are orientable. Then for any element $x \in h^*(D\bar{\gamma}_n, S\bar{\gamma}_n)$ (respectively $x \in h^*(D\gamma_n, S\gamma_n)$) we can consider the corresponding characteristic number of a singular stably almost complex manifold. To be precise we consider the homomorphism

$$\Omega^U_*(D\bar{\gamma}_n, S\bar{\gamma}_n) \otimes h^*(D\bar{\gamma}_n, S\bar{\gamma}_n) \longrightarrow h_*(pt)$$

$$[N,g] \otimes x \longmapsto \langle g^*x, [N,\partial N]_h \rangle,$$

where $[N,\partial N]_h$ denotes the h_*-orientation class of N.

Definition: If M is a closed (un)-oriented manifold, the characteristic numbers of $\chi(M) \in \Omega^U_{2n}(D\bar{\gamma}_n, S\bar{\gamma}_n)$ (resp. $\in \Omega^U_{2n}(D\gamma_n, S\gamma_n)$) are called tangential characteristic numbers of M.

COROLLARY (4.6): Tangential characteristic numbers are SKK-invariant.

Example: As h^*, h_* we can choose (complex) K-theory. If M is a manifold, then an element $x \in K^*(tM, \partial tM)$ can be considered as a symbol of a (pseudo)-differential operator. $\langle x, [tM, \partial tM]_K \rangle$ is then the index of this symbol. An element in $K^*(D\bar{\gamma}_n, S\bar{\gamma}_n)$ (resp. $K^*(D\gamma_n, S\gamma_n)$) can thus be considered as a "universal differential operator" which is defined on all n-dimensional (un)-oriented manifolds. The index of such a "universal operator" is hence an SKK-invariant.

CHAPTER 5: Other SK Concepts

Other SK concepts have been considered in the literature. In this chapter we show how they reduce to the concept of SK used here. For convenience we work in the oriented category; however, the discussion is also valid for manifolds with other structure, e.g., singular manifolds in a space X, manifolds with (B,f)-structure, manifolds with a group action, etc.

A cutting and pasting "relation" will always mean an equivalence relation $\sim$ on the class of manifolds, compatible with disjoint union $+$, and "cancellative." That is, for manifolds M, M', N we require

$$M \sim M' \iff N+M \sim N+M'.$$

Actually, to make our discussion valid also in the equivariant case it is convenient to define a further cutting and pasting relation by adding to the SK-relation that the double $\mathcal{D}M = M \cup_{id} -M$ of any compact manifold be equivalent to zero. Call this relation $\widetilde{SK}$. That is, for the corresponding graded groups,

$$\widetilde{SK}_* = SK_*/J$$

where J is the subgroup generated in SK_* by doubles of closed manifolds.

LEMMA (5.1): In the non-equivariant case $\widetilde{SK} = \overline{SK}$.

Proof: In fact we show this holds for any category of manifolds for which a suitable analog of Theorem (1.1) holds, i.e., bordism is given by surgery, and spheres are doubles of discs.

$\overline{SK}_* = SK_*/I$, where I is the subgroup generated by manifolds which bound, and hence contains J. But by (1.1) I is already generated by spheres, and hence contained in J. Q.E.D.

We consider the relation used by Jänich [14]. This relation is generated by setting any manifold of the form

(1) $$M_0 \cup_{\varphi_0} -M_1 + M_1 \cup_{\varphi_1} -M_2 + M_2 \cup_{\varphi_2} -M_0$$

equivalent to zero. Here $-M_i$ means M_i with reversed orientation and $\varphi_i : \partial M_i \longrightarrow \partial M_{i+1}$ (indices modulo 3) are diffeomorphisms.

THEOREM (5.2): Jänich's relation is the same as $\widetilde{SK}$, and hence the same as $\overline{SK}$ in the non-equivariant case.

