Alg. Variety X/k: separated + of finite type + geom. integral ; complete = proper (so it just means that X/k universally closed)

Group Variety G/k: G variety + $G \times G \xrightarrow{m} G$ (group law) + $i: G \longrightarrow G$ (inverse) + $0_G \in G(k)$ identity

such that: 1) associativity of m

2) 0_G is the identity wrt m

3) i is the inverse

morphism of groups $\mathrm{Hom}(G,H)$

↳ morphism of schemes preserving the group law & sending $e_G \overset{f}{\mapsto} e_H$

Abelian Variety: A/k is a PROPER group variety $/k$

$\mathrm{Hom}(A,B)$

morphism of ab. var. = morphism of groups

notation: additive: $m(a,b) =: a+b$

Rmk: why Ab. Variety not defined as group variety + commutative : to avoid ⇒ E.g. of group varieties: $\mathbb{G}_m = \mathrm{Spec}(k[t,t^{-1}])$ $\mathbb{G}_a = \mathrm{Spec}(k[t])$ (& GL_n)

Theorems on alg. groups: ① q-proj: G alg. group variety $/k$ ⇒ G quasi-projective $/k$. TAG 0BF7

② smooth G alg. group variety $/k$ + $\mathrm{char}(k)=0$ or + $\mathrm{char}(k)=p$ & k perfect field ⇒ G smooth TAG 047N TAG 047P

③ proj: A ab. variety ⇒ A projective. ④ A ab. variety ⇒ A smooth. TAG 0BFC.

Theorem: A/k abelian variety ⇒ m is commutative.

Rigidity Lemma: $f: X \times Y \longrightarrow Z$ all varieties + X proper $/k$

If $\exists\, x_0 \in X$ & $y_0 \in Y$: $f|_{\{x_0\} \times Y}$ & $f|_{X \times \{y_0\}}$ both constants ⇒ f is constant.

Corollary for ab. var. $f: A \longrightarrow B$ ab. varieties: $f(0_A) = 0_B$ ⇒ $f \in \mathrm{Hom}_{a.v.}(A,B)$

proof $f(a+b) = f(a) + f(b) \iff f(a+b) - f(a) - f(b) = 0_B$. Now consider:

$$A \times A \xrightarrow{(f\circ m,\ i\circ f\circ pr_1,\ i\circ f\circ pr_2)} B \times B \times B \xrightarrow{m} B$$

On $A \times \{0_A\}$ & $\{0_A\} \times A$ is $\equiv 0_B$. ✓

Corollary 2 for ab. var.: m commutative

proof: $a+b=b+a \iff a+b-a-b=0_A$: $A\times A \xrightarrow{m,\ i\circ pr_1,\ i\circ pr_2} A\times A\times A \xrightarrow{m} A$ ✓

Rigidity results on line bundles.

THM of the CUBE: $X\times Y\times Z$ varieties s.t. X & Y proper. Let $x_0\in X$, $y_0\in Y$, $z_0\in Z$.

Let $\mathcal{L}$ be a line bundle s.t. $\mathcal{L}\big|_{X\times Y\times\{z_0\}}$ & $\big|_{X\times\{y_0\}\times Z}$ & $\big|_{\{x_0\}\times Y\times Z}$ are all trivial.

$\Rightarrow \mathcal{L}\simeq\mathcal{O}_{X\times Y\times Z}$ trivial as well.

A) Corollary for ab. var. $A\times A\times A$ & $\mathcal{L}\in \mathrm{Pic}(A)$. Let $pr_i: A\times A\times A\to A$ projection on i-th factor.

