

Sample Complexity of Quadratically Regularized Optimal Transport

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Abstract

It is well known that optimal transport suffers from the curse of dimensionality: when the prescribed marginals are approximated by i.i.d. samples, the convergence of the empirical optimal transport problem to the population counterpart slows exponentially with increasing dimension. Entropically regularized optimal transport (EOT) has become the standard bearer in many statistical applications as it avoids this curse. Indeed, EOT has parametric sample complexity, as has been shown in a series of works based on the smoothness of the EOT potentials or the strong concavity of the dual EOT problem. However, EOT produces full-support approximations to the (sparse) OT problem, leading to overspreading in applications, and is computationally unstable for small regularization parameters.

The most popular alternative is quadratically regularized optimal transport (QOT), which penalizes couplings by L^2 norm instead of relative entropy. QOT produces sparse approximations of OT and is computationally stable. However, its potentials are not smooth (do not belong to a Donsker class) and its dual problem is not strongly concave, hence QOT is often assumed to suffer from the curse of dimensionality. In this paper, we show that QOT nevertheless has parametric sample complexity. More precisely, we establish central limit theorems for its dual potentials, optimal couplings, and optimal costs. Our analysis is based on novel arguments that focus on the regularity of the support of the optimal QOT coupling. Specifically, we establish a Lipschitz property of its sections and leverage VC theory to bound its statistical complexity. Our analysis also leads to gradient estimates of independent interest, including $\mathcal{C}^{1,1}$ regularity of the population potentials.

Keywords Optimal Transport; Quadratic Regularization; Sample Complexity; Central Limit Theorem
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1 Introduction

The optimal transport (OT) problem between compactly supported probability measures P, Q on $\mathbb{R}^d$ is

$$\text{OT}(P, Q) := \min_{\pi \in \Pi(P, Q)} \int \frac{1}{2} \|x - y\|^2 d\pi(x, y), \quad (\text{OT})$$

where $\Pi(P, Q)$ denotes the set of couplings between P and Q ; i.e., probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals (P, Q) . A well-known limitation for its application in high-dimensional statistics and data science is the curse of dimensionality suffered by (OT) when

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the marginals are approximated by samples; see [5] for a recent survey. Regularized optimal transport (ROT) introduces a divergence penalty to (OT),

$$\text{ROT}_\varepsilon(P, Q) := \min_{\pi \in \Pi(P, Q)} \int \frac{1}{2} \|x - y\|^2 d\pi(x, y) + \varepsilon \int \varphi \left(\frac{d\pi}{d(P \otimes Q)} \right) d(P \otimes Q), \quad (\text{ROT})$$

where $d\pi/d(P \otimes Q)$ is the Radon–Nikodym density of π w.r.t. the product measure $P \otimes Q$, $\varepsilon > 0$ is the regularization parameter and φ is a convex function. A standard choice is the logarithmic entropy $\varphi(x) = x \log x$, leading to the entropic optimal transport (EOT) problem. EOT admits efficient computation via Sinkhorn’s algorithm [7] and avoids the curse of dimensionality: when (P, Q) are approximated by the empirical measures corresponding to i.i.d. samples, the empirical optimal costs, couplings, and dual potentials all converge to their population counterparts at the parametric rate $n^{-1/2}$ [14, 25, 9]. In addition, central limit theorems for all three quantities have been established [15, 20, 19, 9]; see also the recent surveys [1, 8]. On the flip side, using logarithmic entropy entails that the optimal coupling of (ROT) has full support (equal to the support of $P \otimes Q$), a phenomenon known as overspreading since the true optimal coupling for (OT) is sparse (given by Brenier’s map) as soon as one marginal is absolutely continuous. Overspreading can lead to blurring in image processing [3] or bias in manifold learning [33], for example.

Starting with [26, 3, 12], quadratic regularization has emerged as the most popular alternative to logarithmic entropy. Indeed, the choice $\varphi(x) = \frac{1}{2}x^2$ gives rise to the squared L^2 norm (or χ^2 divergence) as penalty and leads to the quadratically regularized optimal transport (QOT) problem

$$\text{QOT}_\varepsilon(P, Q) := \min_{\pi \in \Pi(P, Q)} \int \frac{1}{2} \|x - y\|^2 d\pi(x, y) + \frac{\varepsilon}{2} \left\| \frac{d\pi}{d(P \otimes Q)} \right\|_{L^2(P \otimes Q)}^2. \quad (\text{QOT})$$

The optimal cost $\text{QOT}_\varepsilon(P, Q)$ approximates $\text{OT}(P, Q)$ at rate $\varepsilon^{2/(d+2)}$ as $\varepsilon \rightarrow 0$ (see [11, 13]) and, in contrast to EOT, the support of the optimal coupling converges to the support of the unregularized optimal transport. This sparsity for small ε has been observed empirically since the initial works (e.g., [3, 12, 24]) and established theoretically more recently in [27, 32, 21]. A separate advantage is that EOT tends to be computationally unstable for small regularization parameters due to the occurrence of exponentially large/small values, as noted for instance in [23]. Such values do not occur for QOT. While QOT has long been effective in computational practice, we also mention the recent theoretical guarantee [17] that gradient descent for the dual problem of QOT converges exponentially fast.

Denoting $(x)_+ = \max\{x, 0\}$, the dual problem of (QOT) reads

$$\begin{aligned} \sup_{(f, g) \in \mathcal{C}(\mathbb{R}^d) \times \mathcal{C}(\mathbb{R}^d)} & \int f(x) dP(x) + \int g(y) dQ(y) \\ & - \frac{1}{2\varepsilon} \int \left(f(x) + g(y) - \frac{\|x - y\|^2}{2} \right)_+^2 d(P \otimes Q)(x, y). \end{aligned} \quad (1)$$

Its optimizers $(f_\varepsilon, g_\varepsilon)$ are called the QOT potentials and describe the density of the optimal coupling π_ε of (QOT) via

$$d\pi_\varepsilon(x, y) = \frac{1}{\varepsilon} \left(f_\varepsilon(x) + g_\varepsilon(y) - \frac{\|x - y\|^2}{2} \right)_+ d(P \otimes Q)(x, y). \quad (2)$$