Proof: By cutting and pasting the above manifold (1) one obtains the union of doubles,

$$M_0 \cup -M_0 + M_1 \cup -M_1 + M_2 \cup -M_2 ,$$

so $\widetilde{SK}$ implies Jänich's relation. On the other hand, putting $M_0 = M$, $M_1 = M_2 = \emptyset$, in (1) shows that $M + (-M) \sim 0$. Now taking $M_0 = M_1 = M_2$, (1) shows that $\mathscr{D}M_0 + \mathscr{D}M_0 + (-\mathscr{D}M_0) \sim 0$, so $\mathscr{D}M_0 \sim 0$. Finally, $M_0 = M_2$ gives $(M_0 \cup_{\varphi_0} - M_1) + (M_1 \cup_{\varphi_1} - M_0) + \mathscr{D}M_0 \sim 0$, whence $M_0 \cup_{\varphi_0} - M_1 \sim M_0 \cup_{\varphi_1} - M_1$. Hence Jänich's relation implies $\widetilde{SK}$. Q.E.D.

If one is interested also in compact manifolds with boundary, the most natural cutting and pasting relation seems to be the one generated by the relations

(2) $$M_0 \cup_{\varphi} - M_1 \sim M_0 + (-M_1),$$

where φ pastes boundary components of M_0 to boundary components of M_1. Call the corresponding graded group A_* (the Grothendieck group of compact manifolds modulo these relations). This is the universal group for "additive" invariants of manifolds.

Clearly, for closed manifolds the above relations only generate the usual SK-relations, so the subgroup of A_* generated by closed manifolds is just SK_*. Now let B_* be the Grothendieck group of closed manifolds which bound, subject only to the relations $M + (-M) = 0$. The torsion subgroup of B_* is thus 2-torsion, generated by bounding manifolds which possess orientation reversing diffeomorphisms. There is an epimorphism $\partial : A_* \longrightarrow B_{*-1}$ given by taking boundaries of manifolds. The following theorem is trivial.

THEOREM (5.3): The sequence

$$0 \longrightarrow SK_* \longrightarrow A_* \longrightarrow B_{*-1} \longrightarrow 0$$

is exact.

Thus "additive" invariants for compact manifolds reduce to the diffeomorphism types of their boundaries together with SK-invariants for closed manifolds. Observe that the above sequence does not split for n even, since $[S^n] = 2[D^n]$ in A_*, but $[S^n]$ is an irreducible element of $SK_* = \operatorname{Ker} \partial$.

Theorem (5.3) is due to Rowlett [17]. Actually Rowlett considers a slightly different relation, namely

(3) $$M_0 \cup_\varphi -M_1 + (-M_0) + M_1 \sim 0.$$

Taking $M_1 = \emptyset$ this implies $M_0 \sim -(-M_0)$, so in particular relation (2) follows, as well as the relation

(4) $$M_0 \cup -M_0 \sim 0,$$

that is, doubles are equivalent to zero. Conversely (2) and (4) clearly imply $M_0 \sim -(-M_0)$, and hence imply (3). Thus Rowlett's relation (3) leads to the same results as relation (2) except that SK_* must be replaced by $\widetilde{SK}_*$.

We now return to a comment of Chapter 1. As remarked in Chapter 1, $SK_n(X)$ is actually <u>equal</u> to the semigroup of singular n-manifolds in X modulo SK-equivalence. To assure this, the definition of SK-equivalence in Chapter I was slightly unnaturally "stabilized" to make sure that it was cancellative. As recently remarked by Ed Miller, this is unnecessary, in fact we have:

<u>THEOREM</u> (5.4): Two closed non-empty oriented singular manifolds (M_1,f_1) and (M_2,f_2) in a connected space X are SK-equivalent and hence represent the same element of $SK_*(X)$ if and only if one is obtainable from the other by a sequence of cutting and pasting operations in X.

Of course the same holds in the unoriented category. To prove Theorem (5.4) let $\sim$ denote the "unstabilized" SK-relation; that is $(M_1,f_1) \sim (M_2,f_2)$ means that (M_2,f_2) results from (M_1,f_1) by a sequence of cutting and pasting

operations in X. It is clearly sufficient to show that the semigroup $\mathfrak{M}_n(X)/\sim$ of singular n-manifolds in X modulo this relation is already a group, and hence equal to $SK_n(X)$.

Firstly, this semigroup has a zero, given by the class of $S^1 \times S^{n-1}$. Indeed, we can cut $S^1 \times S^{n-1}$ along S^{n-1} to get $I \times S^{n-1}$. Now given any $(M,f) \in \mathfrak{M}_n(X)$, we can cut a small disc D^n from M, paste $I \times S^{n-1}$ to this disc as a collar, and paste the result back into M, showing that $(M,f) + (S^1 \times S^{n-1}) \sim (M,f)$.