Let $pr_{ij}=pr_i+pr_j$ & $pr_{123}=pr_1+pr_2+pr_3$. Then:

$$\tilde{\mathcal{L}} := pr_{123}^*\mathcal{L}\otimes(pr_{12}^*\mathcal{L}\otimes pr_{13}^*\mathcal{L}\otimes pr_{23}^*\mathcal{L})^{-1}\otimes pr_1^*\mathcal{L}\otimes pr_2^*\mathcal{L}\otimes pr_3^*\mathcal{L} \quad \text{is TRIVIAL.}$$

B) Corollary 2 for ab. var.: $X\xrightarrow{f,g,h} A$, X any variety. Then:

$$\tilde{\mathcal{L}} = (f+g+h)^*\mathcal{L}\otimes[(f+g)^*\mathcal{L}\otimes(f+h)^*\mathcal{L}\otimes(h+g)^*\mathcal{L}]^{-1}\otimes f^*\mathcal{L}\otimes g^*\mathcal{L}\otimes h^*\mathcal{L} \quad \text{is trivial.}$$

Important Setting: $\mathcal{L}\in\mathrm{Pic}(A)$ & $T \overset{x}{\underset{y}{\rightrightarrows}} A$, T any k-scheme

$$\begin{tikzcd}
A\times_k T = A_T \arrow[r, "(x\circ pr_2{,}\, id)"] & A\times_k A_T \arrow[r, "m"] \arrow[d] & A_T \arrow[r, "pr_T"] \arrow[d, "pr_A"] & T \arrow[d] \\
& A\times A \arrow[r, "m"] & A \arrow[r] & k
\end{tikzcd}$$

Call $\mathcal{L}_T := pr_A^*\mathcal{L}$.

Denote by $t_x = m\circ(x\circ pr_2, id)$ & call it translation by x

Special Case, $T=\mathrm{Spec}\,k$: $A\longrightarrow A\times A\longrightarrow A$, $a\longmapsto(x,a)\longmapsto x+a$

C) Corollary 3 for ab. var. (THM of the SQUARE): Let $T\xrightarrow{(x,y)} A\times A\xrightarrow{m} A$ be $x+y$

$$t_{x+y}^*\mathcal{L}_T\otimes\mathcal{L}_T \simeq t_x^*\mathcal{L}_T\otimes t_y^*\mathcal{L}_T\otimes pr_T^*\left[(x+y)^*\mathcal{L}\otimes x^*\mathcal{L}^{-1}\otimes y^*\mathcal{L}^{-1}\right]$$

If $T=\mathrm{Spec}\,k$: $t_{x+y}^*\mathcal{L}\otimes\mathcal{L}\simeq t_x^*\mathcal{L}\otimes t_y^*\mathcal{L}$

proof: $X = A_T$ & $f = pr_A$, $g = x \circ pr_T$, $h = y \circ pr_T$

COOL COROLL: $\phi_{\mathcal{L}} : A(k) \longrightarrow \mathrm{Pic}(A)$; $\phi_{\mathcal{L}}(x) = t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$ is a group homom.

More generally: $\phi_{\mathcal{L}} : A(T) \longrightarrow \mathrm{Pic}(A_T) / \mathrm{Pic}(A)$ $\quad \phi_{\mathcal{L}}(x) = [t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}]$ is a group homomorphism $\forall t$.

[We'll get back on this later when we define DUALS]

{Elliptic curves} ≡ {abelian varieties of dim 1}

DEF: elliptic curve is a smooth connected projective curve $/k$ of genus 1 together with a point $e \in E(k)$.

The divisor $\mathcal{O}_E(3e)$ has degree $3 \geq 2 \cdot g + 1 = 3 \Rightarrow \mathcal{O}_E(3e)$ is very ample. Moreover $H^0(E, \mathcal{O}_E(e)) = 3 + 1 - 1 = 3$

$\Rightarrow E \hookrightarrow \mathbb{P}^2_k$ closed embedding. $d(d-3) = 2 - 2 = 0 \Rightarrow d = 3 \Rightarrow$ So E is defined by the vanishing of a homogeneous polyn of degree 3.