We observe that the objective function in the dual problem (1) is not strongly (or even strictly) concave. Moreover, the regularity of $(f_\varepsilon, g_\varepsilon)$ does not generally surpass that of the unregularized OT potentials (e.g., [16]). For general divergences φ , the dual has a form similar to (1) but with $\frac{1}{2}(\cdot)_+^2$ replaced by the conjugate $\psi(s) = \sup_{t \geq 0} \{st - \varphi(t)\}$ of φ . For the particular choice of logarithmic entropy, the dual objective is strongly concave and moreover the potentials inherit the smoothness of the transport cost $\frac{1}{2}\|x - y\|^2$. This is crucial for the aforementioned works on sample complexity and central limit theorems for EOT, which can be divided into two approaches: the first is based on empirical process theory and requires uniform regularity of the empirical dual potentials, the second exploits the strong concavity of the dual EOT problem (see [1, 8, 18] for more detailed reviews). Analyzing more general ROT—where the dual is not strongly concave and the potentials are not smooth—through the same approaches, one obtains results as in [2] which are consistent with ROT suffering from the curse of dimensionality. By contrast, and maybe surprisingly, the recent work [18] establishes central limit theorems for a class of ROT problems that are not smooth or strongly concave. Its main condition on the divergence is that the conjugate $\psi(s) = \sup_{t \geq 0} \{st - \varphi(t)\}$ be $\mathcal{C}^2$. This condition fails for QOT, arguably the most interesting example for ROT, where ψ is $\mathcal{C}^{1,1}$ but not $\mathcal{C}^2$. For QOT, the (weak) second derivative is an indicator function rather than a continuous function, and this difference derails the core arguments in [18].

The present work establishes central limit theorems for the dual potentials, optimal costs, and couplings of QOT (Theorems 3.1 to 3.3). While their assertions are analogous to the ones in [18], the derivations are substantially different. In [18], it is assumed that the conjugate ψ is $\mathcal{C}^2$, and this allows for soft arguments—the marginal measures are very general and the transport cost is any $\mathcal{C}^1$ function. For QOT, this assumption fails and finer structures are needed. We focus on quadratic transport cost $\frac{1}{2}\|x - y\|^2$ and marginals with convex supports and bounded Lebesgue densities; these structures are crucial for our regularity results, which are also some of our main technical contributions.

One of those is a Lipschitz regularity property of the sections $\mathcal{S}_x$ of the support of the optimal population coupling. Representing $\mathcal{S}_x$ as the zero-sublevel set of a convex function, we consider more generally the β -sublevel sets which correspond to a thickening or shrinking of $\mathcal{S}_x$, and establish a Lipschitz estimate w.r.t. β (see Proposition 4.1). Both the result and the broader approach are novel to the best of our knowledge. The result is crucial in several steps of the analysis, such as establishing the Fréchet differentiability of the first-order condition for the population potentials (Lemma 5.1). Another consequence is the $\mathcal{C}^{1,1}$ -regularity of the QOT potentials (Corollary 4.2), which is of independent interest and links to Caffarelli’s regularity theory for (OT).

Another key innovation is to employ Vapnik–Chervonenkis (VC) theory. Specifically, we use VC theory to bound the statistical complexity of the sections $\mathcal{S}_x$ of the support of the optimal coupling (and more generally its thickenings or shrinkings). We package this bound into an abstract tool, Proposition 4.4, which is used several times throughout the proof of the central limit theorems; see also Section 3.1 for an overview of the proof methodology. At a high level, this VC-theoretic bound acts as a substitute for the missing regularity of ψ . To the best of our knowledge, VC theory has not been used previously in the sample complexity analysis of optimal transport or regularized optimal transport.

The remainder of the paper is organized as follows. Section 2 details the setting, notation and assumptions, and summarizes background facts about QOT for later reference.

Section 3 states our three central limit theorems—Theorems 3.1 to 3.3 for the dual potentials, the optimal costs, and the optimal couplings—followed by an overview of the proof methodology. In Section 4 we provide regularity results for the potentials and the optimal supports. These results are some of our main technical contributions; they are of independent interest in addition to being key ingredients for proving the central limit theorems. After these preparations, Section 5 proves the central limit theorem for the potentials, Theorem 3.1. Section 6 proves the remaining central limit theorems for the optimal costs and couplings (Theorems 3.2 and 3.3), which follow from Theorem 3.1 and some additional work. Finally, Appendix A details omitted proofs and references for the background facts in Section 2.

2 Setup and Background

This section details our setup and summarizes (mostly) known background material for later reference. Proofs are deferred to Appendix A.

Given $\mathcal{X} \subset \mathbb{R}^d$, we set $\|\mathcal{X}\|_\infty := \sup_{x \in \mathcal{X}} \|x\|$. We write $\mathcal{C}(\mathcal{X})$ for the space of continuous real-valued functions on $\mathcal{X}$ and $\mathcal{C}^{k,\alpha}(\mathcal{X})$ for the functions with k continuous derivatives whose k -th derivatives are Hölder continuous of order α , $0 < \alpha \leq 1$. A random sequence U_n in a separable Banach space $\mathcal{B}$ converges weakly to U if $\mathbb{E}[f(U_n)] \rightarrow \mathbb{E}[f(U)]$ for every bounded and continuous function $f : \mathcal{B} \rightarrow \mathbb{R}$; in that case, we write $U_n \xrightarrow{\mathcal{B}} U$. When $\mathcal{B} = \mathbb{R}$, we also write $\xrightarrow{w}$ instead of $\xrightarrow{\mathbb{R}}$.

2.1 Population setting

Throughout, we impose the following conditions on the given (population) marginal measures.

Assumption 2.1. The probability measures P, Q have compact, convex supports $\mathcal{X}, \mathcal{Y}$ in $\mathbb{R}^d$ and admit bounded densities w.r.t. the Lebesgue measure $\mathcal{L}_d$.

Given $(f, g) \in \mathcal{C}(\mathcal{X}) \times \mathcal{C}(\mathcal{Y})$, we denote by $f \oplus g$ the function $(x, y) \mapsto f(x) + g(y)$. As we will often be interested only in $f \oplus g$ rather than f and g individually, it is useful to define the equivalence relation $(f, g) \sim_\oplus (f', g')$ iff $f \oplus g = f' \oplus g'$, which is equivalent to the existence of $a \in \mathbb{R}$ with $f = f' + a$ and $g = g' - a$. We define the quotient spaces

$$\mathcal{B}_\oplus = (\mathcal{C}(\mathcal{X}) \times \mathcal{C}(\mathcal{Y})) / \sim_\oplus, \quad \mathcal{B}_\oplus^{0,1} = (\mathcal{C}^{0,1}(\mathcal{X}) \times \mathcal{C}^{0,1}(\mathcal{Y})) / \sim_\oplus \quad (3)$$

and denote by $\|\cdot\|_\oplus$ and $\|\cdot\|_{\oplus,1}$ the quotient norms on $\mathcal{B}_\oplus$ and $\mathcal{B}_\oplus^{0,1}$, respectively.