Secondly, the class of S^n has an inverse in this semigroup. Namely let P be the "sphere with two handles" obtained by removing two discs from $S^1 \times S^{n-1}$ and pasting the resulting two boundary components S^{n-1} together. By reversing this construction, clearly $P + S^n \sim S^1 \times S^{n-1}$.

We now have all we need to repeat the proof of Corollary (1.8) and show that if (M_1,f_1) is bordant to (M_2,f_2) in X by a bordism Y, then

$$[M_1,f_1] = [M_2,f_2] - (e(Y) - e(M_1))[S^n]$$

in $\mathfrak{M}_n(X)/\sim$. It follows that any element $[M,f]$ of $\mathfrak{M}_n(X)/\sim$ has an inverse, namely $[-M,f] - e(M)[S^n]$, so $\mathfrak{M}_n(X)/\sim$ is a group, as was to be shown.

<u>Remark</u>: The relation of SK-equivalence as given in Chapter 1 can be simplified in another direction, which is, however, less interesting. Namely, (M_1,f_1) and (M_2,f_2) are SK-equivalent if and only if there exists an (M,f) such that $(M_2,f_2) + (M,f)$ results from $(M_1,f_1) + (M,f)$ by a <u>single</u> cutting and pasting operation. We leave this as an easy exercise for the reader.

CHAPTER 6: Winkelnkemper's "Open Book Theorem"

This chapter was written after the rest of the notes were completed, and discusses some SK-consequences of Elmar Winkelnkemper's "open book theorem" [20]. Maybe the main consequence for SK is the theorem, which strongly supercedes Theorem (2.8) iii):

THEOREM (6.1): For any topological space X and all odd $n \neq 5$, $SK_n(X) = 0$. This is probably also true for $n = 5$.

Let us first recall Winkelnkemper's definition of an "open book." Let V be a manifold with $\partial V \neq \emptyset$ and $h : V \longrightarrow V$ a diffeomorphism with $h|\partial V = id$. Form the mapping torus V_h (see Chapter 1) which has $\partial V_h = S^1 \times \partial V$, and for each $x \in \partial V$ identify the points (t,x), $t \in S^1$, to obtain a closed manifold M called an open book. The fibres of the mapping torus are the "pages" and the image of $S^1 \times \partial V$ under the identification, which is a codimension 2 closed manifold diffeomorphic to ∂V is called the "binding." The binding is the boundary of each page.

In 1923, Alexander [1] proved: every orientable 3-manifold is an open book. Winkelnkemper has extended this to the following powerful structure theorem for manifolds:

THEOREM (6.2) (Open Book Theorem): a) Every orientable closed manifold of dimension $n = 2k+1 \neq 5$ has an open book decomposition.

b) A closed simply connected manifold M of dimension $n = 2k > 6$ has an open book decomposition if and only if $\tau(M) = 0$.

In fact in the simply connected case, $n > 6$, Winkelnkemper shows much more, namely, that the pages and binding can also be chosen simply connected with $H_i(V,\mathbf{Z}) = 0$ for $i > [\frac{n}{2}]$. The latter implies that $h_* : H_i(V,\mathbf{Z}) \longrightarrow H_i(V,\mathbf{Z})$ is the identity for $i < [\frac{n}{2}]$, and Winkelnkemper also gives necessary and sufficient conditions that one can choose it to be the identity also for $i = [\frac{n}{2}]$.

The application to SK is given by the following theorem. We first note a simple lemma:

LEMMA (6.3): Let M^n be a closed connected orientable manifold. Then the following four conditions are equivalent:

i) For any map $f : M \longrightarrow X$ of M into a space X, $[M,f] = 0$ in $\overline{SK}_n(X)$;

ii) $\tau(M) = 0$ and for any map $f : M \longrightarrow X$, $[M,f] = [M,*]$ in $SK_n(X)$;

iii) $[M,id] = 0$ in $\overline{SK}_n(M)$;

iv) $\tau(M) = 0$ and $[M,id] = [M,*]$ in $SK_n(M)$.

THEOREM (6.4): If M^n has an open book decomposition then each of the equivalent conditions of Lemma (6.3) holds.