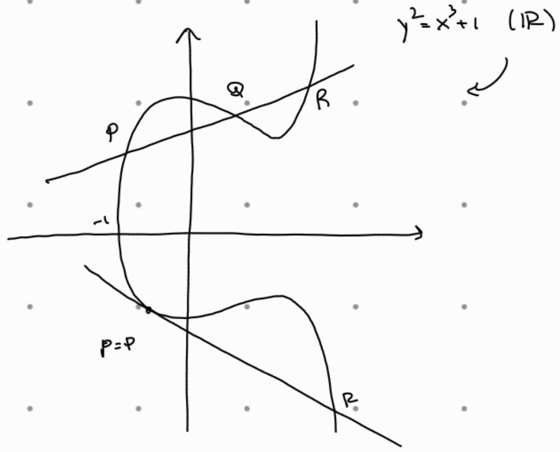
If char$(k) \neq 2$ then this polynomial f can be put into the (Weierstraß) form: $y^2 z = x^3 + a x^2 z + b x z^2 + c z^3 = f(x, z)$

$[0:1:0] = e = \infty$

The non singularity of E translates into $\Delta_E = -4a^3 c + a^2 b^2 + 18abc - 4b^3 - 27c^2 = [(e_1 - e_2)(e_1 - e_3)(e_2 - e_3)]^2$

(Moreover if E is of the form $y^2 z = x^3 + bxz^2 + cz^3$ we get $\Delta_E = -4b^3 - 27c^2$) $\quad$ [e_i are the roots of $f(x,1)$ in $\bar{k}$]

Examples: $y^2 = x^3 + c$ & $c \neq 0$ $/ \mathbb{Q}$ ✓ but since $\Delta_E = -27c^2$ $/ \mathbb{F}_3$ ✗



Bezout theorem: $E \cap L \subseteq \mathbb{P}^2_k$ has degree $= 3$, so they are 3 points counted w/ multiplicity ($L \simeq \mathbb{P}^1_k$).

Group structure on $E(k)$

$E(k) \times E(k) \longrightarrow E(k)$: $\begin{cases} (P,Q) \ \& \ P \neq Q : & \exists!\ L \text{ passing through } P, Q : \#E \cap L = 3 \longrightarrow R \\ (P,P) : & \text{Pick the tangent to } E \text{ at } P \text{ (so at } P \text{ it vanishes with multiplicity} = 2) \to R \end{cases}$

$\Downarrow$

define $P+Q = -R$: $R = (x_R, y_R)$ then $-R = (x_R, -y_R)$

Rmk: there is an explicit formula that expresses coordinates of $-R$ in terms of P & Q.

THM: $(E(k), +, \infty)$ is an abelian group.

Remark: assume you have C abelian variety of dim 1: pick $x \in C$ and consider

$T_x C$ = dashed maps

$$\left\{ \begin{tikzcd} \operatorname{Spec} k(x) \arrow[rr, bend left, "x"] \arrow[dr] \arrow[ddr, bend right, "="'] & & C \arrow[dd] \\ & \operatorname{Spec} k(x)[\varepsilon] \arrow[ur, dashed] \arrow[d] & \\ & \operatorname{Spec} k(x) \arrow[r] & \operatorname{Spec} k \end{tikzcd} \right\}$$

$T_x C \simeq \operatorname{Hom}_{\mathcal{O}_{C,x}}(\Omega_{C,x}, k(x))$

t_x map $\Rightarrow T_0 C \xrightarrow{dt_x} T_x C \Rightarrow$ sections of $\Omega^\vee_{C,0}$ can be transported everywhere $\Rightarrow \Omega^\vee_C$ is free $\Rightarrow g = 1$.

Notice: Product of ab. varieties is an abelian variety. $\left[\begin{array}{c} \text{projective} \\ + \\ \text{smooth} \\ + \\ \text{geom connected} \end{array}\right]$ $\Rightarrow$ product of elliptic curves is an example of ab. variety

Later: Jacobian of curves of genus g = ab. variety of dim $= g$.