Lemma 2.2 (Population potentials). *Let P and Q satisfy Assumption 2.1.*

- (i) *There exists a unique pair $(f_\varepsilon, g_\varepsilon) \in \mathcal{B}_\oplus$, called the population potentials, solving the dual problem (1).*
- (ii) *The functions $(\varphi_\varepsilon, \psi_\varepsilon) \in \mathcal{B}_\oplus$ defined by*

$$\varphi_\varepsilon(x) = \frac{\|x\|^2}{2} - f_\varepsilon(x), \quad \psi_\varepsilon(y) = \frac{\|y\|^2}{2} - g_\varepsilon(y) \quad (4)$$

are called the (transformed) potentials. They are uniquely characterized by the first-order optimality condition

$$\begin{cases} \varepsilon = \int (\langle x, y \rangle - \varphi_\varepsilon(x) - \psi_\varepsilon(y))_+ dQ(y) & \text{for all } x \in \mathcal{X}, \\ \varepsilon = \int (\langle x, y \rangle - \varphi_\varepsilon(x) - \psi_\varepsilon(y))_+ dP(x) & \text{for all } y \in \mathcal{Y}. \end{cases} \quad (5)$$

Moreover, φ_ε and ψ_ε are convex and continuously differentiable, with gradients

$$\nabla \varphi_\varepsilon(x) = \frac{\int_{\mathcal{S}_x} y dQ(y)}{Q(\mathcal{S}_x)} \quad \text{and} \quad \nabla \psi_\varepsilon(y) = \frac{\int_{\mathcal{T}_y} x dP(x)}{P(\mathcal{T}_y)}, \quad \text{where} \quad (6)$$

$$\mathcal{S}_x = \{y \in \mathcal{Y} : \varphi_\varepsilon(x) + \psi_\varepsilon(y) \leq \langle x, y \rangle\}, \quad \mathcal{T}_y = \{x \in \mathcal{X} : \varphi_\varepsilon(x) + \psi_\varepsilon(y) \leq \langle x, y \rangle\}. \quad (7)$$

In particular, φ_ε and ψ_ε are Lipschitz with constants $\|\mathcal{Y}\|_\infty$ and $\|\mathcal{X}\|_\infty$, respectively.

(iii) The sets $\mathcal{S}_x$ and $\mathcal{T}_y$ are convex, have nonempty interior, boundaries given by

$$\partial \mathcal{S}_x = \{y \in \mathcal{Y} : \varphi_\varepsilon(x) + \psi_\varepsilon(y) = \langle x, y \rangle\}, \quad \partial \mathcal{T}_y = \{x \in \mathcal{X} : \varphi_\varepsilon(x) + \psi_\varepsilon(y) = \langle x, y \rangle\} \quad (8)$$

and there exists $\delta > 0$ such that

$$\delta \leq Q(\mathcal{S}_x) \quad \text{and} \quad \delta \leq P(\mathcal{T}_y) \quad \text{for all } x \in \mathcal{X}, y \in \mathcal{Y}. \quad (9)$$

(iv) The primal problem (QOT) has a unique solution $\pi_\varepsilon \in \Pi(P, Q)$. It is related to the potentials by

$$\frac{d\pi_\varepsilon}{d(P \otimes Q)}(x, y) = \frac{1}{\varepsilon} \left(f_\varepsilon(x) + g_\varepsilon(y) - \frac{\|x - y\|^2}{2} \right)_+ = \frac{1}{\varepsilon} (\langle x, y \rangle - \varphi_\varepsilon(x) - \psi_\varepsilon(y))_+ \quad (10)$$

and its support is $\text{spt } \pi_\varepsilon = \{(x, y) \in \mathcal{X} \times \mathcal{Y} : \varphi_\varepsilon(x) + \psi_\varepsilon(y) \leq \langle x, y \rangle\}$, so that $\mathcal{S}_x$ and $\mathcal{T}_y$ can be interpreted as the sections of the support.

We mention that the continuous differentiability of the population potentials will be improved to $\mathcal{C}^{1,1}$ -regularity in Corollary 4.2. We will find it convenient to use both $(f_\varepsilon, g_\varepsilon)$ and $(\varphi_\varepsilon, \psi_\varepsilon)$, and refer to either as potentials. The former pair helps to make the main results more comparable to the related literature whereas the proofs use the latter, in order to benefit from its convexity properties.

2.2 Empirical setting

Next, we turn to the empirical setting. Let $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} P$ and $Y_1, \dots, Y_n \stackrel{\text{iid}}{\sim} Q$ be independent samples, and denote by P_n and Q_n the corresponding empirical measures. For a fixed realization of the samples and hence of the empirical measures, we can again consider the primal and dual problems (QOT) and (1), now with the empirical measures as marginals instead of (P, Q) . The primal problem again has a unique solution, denoted $\pi_n \in \Pi(P_n, Q_n)$. Again, there exists a pair (f_n, g_n) solving the dual and describing the primal optimizer via the analogue of (10) with (P_n, Q_n) instead of (P, Q) . One difference is that, because the marginal

supports are now discrete and hence not connected, (f_n, g_n) are in general non-unique. Another difference is that (f_n, g_n) are a priori only defined on those discrete supports. We may, however, extend them continuously to the population supports $\mathcal{X}$ and $\mathcal{Y}$ (or even all of $\mathbb{R}^d$), in such a way that the extension solves the first-order condition on those sets. Conversely, any such solution defines valid potentials. Passing again to transformed potentials (φ_n, ψ_n) , the latter satisfy convexity and Lipschitz properties that will be useful below.