Proofs: Lemma (6.3): The equivalences i) $\Longleftrightarrow$ ii) and iii) $\Longleftrightarrow$ iv) are clear by observing that $[M,f] = 0$ in $\overline{SK}_n(X)$ implies $[M,*] = 0$ in $\overline{SK}_n(X)$ and applying Theorems (1.1b) and (1.3b). Trivially i) $\Longrightarrow$ iii), and iii) $\Longrightarrow$ i) follows from the fact that $[M,f] \in \overline{SK}_n(X)$ is the image of $[M,id] \in \overline{SK}_n(M)$ under the map $\overline{SK}_n(M) \longrightarrow \overline{SK}_n(X)$ induced by f.

Theorem (6.4): Suppose M has an open book decomposition given by typical page V and diffeomorphism $h : V \longrightarrow V$. We shall prove $[M,id] = 0$ in $\overline{SK}_n(M)$.

Cutting the mapping torus V_h along two fibres to get two copies of $V \times I$ induces a cutting of M (along a manifold diffeomorphic to the double of V) into two pieces N and N', each of which is diffeomorphic to $V \times I/\sim$, where $\sim$ identifies each $x \times I$ $(x \in \partial V)$ to a point (in fact N and N' are still diffeomorphic to $V \times I$). Use a homotopy between $id : V \times I \longrightarrow V \times I$ and $V \times I \xrightarrow{p} V \subset V \times I$, where p is the projection, to slide both N and N' into a single page V of M and re-paste them there to get the double $\mathcal{D}N$ mapping into a page $V \subset M$. This mapping clearly extends to a mapping of $N \times I$ into V if we consider $\mathcal{D}N$ as $\partial(N \times I)$. Hence $[M,id]$ is equivalent by an SK-operation to something which bounds in M, and is hence zero in $\overline{SK}_n(M)$. Q.E.D.

The open book theorem together with (6.4) clearly implies (6.1). There are other interesting implications. Recall that for any connected space X, the augmentation $\varepsilon^X : SK_*(X) \longrightarrow SK_*$ and the map $\eta : SK_*(X) \longrightarrow \text{Ker}\ \varepsilon^X$ given by

$$\eta[M,f] = [M,f] - [M,*]$$

define a direct sum representation

$$SK_*(X) = SK_* \oplus \text{Ker}\ \varepsilon^X.$$

Since SK_* is well understood, it is $\text{Ker}\ \varepsilon^X$, and hence the elements $\eta[M,f]$, which interest us.

As remarked in Chapter 2, if a manifold M is the base of a compact

fibre bundle with structure group G and non-multiplicative signature, and $f : M \longrightarrow BG$ is the classifying map, then $[M,f] \neq [M,*]$ in $SK_*(BG)$, so $\eta[M,f]$ is non-trivial, in fact of infinite order, in $\text{Ker}\ \varepsilon^{BG}$. Thus by Lemma (6.3) $\eta[M,id]$ has infinite order in $\text{Ker}\ \varepsilon^M$. Thus $\eta[M,id] \in \text{Ker}\ \varepsilon^M$ gives an intrinsic obstruction to multiplicativity of signature for arbitrary bundles over M. Two natural questions arise:

<u>Question</u> 1: We have seen that finite order of $\eta[M,id]$ in $\text{Ker}\ \varepsilon^M$ is sufficient for bundles over M to have multiplicative signature. Is it also necessary?

<u>Question</u> 2: By Theorem (6.4) triviality of $[M,id]$ in $\overline{SK}(M)$ (which is equivalent to $\eta[M,id] = 0$ and $\tau(M) = 0$) is necessary for M to have an open book decomposition. Is it also sufficient?

Atiyah's examples show that there are bundles with non-multiplicative signature over any product M of orientable surfaces of sufficiently high genus. Hence $\eta[M,id] \neq 0$ in $\text{Ker}\ \varepsilon^M$, so M has no open book decomposition. Thus the condition $\pi_1(M) = 0$ in the open book theorem cannot be dropped entirely. It was this remark, made by Elmar Winkelnkemper (using a more direct argument) that led to this chapter.

APPENDIX 1: Cutting and Pasting of (B,f)-manifolds

by G. Barthel

Most of the preceding theory can be generalized to cutting and pasting of (B,f)-manifolds, so here we give a summary of the generalization.