Isogenies: $f: A \to B$ homom. of ab. varieties. TFAE:

1) f surjective & $\dim(A) = \dim(B)$

2) $\ker(f)$ is a finite group scheme & $\dim(A) = \dim(B)$

3) f finite, flat, surjective.

Recalls: ① $f: X \to Y$ of varieties. f FLAT $\Rightarrow$ $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y}) + \dim(\mathcal{O}_{X_y, x})$
$f(x) = y$

② $f: X \to Y$ of smooth varieties. then the converse is true.

DEFINITION: if $f: A \to B$ satisfies any of the above 3 properties we say that f is an isogeny

Since it is finite then $f^{\#}: K(B) \xrightarrow{inj} K(A) \rightsquigarrow K(A) / f^{\#}K(B)$ is finite field extension.

Define $\deg(f) = [K(A) : f^{\#}K(B)]$ (even if it is usually denoted by $[K(A) : K(B)]$.

Remark: since f is flat then $\forall b \in B$ $\dim_{K(b)} H^0(A_b, \mathcal{O}_{A_b}) = \deg(f)$ (Ex 1.25 chs Liu's)

We say f is separable if $K(A) / f^{\#}K(B)$ is separable extension. If not it is said inseparable.

If $K(A) / f^{\#}K(B)$ is purely insep. then f is said purely inseparable.

Equivalent definitions. $f: A \to B$ isogeny: TFAE

f separable $\Leftrightarrow$ f étale $\Leftrightarrow$ $\ker(f)$ is an (finite) étale group scheme

f purely insep. $\Leftrightarrow$ f is universally inj $\Leftrightarrow$ $\forall$ field k, $A(k) \xrightarrow{f} B(k)$ is inj $\Leftrightarrow$ $\ker(f)$ connected

Important example of isogeny: multiplication by $n \neq 0$

Theorem: $[n]_A : A \to A$, $a \mapsto \underbrace{a + \dots + a}_{n\text{-times}} = na$ is an isogeny for $n \in \mathbb{Z} \setminus \{0\}$

Moreover if $g = \dim(A)$, then $\deg([n]_A) = n^{2g}$

if $(\mathrm{char}(k), n) = 1$, then $[n_A]$ separable. (actually iff)

Notation

$\mathrm{Ker}([n]_A) = A[n]$

n-torsion of A

lemma for theorem / proposition. $[n]^*\mathcal{L} \simeq \mathcal{L}^{\frac{n^2+n}{2}} \otimes [-1]^*\mathcal{L}^{\frac{n^2-n}{2}}$, $n \in \mathbb{Z}$

(if) $\mathcal{L}$ symmetric (i.e. $\mathcal{L} \simeq [-1]^*\mathcal{L}$) then

$[n]^*\mathcal{L} \simeq \mathcal{L}^{n^2}$

if $\mathcal{L}$ antisymmetric (i.e. $\mathcal{L}^\vee \simeq [-1]^*\mathcal{L}$) then

$[n]^*\mathcal{L} \simeq \mathcal{L}^n$

Proof: check for $n = 1$ & -1 (& $n = 0$); then apply prop. $f, g, h = n, 1, -1$

$\Rightarrow [n+1]^*\mathcal{L} \simeq [n]^*\mathcal{L}^2 \otimes [n-1]^*\mathcal{L}^{-1} \otimes \mathcal{L} \otimes [-1]^*\mathcal{L}$ & then induction on n.

proof theorem: Let $\mathcal{L}$ an ample line bundle (A is proj ✓). $[-1] = i$ inverse: in particular $[-1]^2 = i^2 = \mathrm{id}$