Lemma 2.3 (Empirical potentials). *Given a realization of the empirical marginals (P_n, Q_n) , there exists a pair $(f_n, g_n) \in \mathcal{B}_\oplus$, called empirical potentials, solving the dual problem (1) for (P_n, Q_n) and such that $(\varphi_n, \psi_n) \in \mathcal{B}_\oplus$ defined by*

$$\varphi_n(x) = \frac{\|x\|^2}{2} - f_n(x), \quad \psi_n(y) = \frac{\|y\|^2}{2} - g_n(y) \quad (11)$$

satisfy

$$\begin{cases} \varepsilon = \int (\langle x, y \rangle - \varphi_n(x) - \psi_n(y))_+ dQ_n(y) & \text{for all } x \in \mathcal{X}, \\ \varepsilon = \int (\langle x, y \rangle - \varphi_n(x) - \psi_n(y))_+ dP_n(x) & \text{for all } y \in \mathcal{Y}. \end{cases} \quad (12)$$

Moreover, $\varphi_n : \mathcal{X} \rightarrow \mathbb{R}$ is convex and $\mathcal{L}_d$ -a.e. differentiable with

$$\nabla \varphi_n(x) = \frac{\int_{\hat{\mathcal{S}}_x} y dQ_n(y)}{Q_n(\hat{\mathcal{S}}_x)} \quad \text{for } \mathcal{L}_d\text{-a.e. } x \in \mathcal{X}, \quad (13)$$

where $\hat{\mathcal{S}}_x = \{y \in \mathcal{Y} : \varphi_n(x) + \psi_n(y) \leq \langle x, y \rangle\}$. In particular, φ_n is Lipschitz with constant $\|\mathcal{Y}\|_\infty$. An analogous result holds for ψ_n .

In contrast to Corollary 4.2 for the population potentials, the regularity stated above cannot be improved for the empirical potentials: as $x \mapsto Q_n(\hat{\mathcal{S}}_x)$ is integer-valued, the gradient (13) is discontinuous and fails to exist at certain points (except in trivial cases).

The empirical potentials (f_n, g_n) , or equivalently (φ_n, ψ_n) , are random and possibly non-unique. For the remainder of the paper, we choose and fix one pair; all subsequent statements are valid as long as the selection is measurable and satisfies (12).

The last result of this section is the consistency of the empirical potentials towards the population counterpart.

Lemma 2.4 (Consistency). *We have $\lim_{n \rightarrow \infty} \|(\varphi_n, \psi_n) - (\varphi_\varepsilon, \psi_\varepsilon)\|_\oplus = 0$ a.s.*

3 The central limit theorems

This section states our three central limit theorems for the dual potentials, optimal costs and couplings. The theorem for the potentials is stated in the Banach space $\mathcal{B}_\oplus$ defined in (3) and the limit is described using the linear operator $[\mathbb{L}]_\oplus$, defined as the composition of

$$\begin{aligned} \mathbb{L} : \mathcal{B}_\oplus &\rightarrow \mathcal{C}(\mathcal{X}) \times \mathcal{C}(\mathcal{Y}) \\ \begin{pmatrix} f \\ g \end{pmatrix} &\mapsto \begin{pmatrix} f \\ g \end{pmatrix} + \begin{pmatrix} \frac{1}{Q(\mathcal{S}_{(\cdot)})} \int_{\mathcal{S}_{(\cdot)}} g(y) dQ(y) \\ \frac{1}{P(\mathcal{T}_{(\cdot)})} \int_{\mathcal{T}_{(\cdot)}} f(x) dP(x) \end{pmatrix} \end{aligned} \quad (14)$$

and the quotient map $[\cdot]_{\oplus} : \mathcal{C}(\mathcal{X}) \times \mathcal{C}(\mathcal{Y}) \rightarrow \mathcal{B}_{\oplus}$. We write $\mathcal{S}_{(\cdot)}$ and $\mathcal{T}_{(\cdot)}$ for the set-valued mappings $x \mapsto \mathcal{S}_x$ and $y \mapsto \mathcal{T}_y$ defined in (7); moreover, we use the shorthand

$$\xi_{\varepsilon}(x, y) := f_{\varepsilon}(x) + g_{\varepsilon}(y) - \frac{1}{2}\|x - y\|^2.$$

Theorem 3.1 (CLT for potentials). *The operator $[\mathbb{L}]_{\oplus} : \mathcal{B}_{\oplus} \rightarrow \mathcal{B}_{\oplus}$ admits a bounded inverse $[\mathbb{L}]_{\oplus}^{-1}$ and we have*

$$\sqrt{n} \begin{pmatrix} f_n - f_{\varepsilon} \\ g_n - g_{\varepsilon} \end{pmatrix} \xrightarrow{\mathcal{B}_{\oplus}} -[\mathbb{L}]_{\oplus}^{-1} \left[\begin{pmatrix} \frac{\mathbf{G}_Q}{Q(\mathcal{S}_{(\cdot)})} \\ \frac{\mathbf{G}_P}{P(\mathcal{T}_{(\cdot)})} \end{pmatrix} \right]_{\oplus},$$

where $(\mathbf{G}_Q, \mathbf{G}_P) \in \mathcal{C}(\mathcal{X}) \times \mathcal{C}(\mathcal{Y})$ is the unique-in-law pair of independent, tight, centered Gaussian processes with covariances

$$\begin{aligned} \mathbb{E}[\mathbf{G}_Q(x)\mathbf{G}_Q(x')] &= \mathbb{E}[(\xi_{\varepsilon}(x, Y) - \mathbb{E}[\xi_{\varepsilon}(x, Y)])(\xi_{\varepsilon}(x', Y) - \mathbb{E}[\xi_{\varepsilon}(x', Y)])], \\ \mathbb{E}[\mathbf{G}_P(y)\mathbf{G}_P(y')] &= \mathbb{E}[(\xi_{\varepsilon}(X, y) - \mathbb{E}[\xi_{\varepsilon}(X, y)])(\xi_{\varepsilon}(X, y') - \mathbb{E}[\xi_{\varepsilon}(X, y')])]. \end{aligned}$$

for $X \sim P$ and $Y \sim Q$.

The proof of Theorem 3.1 is given in Section 5. Next, we state the central limit theorem for the optimal costs.

Theorem 3.2 (CLT for costs). *We have*

$$\sqrt{n}(\text{QOT}(P_n, Q_n) - \text{QOT}(P, Q)) \xrightarrow{w} N(0, \sigma^2),$$

where the variance σ^2 is that of the random variable

$$f_{\varepsilon}(X) + g_{\varepsilon}(Y) - \frac{1}{2\varepsilon} \left(\int (\xi_{\varepsilon}(x, Y))_+^2 dP(x) + \int (\xi_{\varepsilon}(X, y))_+^2 dQ(y) \right) \quad (15)$$

for $(X, Y) \sim P \otimes Q$.