Let us briefly recall the definition of a (B,f)-structure on a manifold as given by Lashof [3] (see also Stong [7]). Let $(B,f) = (B_k, f_k)$ be a sequence of fibrations $f_k : B_k \longrightarrow BO_k$ and maps $g_k : B_k \longrightarrow B_{k+1}$ such that all diagrams

$$\begin{tikzcd} B_k \arrow[r, "g_k"'] \arrow[d, "f_k"'] & B_{k+1} \arrow[d, "f_{k+1}"] \\ BO_k \arrow[r, "j_k"'] & BO_{k+1} \end{tikzcd}$$

commute (j_k is the usual inclusion).

Any smooth imbedding $i_{k_0} : M^n \longrightarrow \mathbb{R}^{n+k_0}$ of a compact smooth n-manifold yields imbeddings $i_k : M^n \longrightarrow \mathbb{R}^{n+k}$, $k \geq k_0$, by the inclusion of $\mathbb{R}^{n+k_0}$ into $\mathbb{R}^{n+k}$. The geometric normal maps $\nu_k : M \longrightarrow BO_k$ (taking BO_k as an infinite Grassman manifold) of these imbeddings are related by $\nu_{k+1} = j_k \cdot \nu_k$. Given a (B_{k_0}, f_{k_0})-structure on (M, i_{k_0}) (i.e., a homotopy class of liftings

$$\begin{tikzcd} & B_{k_0} \arrow[d] \\ M \arrow[ur, "\xi_{k_0}"] \arrow[r, "\nu_{k_0}"'] & BO_{k_0} \end{tikzcd}$$

of the normal map to B_{k_0}), one obtains a unique sequence $\xi = (\xi_k)_{k \geq k_0}$ of (B_k, f_k)-structures on (M, i_k).

Provided that k is sufficiently large, any two imbeddings i_k and i'_k

of M^n into $\mathbb{R}^{n+k}$ are regularly homotopic and any two regular homotopies are homotopic through regular homotopies of the given imbeddings. The induced homotopy of the normal maps yield by the homotopy lifting property for the maps f_k a one-one correspondence between (B_k, f_k)-structures on (M, i_k) and (M, i'_k).

Two sequences $\xi = (\xi_k)_{k \geq k_0}$ and $\zeta = (\zeta_\ell)_{\ell \geq \ell_0}$ belonging to embeddings $i_{k_0} : M^n \longrightarrow \mathbb{R}^{n+k_0}$ and $i'_{\ell_0} : M^n \longrightarrow \mathbb{R}^{n+\ell_0}$ will be called equivalent if ξ_r and ζ_r correspond by the above correspondence for some r. A (B,f)-structure on M is then defined to be an equivalence class of such sequences of (B_k, f_k)-structures, and a manifold M together with a (B,f)-structure ζ is called a (B,f)-manifold.

If $\varphi : M' \longrightarrow M$ is a diffeomorphism, any (B,f)-structure on M induces one on M'. An isomorphism of (B,f)-manifolds is a diffeomorphism inducing the given structure on the source M'. This notion of induced structure and of (B,f)-morphism can be extended to immersions with trivialized normal bundle, see Stong [7], p. 16, for details.

Let W^{n+1} be a (B,f)-manifold with boundary. Imbed W^{n+1} in $\mathbb{R}^{n+k} \times \mathbb{R}_+$ such that ∂W lies in $\mathbb{R}^{n+k} \times \{0\}$ and W meets $\mathbb{R}^{n+k} \times \{0\}$ orthogonally along ∂W. Then the (B_k, f_k)-structure on W induces one on ∂W by restriction, called the boundary structure. For a closed (B,f)-manifold M, the boundary structure on $\partial(M \times I)$ induces the given structure on $M = M \times \{0\}$ and a structure on $M = M \times \{1\}$ called the opposite structure, briefly denoted by $-M$.

Two closed (B,f)-manifolds M and M' are called bordant if $M + (-M')$ is a (B,f)-boundary. The (B,f)-bordism classes of closed n-dimensional (B,f)-manifolds form an abelian group $\Omega_n^{(B,f)}$ called the n^{th} (B,f)-bordism group.