$\Rightarrow [-1]$ is an isom. $\Rightarrow [-1]^*\mathcal{L}$ ample $\Rightarrow [-1]^*\mathcal{L} \otimes \mathcal{L}$ ample & $[-1]^*([-1]^*\mathcal{L} \otimes \mathcal{L}) = \mathcal{L} \otimes [-1]^*\mathcal{L}$

so it is symmetric. $\Rightarrow$ wlog pick $\mathcal{L}$ ample & symm

Then $[n]_A^*\mathcal{L} \simeq \mathcal{L}^{\otimes n^2}$.

$$\begin{tikzcd} \mathrm{Ker}[n] \arrow[r] \arrow[d] & A \arrow[d, "{[n_A]}"] \\ 0_A \arrow[r] & A \end{tikzcd}$$

$[n_A]^*\mathcal{L}\big|_{\mathrm{Ker}} \simeq \mathcal{O}_{\mathrm{Ker}}$ but it is also ample since $\mathcal{L}^{\otimes n^2}$ is

$\Rightarrow$ Ker proper + trivial ample line bundle $\Rightarrow \mathrm{Ker}([n]_A)$ finite

$\Rightarrow [n]_A$ isogeny $\Rightarrow [n_A]$ is surjective.

Intersection theory recall

$V \subseteq A$ closed & integral

f proper: $f_*([V]) = \begin{cases} \deg(K(V)/K(W))\,[W] & \text{if } f(V)=W \text{ \& same dim} \\ 0 & \text{otherwise} \end{cases}$

f flat of rel. dim = n : $f^*([V]) = [f^{-1}(V)]$

rmk: $f: X \to Y$ flat, Y irred. X equid. of dim $\dim(Y)+n$ $\Rightarrow$ f rel. dim $= n$.

Projection formula: $f: A \to B$ proper, $D \subseteq B$ divisor & $\alpha \in CH_k(A)$; then $f_*(f^*D \cdot \alpha) = D \cdot f_*\alpha$

Particular case: $f: A \to B$ proper varieties & f finite of degree d. Then $g = \dim(A) = \dim(B)$

$$f_*(f^*D_1 \cdot \ldots \cdot f^*D_g) = D_1 \cdot \ldots \cdot D_{g-1}\, f_*(f^*D_g \cdot [A]) = D_1 \cdots D_{g-1} \cdot \deg(A/B)\,[B]$$

$f = [n]_A$: $\quad [n]_{A*}([n]_A^*D \cdot \ldots \cdot [n]_A^*D) = D^g \cdot \deg([n]_A)$

D is the (very) symm ample div. $c_1(\mathcal{L})$. $\quad [n]_A^*D = n^2 D \Rightarrow [n_A]_*(n^{2g} \cdot D^g)$

Recall: $\deg(\alpha) = \deg(\alpha \cdot D^{g-|\alpha|})$ $\Rightarrow$ $\deg(D^g) \cdot \deg([n]_A) = n^{2g} \deg([n_A]_*(D^g))$

$D^g = \sum m_i [P_i]$ & $[n_A]_*^g D = \sum m_i \deg(P_i/n_A(P_i))\,[n_A(P_i)]$ so same degree

$\Rightarrow \deg([n]_A) = n^{2g} = [K(A)/n^*K[A]]$. If $\mathrm{char}(k)=0$ $\Rightarrow$ separable ✓

$(\mathrm{char}(k), n) = 1 \Rightarrow \mathrm{char}(k) \nmid [k(A) : n^*k(A)] \Rightarrow$ separable ✓

ISOGENIES & HOMOMORPHISM for Elliptic Curves.

Consider $f: E_1 \to E_2$ a homomorphism of gr. schemes. Since E_i are proper$/k$ $\Rightarrow$ f is proper.