The proof is given in Section 6. Finally, we state the central limit theorem for the optimal couplings; the proof is also reported in Section 6. To describe the limiting variance in a more compact form, we use the notation $\oplus(f, g) := f \oplus g$.

Theorem 3.3 (CLT for couplings). *For any bounded measurable function $\eta : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$,*

$$\sqrt{n} \left(\int \eta d(\pi_n - \pi_{\varepsilon}) \right) \xrightarrow{w} N \left(0, \frac{\sigma^2(\eta)}{\varepsilon^2} \right)$$

with $\sigma^2(\eta) = \text{Var}(V_X + V_Y)$ where, for $Z \in \{X, Y\}$,

$$V_Z := \mathbb{E} \left[\int U(x, y, X, Y) \bar{\eta}(x, y) (\xi_{\varepsilon}(x, y))_+ d(P \otimes Q)(x, y) - \bar{\eta}(X, Y) (\xi_{\varepsilon}(X, Y))_+ \middle| Z \right]$$

for $(X, Y) \sim P \otimes Q$ and $\bar{\eta} := \eta - \int_{\xi_{\varepsilon} \geq 0} \eta d(P \otimes Q)$ and

$$U(\cdot, \cdot, X, Y) := \oplus \left([\mathbb{L}]_{\oplus}^{-1} \left[\begin{pmatrix} \frac{(\xi_{\varepsilon}(\cdot, Y))_+}{Q(\mathcal{S}_{(\cdot)})} \\ \frac{(\xi_{\varepsilon}(X, \cdot))_+}{P(\mathcal{T}_{(\cdot)})} \end{pmatrix} \right] \right)_{\oplus}.$$

3.1 Proof methodology

The key result is Theorem 3.1 on the potentials; Theorems 3.2 and 3.3 on the costs and couplings are derived from it. We follow the scheme of Z-estimation (see [30, Chapter 3.3]), as in [18], based on the fact that the population and empirical potentials are solutions to certain equations. Specifically, the first-order conditions (5) and (12), which we represent with nonlinear operators Φ and Φ_n on $\mathcal{B}_\oplus$ as

$$\begin{pmatrix} \varepsilon \\ \varepsilon \end{pmatrix} = \Phi \begin{pmatrix} \varphi_\varepsilon \\ \psi_\varepsilon \end{pmatrix}, \quad \begin{pmatrix} \varepsilon \\ \varepsilon \end{pmatrix} = \Phi_n \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix}.$$

Following the Z-estimation methodology, we need to establish:

- (i) $\Phi_n(\varphi_\varepsilon, \psi_\varepsilon) - \Phi(\varphi_\varepsilon, \psi_\varepsilon)$ satisfies a central limit theorem in $\mathcal{B}_\oplus$,
- (ii) Φ is Fréchet differentiable at $(\varphi_\varepsilon, \psi_\varepsilon)$ with invertible derivative,
- (iii) $\|(\varphi_n, \psi_n) - (\varphi_\varepsilon, \psi_\varepsilon)\|_\oplus \rightarrow 0$ in probability,
- (iv) the following expansion holds,

$$\begin{aligned} & \left\| \Phi_n \begin{pmatrix} \varphi_\varepsilon \\ \psi_\varepsilon \end{pmatrix} - \Phi \begin{pmatrix} \varphi_\varepsilon \\ \psi_\varepsilon \end{pmatrix} - \left(\Phi_n \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} - \Phi \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} \right) \right\|_\oplus \\ &= o_{\mathbb{P}} \left(\left\| \begin{pmatrix} \varphi_n - \varphi_\varepsilon \\ \psi_n - \psi_\varepsilon \end{pmatrix} \right\|_\oplus + n^{-\frac{1}{2}} \right). \end{aligned}$$

Item (i) follows easily from the fact that

$$\begin{aligned} & \Phi_n \begin{pmatrix} \varphi_\varepsilon \\ \psi_\varepsilon \end{pmatrix} - \Phi \begin{pmatrix} \varphi_\varepsilon \\ \psi_\varepsilon \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{n} \sum_{i=1}^n (\langle \cdot, Y_i \rangle - \varphi_\varepsilon(\cdot) - \psi_\varepsilon(Y_i))_+ - \int (\langle \cdot, y \rangle - \varphi_\varepsilon(\cdot) - \psi_\varepsilon(y))_+ dQ(y) \\ \frac{1}{n} \sum_{i=1}^n (\langle X_i, \cdot \rangle - \varphi_\varepsilon(X_i) - \psi_\varepsilon(\cdot))_+ - \int (\langle x, \cdot \rangle - \varphi_\varepsilon(x) - \psi_\varepsilon(\cdot))_+ dP(x) \end{pmatrix} \end{aligned}$$

is an average of an i.i.d. sequence of uniformly Lipschitz random functions. For (ii), the Fréchet differentiability (shown in Lemma 5.1) hinges on a novel Lipschitz regularity property of the sections $\mathcal{S}_x$ of the optimal support, formalized in Proposition 4.1. More specifically, recognizing that $\mathcal{S}_x$ is the zero-sublevel set of a convex function, we consider the β -sublevel sets which correspond to a thickening or shrinking of $\mathcal{S}_x$, and establish a Lipschitz estimate w.r.t. β . This property is one of the key technical innovations of the paper. The invertibility of the derivative is shown in Lemma 5.2 via the Fredholm alternative, completing (ii). The consistency (iii) of the potentials was already stated in Lemma 2.4; it follows by a standard Arzelà–Ascoli argument. The main task is to establish (iv). This property follows from two estimates that we state in Lemma 5.3 and Lemma 4.6, respectively:

$$\begin{aligned} & \left\| \Phi_n \begin{pmatrix} \varphi_\varepsilon \\ \psi_\varepsilon \end{pmatrix} - \Phi \begin{pmatrix} \varphi_\varepsilon \\ \psi_\varepsilon \end{pmatrix} - \left(\Phi_n \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} - \Phi \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} \right) \right\|_\oplus \\ &= o_{\mathbb{P}} \left(\|\nabla \varphi_n - \nabla \varphi_\varepsilon\|_\infty + \|\nabla \psi_n - \nabla \psi_\varepsilon\|_\infty + n^{-\frac{1}{2}} \right) \end{aligned}$$

and

$$\|\nabla\varphi_n - \nabla\varphi_\varepsilon\|_\infty + \|\nabla\psi_n - \nabla\psi_\varepsilon\|_\infty = \mathcal{O}_{\mathbb{P}}\left(\|(\varphi_n, \psi_n) - (\varphi_\varepsilon, \psi_\varepsilon)\|_\oplus + n^{-\frac{1}{2}}\right).$$