We remark that these groups are isomorphic to certain stable homotopy groups of appropriate Thom spaces (see [3], [7] for details). Furthermore, if a multiplicative structure is given (defined by maps $B_r \times B_s \longrightarrow B_{r+s}$ such that the projections f_k preserve products up to homotopy, $BO_r \times BO_s \dashrightarrow BO_{r+s}$ being the usual multiplication), we get a graded ring structure on $\Omega_*^{(B,f)}$, and the homomorphism $\Omega_*^{(B,f)} \longrightarrow \mathfrak{N}_*$ is a homomorphism of graded rings.

Suppose that a closed manifold M is the union of two bounded manifolds N and N' pasted along the common boundary $\partial N = \partial N'$. Then a given (B,f)-structure on M induces (B,f)-structures on N and N' such that the boundary structures on ∂N and $\partial N'$ are opposite to each other. If $\varphi : \partial N \longrightarrow -\partial N'$ is a (B,f)-isomorphism, the pieces N and N' may be pasted by φ to give a new (B,f)-manifold M', and we say M' has been obtained from M by an SK-operation. Note that in general <u>the</u> (B,f)-<u>structure on</u> M' <u>is not uniquely determined by the</u> (B,f)-<u>manifolds</u> N, N' <u>and by</u> φ.

As in Chapter 1, one defines an SK-group $SK_n^{(B,f)}$ as the Grothendieck group of closed n-dimensional (B,f)-manifolds modulo the relations given by SK-operations. $\overline{SK}_n^{(B,f)}$ is then defined by factoring $SK_n^{(B,f)}$ by the bordism relation. If the (B,f)-structure is multiplicative, then $SK_*^{(B,f)}$ and $\overline{SK}_*^{(B,f)}$ are graded rings, and the natural epimorphisms

$$SK_*^{(B,f)} \longrightarrow \overline{SK}_*^{(B,f)}$$

and

$$\Omega_*^{(B,f)} \longrightarrow \overline{SK}_*^{(B,f)}$$

are graded ring homomorphisms.

We first remark that without loss of generality we can assume the spaces B_k to be connected. Collapsing the connected components of the fibres of $B_k \longrightarrow BO_k$ to points yields a connected covering of BO_k, which must be either

the trivial covering $BO_k \longrightarrow BO_k$, or the universal covering $BSO_k \longrightarrow BO_k$. Thus the fibres of B_k have at most two components, so there are at most two (B,f)-structures on a point, and they are opposite to each other. The same holds for the spheres S^n with boundary structures induced from the disc D^{n+1}. These structures on the sphere are isomorphic by an orientation reversing diffeomorphism, so in fact there is only one such structure induced from the disc; we call it the <u>point structure</u>.

Corresponding to Theorems (1.1) and (1.2) of Chapter 1 we have the following results:

<u>THEOREM</u> 1: There is an exact sequence

$$0 \longrightarrow I_n^{(B,f)} \longrightarrow SK_n^{(B,f)} \longrightarrow \overline{SK}_n^{(B,f)} \longrightarrow 0,$$

where $I_n^{(B,f)}$ is the cyclic subgroup of $SK_n^{(B,f)}$ generated by the class $[S^n]$ of the sphere S^n with the point structure, and

$$I_n^{(B,f)} \cong \mathbb{Z}, \qquad n \equiv 0 \pmod 2)$$

$$I_n^{(B,f)} \cong 0 \text{ or } \mathbb{Z}_2, \quad n \equiv 1 \pmod 2).$$

If the fibres of B_k have two connected components, then the sequence splits for n even.

<u>THEOREM</u> 2: Let $F_n^{(B,f)}$ be the subgroup of $\Omega_n^{(B,f)}$ of all elements representable by a manifold which fibres over S^1. Then

$$0 \longrightarrow F_n^{(B,f)} \longrightarrow \Omega_n^{(B,f)} \longrightarrow \overline{SK}_n^{(B,f)} \longrightarrow 0$$

is exact.