Then $f(E_1) = E_2$ or $f(E_1) = $ one point. Since homom. $\Rightarrow f(E_1) = 0_{E_2}$.

f surj & same dim $\Rightarrow$ **f isogeny or is the constant map**. $\Rightarrow \mathrm{Hom}_{gr}(E_1, E_2) = \mathrm{Isogenies}(E_1, E_2) \amalg \{0_2\}$

Rmk: $\mathrm{char}(k) = 0$: $\mathbb{Z} \longrightarrow \mathrm{End}(E) \quad n \mapsto [n_A]$ is injective

If $\mathrm{End}(E_{\bar{k}})$ strictly contains $\mathbb{Z}$ then E has CM, complex multiplication.

Corollary. $A(\bar{k})$ divisible ab. group $\Rightarrow$ $A(\bar{k}) \simeq \mathbb{Q}^{\oplus rk_0} \oplus \left(\bigoplus_{p \text{ prime}} \mathbb{Z}[p^\infty]^{\oplus rk_p} \right)$

Prüfer group.

Corollary 2: If $(\mathrm{char}(k), n) = 1$, then $A[n](\bar{k}) \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$.

proof: $A[n]$ étale & $\dim_k H^0(A[n], \mathcal{O}_{A[n]}) = n^{2g}$

$A(n) = \coprod_{i=1}^{m} \mathrm{Spec}(K_i)$ K_i/k separable & $\sum_{i=1}^{n} [K_i : k] = n^{2g}$.

Given $\mathrm{Spec}(\bar{k}) \to A[n]$ $\Leftrightarrow$ $K_i \hookrightarrow \bar{k}$ k-morphism. Since separable: #embedding = degree

$\Rightarrow$ $\# A[n](\bar{k}) = n^{2g}$. Moreover $\forall\, d \mid n$ the subgroup killed by d is $A[d](\bar{k})$.

Use structure of finite groups.

§ Dual & Polarization (see ch8 of Neron Model for proofs)

Let A be an abelian variety: in particular A is a proper integral scheme $/k$

+ $0_A \in A(k)$ section of $A \to k$. For any scheme T/k define $\mathrm{Pic}_{A/k}(T) := \mathrm{Pic}(A \times_k T) / pr_T^* \mathrm{Pic}(T)$.

We call it the relative Picard functor. It has a group structure induced by $\otimes$

Consider the connected component of $[\mathcal{O}_A] \in \mathrm{Pic}_{A/k}$ & denote it by $\mathrm{Pic}^0_{A/k}$. Analogously:

Theorem. $\mathrm{Pic}^0_{A/k}$ is an abelian variety of dimension $\dim H^1(A, \mathcal{O}_A)$ (this is because $T_0 \mathrm{Pic}^0_{A/k} \simeq H^1(A, \mathcal{O}_A)$)

It is called the dual ab. variety $A^\vee$ of A. Moreover $A \xrightarrow{\sim} \hat{\hat{A}}$.

Proposition. $\phi_{\mathcal{L}} : A \longrightarrow \mathrm{Pic}_{A/k}$ factors through $\mathrm{Pic}^0_{A/k}$ $(0 \mapsto [\mathcal{O}_A])$.

If $\mathcal{L}$ is ample then $\ker(\phi_{\mathcal{L}})$ finite. Notation: $K(\mathcal{L}) := \ker(\phi_{\mathcal{L}})$.

Corollary: $\phi_{\mathcal{L}}$ is an isogeny:

proof $\phi_{\mathcal{L}}$ finite kernel $\Rightarrow$ $\dim \mathrm{Pic}^0_{A/k} \geq g \overset{\text{FACT}}{\geq} \dim_k H^1(A, \mathcal{O}_A) = \dim \mathrm{Pic}^0_{A/k}$. So same dim.

Remarks:

$\hat{A} \xrightarrow{id} \hat{A}$ corresponds to the universal bundle class $[\mathcal{P}] \in \mathrm{Pic}(A \times \hat{A}) / \mathrm{Pic}(\hat{A})$

Call $\mathcal{P}$ Poincaré bundle the representant s.t. $\mathcal{P}|_{\{0\} \times \hat{A}} \simeq \mathcal{O}_{\hat{A}}$.

