DIEUDONNÉ MODULES AND GROUP SCHEMES

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1. Overview

We follow [Dem23, §10], with occasional references scattered throughout. It is important to note here that we are working over finite fields.

The Tate module is defined away from the distinguished prime p, the characteristic of the base field k. We cannot satisfactorily proceed the same way at this prime, ultimately due to the presence of the Frobenius. Instead, one can work over the Witt vectors (resp. their fraction field) $W(k)$ (resp,. $W(k)[p^{-1}]$). Working with the field versions resets the characteristic to 0. The main theorem of today is that there is a correspondence M which takes finite commutative group schemes of order p^h to (left!) modules over a certain $W(k)$-algebra of length h. Then, we can use the procedure used to define the Tate module on the objects $M(\cdot)$.

2. Chapter 10, Dembele

2.1. **10.1.** A *p-divisible group* relativizes the construction of the Tate module: such an object over S is just an inductive system over S where each G_n is p^n-torsion and $G_n \simeq G_{n+1}[p^n]$. The natural p-group associated to an abelian variety A over $S = \mathrm{Spec}(k)$ is where $G_n = A[p^n]$ for some abelian variety A.

The *height* of a p-divisible group is the exponent of p in the rank of G_1. The Tate module associated to G is the inverse limit you can get from the aforementioned inductive system (multiplication by p).

2.2. **10.2, 10.4.** Of the two sources on Dieudonné theory given by Dembele, the proof of the following proposition seems to be written in French by Fontaine [Fon77].

Proposition 2.1. *Let k be a field with characteristic p. Let G be a flat k-group scheme. Then, there exists $V_{G/k} : G^{(p)} \to G$ such that $VF = FV = p$. Also, they interchange roles on the dual.*

Definition 2.2. Mimicking the relations above, the Dieudonné ring $D_k := W(k)[F, V]$ satisfies the relations above and $F\alpha = \sigma(\alpha)F$ and $\alpha V = V\sigma(\alpha)$, where σ is the Frobenius on $W(k)$.

Remark 2.2.1. The second condition seems to just be an artifact of Dieudonné theories. This is a line taken from [CCO13], but this is also immediate from the theory of difference rings [Ked22, Chapter 14, p. 262]. This reference reproduces the Dieudonné-Manin classification theorem for difference modules over $W(k)[p^{-1}]$ but only when k, is among other things, algebraically closed. The key feature to extract here, though, is the semi-linearity condition on F, and dually for V.

Theorem 2.3. *There is an additive anti-equivalence of categories M as stated in the overview. The following properties are satisfied:*

(1) M is functorial in the base field (base change of fields gives base change of Witt vectors).

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(2) The σ-semilinear action on $M(G)$ induced by $M(F)$ through the isomorphism $M(G^{(p)}) \simeq \sigma(M(G))$ equals the action of the (Dieudonné) F. According to Dembele, similarly for V, presumably by taking the dual.
(3) The Cartier dual of G has Dieudonné module naturally isomorphic to the $(W(k)[p^{-1}])/W(k)$-dual of $M(G)$.
(4) The quotient $M(G)/F(M(G))$ is naturally isomorphic to the dual of the tangent space of G at the identity.

As in the overview, T_pA will be defined using the inverse system of $M(A[p^n])$, and $V_p = \mathbb{Q}_p \otimes T_p$. We will discuss, after an example, that this gives us a rank $2g$ $W(k)$-module. We also get a version of Tate's theorem. Regarding the proof of the latter, injectivity follows from essentially the same argument as before (in fact, easier!). For surjectivity, we can reduce to the case when A is k-simple. It will be enough to show that the map $\mathbb{Q}_p \otimes_{\mathbb{Z}} \mathrm{End}_k(A) \to \mathrm{End}_{D_k[p^{-1}]}(V_p(A))^{\mathrm{opp}}$ is surjective.

2.3. **10.3: Height 1 of this Correspondence.** See the corresponding section in Dembele's notes. The dimension 1 case will be presented at the seminar.

2.4. **Proving the End of §2.2.** The proof of what was introduced in §2.2 (Tate's theorem) will require us to be familiar with cyclic algebras.

2.4.1. *Some Facts About Cyclic Algebras.* We need some results on central simple algebras, particularly cyclic ones.