In addition to the aforementioned Lipschitz regularity property of the sections of the optimal support (Proposition 4.1), both estimates rely on another key innovation of this paper, namely to utilize VC theory to bound the complexity of $\mathcal{S}_x$ (and its generalizations by thickening); this gives rise to the term $\mathcal{O}_{\mathbb{P}}(n^{-\frac{1}{2}})$. As similar VC arguments are used several times throughout the proof of the central limit theorems, we formulate a slightly more general estimate as an abstract tool (Proposition 4.4), with the benefit of streamlining the later proofs. To the best of our knowledge, VC theory has not been used previously in the sample complexity analysis of optimal transport or regularized optimal transport.

4 Regularity estimates

This section provides several estimates which are key to the proof of the central limit theorems, but also of independent interest.

4.1 Regularity of the population support and potentials

The first result studies the section $\mathcal{S}_x$ of the optimal support; it can be seen as the zero-sublevel set of a function involving the population potentials $\varphi_\varepsilon, \psi_\varepsilon$ of Lemma 2.2. The proposition asserts a Lipschitz regularity with respect to the level β which acts as a parameter to define a thickening (or shrinking) of $\mathcal{S}_x$.

Proposition 4.1. *Define the set-valued mapping*

$$\mathbb{R} \ni \beta \mapsto \mathcal{S}_x(\beta) := \{y \in \mathcal{Y} : \varphi_\varepsilon(x) + \psi_\varepsilon(y) \leq \langle x, y \rangle + \beta\}. \quad (16)$$

There exist $L, \beta_0 > 0$ such that

$$\sup_{\|g\|_\infty \leq 1} \sup_{x \in \mathcal{X}} \left| \int_{\mathcal{S}_x(\alpha)} g(y) dQ(y) - \int_{\mathcal{S}_x(\beta)} g(y) dQ(y) \right| \leq L|\alpha - \beta| \quad \text{for all } \alpha, \beta \in (-\beta_0, \beta_0),$$

where the outer supremum is taken over measurable functions $g : \mathcal{Y} \rightarrow \mathbb{R}$.

Before proving the proposition, we use it to show $\mathcal{C}^{1,1}$ -regularity of the population potentials. While this will not be the only application of Proposition 4.1, it is a good illustration of how this abstract result can be used.

Corollary 4.2. *The population potential φ_ε belongs to $\mathcal{C}^{1,1}(\mathcal{X})$; that is, its gradient*

$$x \mapsto \nabla\varphi_\varepsilon(x) = \frac{\int_{\mathcal{S}_x} y dQ(y)}{Q(\mathcal{S}_x)} \quad (17)$$

is Lipschitz continuous. Similarly, the mass $x \mapsto Q(\mathcal{S}_x)$ is Lipschitz continuous. The analogue holds for ψ_ε and $y \mapsto P(\mathcal{T}_y)$.

Proof. Recall that φ_ε is ℓ -Lipschitz (with $\ell = \|\mathcal{Y}\|_\infty$) and note that this implies

$$\mathcal{S}_x(-\ell\|x - z\|) \subset \mathcal{S}_z \subset \mathcal{S}_x(\ell\|x - z\|) \quad \text{for all } x, z \in \mathcal{X}.$$

Let $L, \beta_0 > 0$ be as in Proposition 4.1. Then Proposition 4.1 with $\alpha = 0$, $\beta = \mp\ell\|x - z\|$ and $g \equiv 1$ implies for all x, z with $\ell\|x - z\| \leq \beta_0$ that

$$Q(\mathcal{S}_x) - L\ell\|x - z\| \leq Q(\mathcal{S}_x(-\ell\|x - z\|)) \leq Q(\mathcal{S}_z) \leq Q(\mathcal{S}_x(\ell\|x - z\|)) \leq Q(\mathcal{S}_x) + L\ell\|x - z\|.$$

That is, $x \mapsto Q(\mathcal{S}_x)$ is Lipschitz with constant $L\ell$. Similarly, Proposition 4.1 yields that

$$x \mapsto \int_{\mathcal{S}_x} y_i dQ(y)$$

is Lipschitz with constant $L\ell\|\mathcal{Y}\|_\infty$, where y_i denotes the i -th component of $y \in \mathcal{Y}$. Recalling the uniform lower bound (9) for the denominator in (17), it follows that $\nabla\varphi_\varepsilon$ is Lipschitz, which was the claim. $\square$

Remark 4.3. Corollary 4.2 links to the regularity theory for unregularized optimal transport, where the gradient of the potential is the optimal transport map. The standard Caffarelli–Urbas conditions [4, 28] for Lipschitz continuity of the transport map require in particular that the marginal densities be (bounded and) bounded away from zero and the supports be convex with Lipschitz boundaries. The condition that densities be bounded away from zero is essential, as illustrated by Wang’s counterexamples [31].

In Corollary 4.2, the regularization parameter ε is positive and fixed. In light of the aforementioned results, one cannot expect the constant L to be uniform in ε , as the estimate would then extend to the unregularized limit. Obtaining such a uniform estimate (under stronger conditions) is an interesting open problem. It is equivalent to a uniform bound for the Hessian of φ_ε and closely related to establishing the sharp rate $\varepsilon^{\frac{1}{d+2}}$ for the diameter of $\mathcal{S}_x$ as $\varepsilon \rightarrow 0$ (which was shown in [16] for the case $d = 1$).

Proof of Proposition 4.1. Fix $x_0 \in \mathcal{X}$. Recall from Lemma 2.2 that $\mathcal{S}_{x_0}$ is convex, that $\text{int } \mathcal{S}_{x_0} \neq \emptyset$, and the representation (8). Changing the coordinate system, we can assume without loss of generality that

$$0 \in \text{int } \mathcal{S}_{x_0} = \{y \in \mathcal{Y} : \varphi_\varepsilon(x_0) + \psi_\varepsilon(y) - \langle x_0, y \rangle < 0\}.$$

As φ_ε and ψ_ε are Lipschitz, this representation yields that there exists $\delta_0 > 0$ such that $0 \in \text{int } \mathcal{S}_x(\delta)$ for all $(x, \delta) \in \mathcal{X} \times \mathbb{R}$ with $|\delta| + \|x - x_0\| \leq \delta_0$.