$A \xrightarrow{\phi_{\mathcal{L}}} \hat{A}$ $\Rightarrow$ line bundle class $[\Lambda(\mathcal{L})] \in \mathrm{Pic}(A \times A) / \mathrm{Pic}(A)$

Let $\Lambda(\mathcal{L}) := (id_A \times \phi_{\mathcal{L}})^* \mathcal{P}$ and call it the Mumford bundle

Proposition: $\Lambda(\mathcal{L}) \simeq m^*\mathcal{L} \otimes pr_1^* \mathcal{L}^{-1} \otimes pr_2^* \mathcal{L}^{-1}$ where $pr_i : A \times A \longrightarrow A$

POLARIZATION: is an isogeny $\lambda : A \to \hat{A}$ s.t. on $A(\bar{k}) \to \hat{A}(\bar{k})$ is given by $\phi_{\mathcal{L}}$ where $\mathcal{L} \in \mathrm{Pic}(A_{\bar{k}})$ ample.

$\deg(\lambda)$ = degree of the polarization. (A, λ) is a polarized abelian variety

$\deg(\lambda) = 1 \Rightarrow (A, \lambda)$ is said to be a principally polarized ab. variety.

Morphism of polarized ab. var.: $f : (A, \lambda_A) \to (B, \lambda_B)$ s.t.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\lambda_A"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "\lambda_B"] \\ \hat{A} & \hat{B} \arrow[l, "\hat{f}"] \end{tikzcd}$$

A = elliptic curve: $\mathrm{Pic}^0_{E/k}$: $\mathcal{L} = \mathcal{O}_E(e)$ ample. Let $x \in E(\bar{k})$. $\phi_{\mathcal{L}}(x) = t_x^* \mathcal{O}_E(e) \otimes \mathcal{O}_E(-e) \simeq \mathcal{O}_E((-x)) \otimes \mathcal{O}_E(-e)$

$E(\bar{k}) \longrightarrow \mathrm{Pic}^0(E_{\bar{k}}) \qquad x \mapsto \mathcal{O}_E((-x) - e) = 0_E - (-x)$

Usually they do the following

$$\begin{tikzcd} E \arrow[r, "\phi_{\mathcal{L}}"] \arrow[d, "i"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Pic}^0 \arrow[d, "i"] \\ E \arrow[r, "\phi_{\mathcal{L}}"] & \mathrm{Pic}^0 \end{tikzcd} \Rightarrow \begin{tikzcd} x \arrow[r, mapsto] \arrow[d] & x - 0_E \\ -x \arrow[r] & 0_E - x \arrow[u] \end{tikzcd}$$

EXTRAS:

(1) Factorization into sep & purely insep.

Recall: any extension can be factored as $\underset{\uparrow \text{ separable}}{\overset{f^\# K(B)}{\overset{\|}{K}}} \subseteq K^{\text{sep}\,F} \subseteq \underset{\uparrow \text{ purely inseparable}}{\overset{K(A)}{\overset{\|}{F}}}$

for abelian varieties.

Proposition: $f: A \longrightarrow B$ isogeny : then $\exists$:
- C ab. variety
- h, g: $A \xrightarrow{g} C \xrightarrow{h} B$ s.t. h separable isogeny & g purely inseparable

s.t. $h \circ g = f$. This factorization is unique up to isom.

pay attention: in the lecture they omit it.

Corollary: $A(\bar{k})[n] = \left[\bigoplus_{p \mid n} \mathbb{Z}[p^\infty]^{\oplus rk_p} \right] [n]$. Pick n a prime: $A(\bar{k})[p] = \mathbb{Z}[p^\infty]^{\oplus rk_p} [p] = (\mathbb{Z}/p\mathbb{Z})^{\oplus rk_p}$

$\ell \neq \mathrm{char}(k)$:

$A(\bar{k})[\ell] = (\mathbb{Z}/\ell\mathbb{Z})^{\oplus rk_\ell} = (\mathbb{Z}/\ell\mathbb{Z})^{2g} \Rightarrow rk_\ell = 2g$. If $\ell = p \Rightarrow$ then there are less p-torsion points.