Theorem 2.4. *Let $h \in \mathbb{Q}_p[x]$ be monic and irreducible of degree n with non-zero constant term. Let L be the splitting field of h. Then, $D' := D_k[p^{-1}]/(h(F^n))$ is a central simple L-algebra. Its corresponding division algebra is the cyclic L-algebra $(W(k)[p^{-1}]L/L, \sigma^r, \pi)$, where L is as in the lemma.*

Corollary 2.4.1. *With notation as in the theorem, one has*

$$\dim_{K_0} \mathrm{End}_{W(k)[p^{-1}, F^n]}(M) = \dim_{\mathbb{Q}_p} \mathrm{End}_{D'}(M).$$

2.4.2. *The Proof.* Recall the goal is to prove surjectivity of the Tate map. A different version of Albert's classification in [Gar16] tells us that the center of $D := \mathrm{End}_k^0(A)$ (for A a simple abelian variety) is $\mathbb{Q}[\pi_A]$. Before proceeding, we will need one final standard fact about central simple algebras.

Lemma 2.5. *Let A be a central simple K-algebra of finite degree, and L a semi-simple commutative sub-algebra. Then, the following are equivalent:*
(1) L is a maximal commutative sub-algebra of A.
(2) The L-module A is free of dimension $[L : K]$.
(3) $[A : K] = [L : K]^2$.

Proof. See [Dem23, Proposition C.13]. □

Now, consider the algebra $D' = \mathrm{End}_{D_k[p^{-1}]}(V_p(A))^{\mathrm{opp}}$, which is just the usual endomorphism ring tensored by $\mathbb{Q}_p$. By the functoriality of Dieudonné modules, $V_p(\pi_A) = F^n$. Therefore,

$$Z(D') = \mathbb{Q}_p[F^n]/(g_A(F^n)) = Z \otimes_{\mathbb{Q}} \mathbb{Q}_p.$$

Decompose the right-hand side over all places over p:

$$\mathbb{Q}_p \otimes_{\mathbb{Q}} Z = \prod_{v|p} \mathbb{Q}_p[x]/(g_{A,v}).$$

We have a similar decomposition for $V_p(A)$: the group $G := A[p^\infty]$ decomposes into $\prod G_v$, where the product terms are idempotents of Z_v. This tells us that $V_p(G_v)$ is a left module over the quotient D'_v.

It is enough to show equality of $\mathbb{Q}_p$-dimension for each place v. By construction,

$$L_v = W(k)[p^{-1}, x]/(g_{A,v}(x))$$

is a commutative sub-algebra of both domain and codomain at v with dimension $[D'_v : F_v]^{1/2}$, and so we can show equality of L_v-dimension instead. From Corollary 2.6.1 and Lemma 2.7, we're done.

2.5. **10.5.** In the last section, we didn't decompose D into D_v's. We can do this as well, and the form is expected at this point: $D_k[p^{-1}]/(g_{A,v}(F^n))$. By Theorem 2.6 and [Dem23, Theorem C.24], D_v is Brauer equivalent to $(W(k)[p^{-1}]Z_v/Z_v, \sigma', \pi^{f_v/g_v})$, where $f_v = f(v \mid p)$ and $g_v = \gcd(f_v, n)$ and has $\mathrm{inv}_v(D_v) = \frac{f_v}{n} v(\pi) \in \mathbb{Q}/\mathbb{Z}$.

3. Extras

As stated in §1, the jump from k to $W(k)[p^{-1}]$ means shifting from positive to zero characteristic. This must remind one of tilting, the natural setting for which is perfectoid rings. A quick Google search for Dieudonné modules over perfectoid rings yields both what is wanted and also a prismatic version of the theory. One sample statement comes from a seminar at Bonn from the Winter semester in the academic year 2019-20.

Theorem 3.1. *Let R be a quasisyntomic ring that is flat over $\mathbb{Z}/p^n\mathbb{Z}$ for some n or over $\mathbb{Z}_p$. Then, the category of p-divisible groups over R is equivalent to a certain category of locally free sheaves on the prismatic site of R equipped with filtrations and a Frobenius.*

References

[CCO13] Ching-Li Chai, Brian Conrad, and Frans Oort, *Complex multiplication and lifting problems*, vol. 195, American Mathematical Soc., 2013.

[Dem23] Lassina Dembélé, *Abelian varieties over finite fields: Honda-tate's theorem.*

[Fon77] Jean-Marc Fontaine, *Groupes p-divisibles sur les corps locaux.*

[Gar16] Elisa Lorenzo García, *Abelian varieties. lecture 5.*

[Ked22] Kiran S Kedlaya, *p-adic differential equations*, Cambridge University Press, 2022.