Consider such (x, δ) and let $g : \mathbb{R}^d \rightarrow \mathbb{R}$ be bounded and measurable with $\|g\|_\infty \leq 1$. By the co-area formula,

$$\int_{\mathcal{S}_x(\delta)} g(y) dQ(y) = \int_{\mathbb{S}^{d-1}} \int_0^{t(x,v,\delta)} g(rv) q(rv) dr d\mathcal{H}^{d-1}(v),$$

where $t(x, v, \delta)$ is the unique number $t > 0$ such that $tv \in \partial\mathcal{S}_x(\delta)$, q denotes the Lebesgue density of Q (which is bounded by Assumption 2.1) and $\mathcal{H}^{d-1}$ denotes $(d-1)$ -dimensional Hausdorff measure on $\mathbb{S}^{d-1}$. Next, we show that $t(x, v, \delta)$ is locally Lipschitz. By [10,

Theorem 5.4], there exists a convex Lipschitz function h_Q such that $\mathcal{Y} = \{y : h_Q(y) \leq 0\}$ and $\partial\mathcal{Y} = \{y : h_Q(y) = 0\}$. We then introduce the function

$$\gamma(t, x, v, \delta) := \max \{ \varphi_\varepsilon(x) + \psi_\varepsilon(tv) - t\langle v, x \rangle + \delta, h_Q(tv) \}.$$

Note that $t \mapsto \gamma(t, x, v, \delta)$ is convex, that $0 \in \text{int } \mathcal{S}_x(\delta)$ translates to $\gamma(0, x, v, \delta) < 0$, and

$$\gamma(t(x, v, \delta), x, v, \delta) = 0.$$

We can see this equation as an implicit definition of $t(x, v, \delta)$. Indeed, fix $v_0 \in \mathbb{S}^{d-1}$ and note that $\gamma(t(x_0, v_0, 0), x_0, v_0, 0) = 0$ and $\gamma(0, x_0, v_0, 0) < 0$ imply that $t(x_0, v_0, 0)$ cannot be a minimum of $t \mapsto \gamma(t, x_0, v_0, 0)$. As a consequence, zero is not in the subdifferential,

$$\begin{aligned} 0 &\notin \partial_1 \gamma(t(x_0, v_0, 0), x_0, v_0, 0) \\ &= \{t \in \mathbb{R} : \gamma(t(x_0, v_0, 0), x_0, v_0, 0) \leq \gamma(s, x_0, v_0, 0) + t \cdot (t(x_0, v_0, 0) - s), \forall s \in \mathbb{R}\}. \end{aligned}$$

As $\gamma(t, x, v, \delta)$ is jointly Lipschitz in a neighborhood of $(t(x_0, v_0, 0), x_0, v_0, 0)$, Clarke's implicit function theorem (more precisely, the corollary on [6, p. 256]) then yields a neighborhood of $(x_0, v_0, 0)$ where the function $t(x, v, \delta)$ is Lipschitz with some constant ℓ . By a compactness argument, the constant ℓ can be taken uniformly in the variable $v \in \mathbb{S}^{d-1}$. As a consequence, there exists δ_0 such that $|t(x, v, \delta) - t(x, v, \beta)| \leq \ell|\delta - \beta|$ for all x, δ, β with $\|x - x_0\| < \delta_0$ and $\delta, \beta \in (-\delta_0, \delta_0)$ and all $v \in \mathbb{S}^{d-1}$. Since g and q are bounded, it follows for all such x, δ, β that

$$\left| \int_{\mathcal{S}_x(\delta)} g(y) dQ(y) - \int_{\mathcal{S}_x(\beta)} g(y) dQ(y) \right| \leq \ell \|g\|_\infty \|q\|_\infty |\beta - \delta|. \quad (18)$$

Here ℓ and δ_0 depend on x_0 , and (18) holds only for x in a ball around x_0 . However, the compact set $\mathcal{X}$ is covered by finitely many such balls, hence (18) holds uniformly in $x \in \mathcal{X}$ after possibly changing ℓ and δ_0 . $\square$

4.2 VC bounds for statistical complexity of the optimal support

The next proposition encapsulates a tool that will be used in several of the most important steps towards the central limit theorems. It uses VC theory to bound the statistical complexity of the sublevel sets $\mathcal{S}_x(\cdot)$ from (16) to guarantee that a given class of functions satisfies a Donsker-type property.

Proposition 4.4. *Let $g : \mathcal{Y} \rightarrow \mathbb{R}$ be bounded measurable and let Λ be any index set. For each $\lambda \in \Lambda$ and $x \in \mathcal{X}$, consider two measurable random functions $V_{n,\lambda,x}, W_{n,\lambda,x} : \mathcal{Y} \rightarrow \mathbb{R}$ such that*

$$\mathbf{1}_{\mathcal{S}_x(\alpha_n)} \leq V_{n,\lambda,x}, W_{n,\lambda,x} \leq \mathbf{1}_{\mathcal{S}_x(\beta_n)} \quad Q\text{-a.s. for all } n \in \mathbb{N}$$

for some real-valued random variables α_n, β_n with $\alpha_n \rightarrow 0$ and $\beta_n \rightarrow 0$ in probability $\mathbb{P}$. Then

$$\sup_{\lambda \in \Lambda, x \in \mathcal{X}} \left| \int V_{n,\lambda,x}(y) g(y) dQ(y) - \int W_{n,\lambda,x}(y) g(y) dQ_n(y) \right| = \mathcal{O}_{\mathbb{P}} \left(n^{-\frac{1}{2}} + |\alpha_n - \beta_n| \right).$$

Proof. Decomposing $g = (g)_+ - (g)_-$ and using the triangle inequality, we may assume w.l.o.g. that $g \geq 0$. Set

$$D_{n,\lambda,x} := \int V_{n,\lambda,x}(y)g(y)dQ(y) - \int W_{n,\lambda,x}(y)g(y)dQ_n(y).$$

As $g \geq 0$, the assumed inequalities yield

$$D_{n,\lambda,x} \leq \int_{\mathcal{S}_x(\beta_n)} g(y)dQ(y) - \int_{\mathcal{S}_x(\alpha_n)} g(y)dQ_n(y)$$

and hence

$$D_{n,\lambda,x} \leq \left| \int_{\mathcal{S}_x(\beta_n)} g(y)dQ(y) - \int_{\mathcal{S}_x(\alpha_n)} g(y)dQ(y) \right| + \left| \int_{\mathcal{S}_x(\alpha_n)} g(y)d(Q - Q_n)(y) \right|. \quad (19)$$

Let $L, \beta_0 > 0$ be the constants from Proposition 4.1. Then by Proposition 4.1, the first term on the right-hand side of (19) is bounded by $\|g\|_\infty L|\alpha_n - \beta_n|$ on the event $\{|\alpha_n|, |\beta_n| \leq \beta_0\}$ which satisfies $\mathbb{P}(|\alpha_n|, |\beta_n| \leq \beta_0) \rightarrow 1$ as $n \rightarrow \infty$.