The proofs are as in Chapter 1, with the following reservations: the connection between SK and surgery discussed in Chapter 1 goes through without change to prove Theorem 1, however, the cutting and pasting Lemma (1.5) needs additional conditions:

LEMMA 3: i) If the (B,f)-manifold M fibres over S^1 then $[M] = 0$ in $SK_*^{(B,f)}$.

ii) If M fibres over S^n with typical fibre F then $[M] = [S^n \times F]$ in $SK_*^{(B,f)}$, where the structure on $S^n \times F$ is induced from $D^{n+1} \times F$. If the theory is multiplicative, then F can be given a (B,f)-structure such that $[M] = [S^n][F]$ in $SK_*^{(B,f)}$.

iii) If the (B,f)-structure is multiplicative and if there are (B,f)-structures on $P_n\mathbf{C}$ for all n, then for any (B,f)-manifold M fibred over $P_n\mathbf{C}$ with fibre F,

$$[M] = [P_n\mathbf{C}][F]$$

holds in $SK_*^{(B,f)}$, for a suitable (B,f)-structure on F.

iv) The same as iii) with $P_n\mathbf{R}$ instead of $P_n\mathbf{C}$.

COROLLARY 4. Under the assumption of part iii) above, $[S^{2n+1}] = 0$ in $SK_{2n+1}^{(B,f)}$, so $I_{2n+1}^{(B,f)} = 0$.

Theorem 2 is proved as in Chapter 1, by showing that the (B,f)-bordism classes of two manifolds related by a single SK-operation differ by the class of a manifold which fibres over the circle. Note that two SK-operations may yield the same manifold fibering over the circle but with different (B,f)-structures, due to the non-uniqueness of (B,f)-structures under cutting and pasting mentioned earlier.

This means that the calculation of the SKK-groups of Chapter 4 is not the same in the (B,f)-case: (B,f)-bordism class needn't be an SKK-invariant. However, the class in $\Omega_*^{(B,f)}/J$, where J is the subgroup generated by all (B,f)-structures on manifolds of the form $M \times S^1$, is an SKK-invariant, and the discussion of Chapter 4 goes through using this group in place of $\Omega_*^{(B,f)}$.

As an example of (B,f)-SK we now calculate the SK-groups for weakly complex manifolds, obtaining the following result.

THEOREM 5: The rings SK_*^U and $\overline{SK}_*^U$ are isomorphic to SK_* and $\overline{SK}_*$ by the obvious homomorphisms.

Proof: By Lemma 3 and Corollary 4 we know that SK_{2n+1}^U is isomorphic to $\overline{SK}_{2n+1}^U$, which is a quotient of Ω_{2n+1}^U. Now Ω_*^U is known, namely, it is the integral polynomial ring $\mathbf{Z}[Y_0, Y_1, Y_2, \ldots]$ on $2i$-dimensional generators Y_i that can be represented by certain linear combinations of products of complex projective spaces $P_n\mathbf{C}$ and hypersurfaces $H_{r,t}$ in $P_r\mathbf{C} \times P_t\mathbf{C}$ (Milnor, Novikov, Hirzebruch [4], [5], [6], [1]).

Hence the Ω_{2n+1}^U, and thus also the $SK_{2n+1}^U = \overline{SK}_{2n+1}^U$ are zero, proving the theorem for odd dimensions.

In the even dimensional case we see that $\overline{SK}_{2n}^U \longrightarrow SK_{2n}$ is onto, as it maps generators onto generators. By Lemma 3 iii) these generators may be chosen as products of complex projective spaces. Now one sees that Jänich's proof that $[P_{n+2}\mathbf{C}] = [P_n\mathbf{C}][P_2\mathbf{C}]$ in $\overline{SK}_*$ (given in [2], 2., (4a)) holds also in $\overline{SK}_*^U$ (where $P_n\mathbf{C}$ has its usual weakly complex structure). Thus $\overline{SK}_{4k+2}^U$ is generated by products with at least one factor $P_1\mathbf{C}$ and is hence zero, while $\overline{SK}_{4k}^U$ is generated by

$P_2\mathbb{C} \times \dots \times P_2\mathbb{C}$ (k times) and is hence isomorphic to $\mathbb{Z}$ by signature. Thus $\overline{SK}^U_* = \overline{SK}_*$, and the 5-lemma on

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_n^U & \longrightarrow & SK_n^U & \longrightarrow & \overline{SK}_n^U & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_n & \longrightarrow & SK_n & \longrightarrow & \overline{SK}_n & \longrightarrow & 0 \end{array}$$

completes the proof.

Theorems 2 and 5 yield the characterization of weakly complex manifolds which fibre over the circle up to unitary bordism, namely, that signature vanishes.

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