(see later $\# A[p](\bar{k}) \leq p^g$)

Quotients of ab. variety. A is projective. G finite group scheme acting freely on $A \Rightarrow A/G$ is a scheme

CRITERION for when A/G ab. variety.

Corollary 3: $f: A \to B$ isogen. $\deg(f) = n$. There $\exists$ $g: B \to A$ isog. s.t. $g \circ f = [n]$ & $f \circ g = [n]$.

proof: $\deg(f) = n \Rightarrow n \cdot \ker(f) = 0_A \Rightarrow \ker(f) \subseteq \ker([n])$

$$\Rightarrow \begin{tikzcd} A \arrow[rr, "n"] \arrow[dr, "\pi"'] & & A \\ & A/\ker(f) \cong_f B \arrow[ur] & \end{tikzcd} \Rightarrow \begin{tikzcd} A \arrow[r, "n"] \arrow[d, "f"'] & A \arrow[r, "n"] \arrow[dr, "f"] & A \\ B \arrow[ur, "g"] \arrow[rr, "n_B"'] & & B \arrow[u, "g"'] \end{tikzcd}$$

$g \circ n_B \circ f = g \circ (f \circ g) \circ f = [n^2]_A$

$\Rightarrow f \circ g = [n]_B$.

& commutes

Lemma:

$$\begin{tikzcd} C \arrow[d, "g"'] & D \\ A \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] & B \arrow[u, "h"'] \end{tikzcd}$$

all ab. varieties. & g, h isogenies.

Assume $h \circ f_1 \circ g = h \circ f_2 \circ g$. Then $f_1 = f_2$

g flat + surj $\Rightarrow$ faithfully flat $\Rightarrow$ epimorphism of schemes $\Rightarrow$ $h \circ f_1 = h \circ f_2$ $\Rightarrow$ $h \circ (f_1 - f_2)$

$f_1 - f_2 : A \to \ker(h)$ finite $\Rightarrow$ $f_1 - f_2$ goes to $\ker(h)^{\circ B}_{red}$ $\Rightarrow$ $f_1 - f_2$ constant.

$\Uparrow$

A connected
& reduced.

Proposition: $A \xrightarrow{f} B$ hom. $\Rightarrow$ $\exists$ an induced hom. $f^\vee : B^\vee \longrightarrow A^\vee$ called the dual or the traspose.

$f^\vee$ is the unique homomorphism : $(id_A \times f^\vee)^* \mathcal{P}_A = (f \times id_B)^* \mathcal{P}_B$

Recall: $\ker(f)$ dim 0 & noeth. k-scheme $\Rightarrow$ $\ker(f) = \operatorname{Spec}(A)$ A artinian $= \operatorname{Spec}(\prod A_p)$ and

$= \coprod \operatorname{Spec}(A_p)$. A_p local & artinian

$\boxed{\ker(f) \text{ connected}}$ $\Leftrightarrow$ $\operatorname{Spec}(A_p)$ (which is topologically just a point).

$\boxed{\ker(f) \text{ étale}}$ $\Leftrightarrow$ all A_p are fields & thes are separable extension of k.

Thm: $\operatorname{Pic}^0_{A/k}(T) = \{ \mathcal{L} \in \operatorname{Pic}(A_T) : \forall\, k(t) \xrightarrow{t} T \ \deg(\mathcal{L}|_{A_t}) = 0 \} / pr_T^* \operatorname{Pic}(T)$

X/k projective variety $\deg(\mathcal{L}) = \deg(c_1(\mathcal{L}) \cap c_1(\mathcal{O}_X(1))^{g-1})$