01/16/24 **TODAY:**

- Course Outline from Dembélé, from Li
- Comparison w/ AWS24 program
- Split of the material

First course outline: Abelian varieties over finite fields.

Motivation: interests in Ab. Var / $\mathbb{Q}$. → pass mod p & study the variety there.

Main result: classification of ab. varieties / $\mathbb{F}_q$ up to isogeny [Honda-Tate thm].

Lec 1: def of Ab. variety – rigidity results – isogenies – dual $A^\vee$ & polarization (§1-2 ; 45 min.)

+ Elliptic curve (§4 - def, group law ; 15 min) → 30min left

Lec 2: properties of $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$, $\mathrm{Hom}(A,B) \otimes_{\mathbb{Z}} \mathbb{Q}$, Tate-module $T_\ell(A)$ (§3-5.1-5.2-5.3 ; 50 min)

+ prop. 5.12 & cor. 5.13 & prop. 5.14 (from 5.4 ; 15 min) (no proof but examples would be cool)

+ Albert classification (structure thm on $\mathrm{End}^\circ_k(A)$) (§7.1 ; 15 min) (from Lec. 4)

Lec 3: Tate's isogeny theorem : $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \xrightarrow{\sim} \mathrm{End}(T_\ell(A))$ ($\ell \neq p$, $A/\mathbb{F}_q$).

Frobenii maps, Tate's thms like (§6 ; 60 min) → 40 min left

↳ $\pi_A : A \longrightarrow A$ $/\mathbb{F}_q$ id on top. space $f \mapsto f^q$ on sheaves : Prop: $\mathbb{Q}(\pi_A) \subseteq \mathrm{End}^\circ_k(A)$ is a #-field.

Lec 4. Weil conj for ab. varieties (§7.2 ; 40 min), Jacobian of curves (§8 ; 30 min)

Weil conj comp. for curves (§9.1 ; 15 min).

Lec 5. Dieudonné modules attached to A & Tate's thm analogous. (§10.4 : 10 min)

[$\ell \neq p$ → Tate modules;

case $\ell = p$ → Dieudonné mod]

↑ (I'd skip surj. proof part.)

divisible group, Verschiebung map, Dieudonné ring D_k, eq. $G \mapsto M(G)$ of cat. ($\S 10.1$-10.2: 40min)

local invariants for ab. var. (namely decomposition in terms of places $v|p$ of $\mathrm{End}_k^0(A) \otimes_{\mathbb{Q}} \mathbb{Q}_p$)

($\S 10.5$: 10min)

→ 30 min. left

Lec 6: classification.- Honda-Tate's thm: $\left\{ \begin{matrix} \text{simple ab.} \\ \text{var. } / \mathbb{F}_q \end{matrix} \right\} / \text{isog} \xrightarrow[1:1]{A \mapsto \pi_A} \{\text{Weil } q\text{-num.}\} / \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\text{-conj.}$

namely, π_A algebraic integer s.t $\forall$ embedding $\varphi : \mathbb{Q}(\pi_A) \longrightarrow \mathbb{C}$ $|\varphi(\pi_A)| = \sqrt{q}$

See examples & proof ($\S 11$: 70min) → 20min left

Rule: skipped: □ Weil's pairing $e_n^\lambda : A[n] \times A[n] \longrightarrow \mathbb{G}_m[n] \Rightarrow e^\lambda : T_\ell(A) \times T_\ell(A) \longrightarrow T_\ell(\mathbb{G}_m) \left(\cong \mathbb{Z}_\ell \text{ as group} \right)$

($\S 5.4$; 15 min)

□ Appendices (Appendix C in particular is crucial to review: (semi simple modules, div. alg, Brauer group.)

Second course outline: Elliptic curves with Complex multiplication. each lec. ~ 60min ⇒ so we need to skip stuff.

Lec 1: def. elliptic curve, $Pic^0(E)$, isogenies, duals, structure $End(E_{\bar{k}})$

DEF if $char(k)=0$ & $\mathbb{Z} \subsetneq End(E_{\bar{k}})$ then we say E has complex multiplication.

Lec 2: elliptic curves / $\mathbb{C}$ as tori (lattices Λ, $E(\mathbb{C}) = \mathbb{C}/\Lambda$, Weierstrass $\wp$-function)

& nec. & suff. condition on Λ s.t. $E : E(\mathbb{C}) = \mathbb{C}/\Lambda$ has CM.

Lec 3: modular form, j-inv & j-inv. of a CM ell. curve is an alg. number, CM ell curves defined over

field & everywhere potentially good reduction.

Lec 4: How CM Ell. curve help? ▫ construct rational points (Heegner pts)

it contains nice thms but I would avoid the actual construction

Lec 5: more on field of def. for E w/ CM & same for End(E). (I would skip this)

Lec 6: reduction of E to finite field & set of primes: $End(E_p)$ is bigger (supersingular primes)

Missing but useful: Newton polygons of isogeny classes, curves