We can thus focus on the last term of (19); our goal is to show that it is of order $\mathcal{O}_{\mathbb{P}}(n^{-\frac{1}{2}})$. Note that

$$\left| \int_{\mathcal{S}_x(\alpha_n)} g(y)d(Q - Q_n)(y) \right| \leq \sup_{\delta \in \mathbb{R}} \left| \int_{\mathcal{S}_x(\delta)} g(y)d(Q - Q_n)(y) \right|.$$

We use VC theory (see [30, Section 2.6]) to bound the right-hand side. Define the VC dimension (or index) of a collection $\mathcal{C}$ of subsets of a set $\mathcal{X}$ as in [30, Section 2.6.1], namely

$$\mathcal{V}(\mathcal{C}) := \inf \left\{ n \in \mathbb{N} : \max_{y_1, \dots, y_n \in \mathcal{X}} \Delta_n(\mathcal{C}, y_1, \dots, y_n) < 2^n \right\}$$

where

$$\Delta_n(\mathcal{C}, y_1, \dots, y_n) := \# \{C \cap \{y_1, \dots, y_n\} : C \in \mathcal{C}\}.$$

Recall also that the VC dimension of a class $\mathcal{F}$ of functions $f : \mathcal{X} \rightarrow \mathbb{R}$ is defined as the VC dimension of the collection of all its subgraphs $\{(y, t) \in \mathcal{X} \times \mathbb{R} : t < f(y)\}$.

Consider the collection

$$\mathcal{C} := \{\mathcal{S}_x(\delta) : \delta \in \mathbb{R}, x \in \mathcal{X}\}$$

of subsets of $\mathcal{Y}$. To verify that $\mathcal{V}(\mathcal{C}) < \infty$, we consider the following linear space of functions on $\mathcal{Y}$,

$$\mathcal{G} := \left\{ y \mapsto \beta_1 \psi(y) - \langle x, y \rangle + \beta_2 : (x, \beta_1, \beta_2) \in \mathbb{R}^{d+2} \right\}$$

and note that $\mathcal{G}$ has algebraic dimension $d + 2$. By [30, Lemma 2.6.15], it follows that $\mathcal{V}(\mathcal{G}) \leq d + 4 < \infty$. Given a function class of finite VC dimension, the class of its zero-sublevel sets has finite VC dimension by [30, Lemmas 2.6.17 and 2.6.18]. Noting

$$\mathcal{C} \subset \{g \in \mathcal{G} : g \leq 0\},$$

it follows that $\mathcal{V}(\mathcal{C}) < \infty$. Using [30, Lemma 2.6.18], this implies that the function class

$$\mathcal{F} := \{g \mathbf{1}_C : C \in \mathcal{C}\}$$

has finite VC dimension. Hence, [30, Theorem 2.6.8] yields that $\mathcal{F}$ is Q -Donsker. (The condition of [30, Theorem 2.6.8] holds due to the uniform boundedness of the functions in $\mathcal{F}$, as observed in the proof of that theorem in [30].) We conclude by [30, Corollary 2.3.12] that

$$\sup_{x \in \mathcal{X}, \delta \in \mathbb{R}} \left| \int_{\mathcal{S}_x(\delta)} g(y) d(Q - Q_n)(y) \right| = \mathcal{O}_{\mathbb{P}} \left(n^{-\frac{1}{2}} \right).$$

So far, we have shown that $\sup_{\lambda, x} D_{n, \lambda, x} \leq \mathcal{O}_{\mathbb{P}}(n^{-\frac{1}{2}} + |\alpha_n - \beta_n|)$. Analogously, we obtain a lower bound of the same order by exchanging the roles of α_n and β_n in the above argument, completing the proof. $\square$

Next, we record a version of Proposition 4.4 on the product space $\mathcal{X} \times \mathcal{Y}$ instead of $\mathcal{Y}$; it will be used in the proof of the central limit theorem for the optimal couplings (Theorem 3.3). The proof is analogous to the one of Proposition 4.4 and hence omitted.

Proposition 4.5. *Define*

$$\mathcal{M}(\beta) := \{(x, y) \in \mathcal{X} \times \mathcal{Y} : \varphi_{\varepsilon}(x) + \psi_{\varepsilon}(y) \leq \langle x, y \rangle + \beta\} \quad \text{for } \beta \in \mathbb{R}. \quad (20)$$

Let $g : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ be bounded measurable and let Λ be any index set. For each $\lambda \in \Lambda$, consider two measurable random functions $V_{n, \lambda}, W_{n, \lambda} : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ such that

$$\mathbf{1}_{\mathcal{M}(\alpha_n)} \leq V_{n, \lambda}, W_{n, \lambda} \leq \mathbf{1}_{\mathcal{M}(\beta_n)} \quad P \otimes Q\text{-a.s. for all } n \in \mathbb{N}$$

for some real-valued random variables α_n, β_n with $\alpha_n \rightarrow 0$ and $\beta_n \rightarrow 0$ in probability $\mathbb{P}$. Then

$$\begin{aligned} \sup_{\lambda \in \Lambda} \left| \int V_{n, \lambda}(x, y) g(x, y) d(P \otimes Q)(x, y) - \int W_{n, \lambda}(x, y) g(x, y) d(P_n \otimes Q_n)(x, y) \right| \\ = \mathcal{O}_{\mathbb{P}} \left(n^{-\frac{1}{2}} + |\alpha_n - \beta_n| \right). \end{aligned}$$

4.3 Statistical gradient estimate

Our first application of Proposition 4.4 is a gradient estimate of statistical nature: we bound the difference $\nabla \varphi_n - \nabla \varphi_{\varepsilon}$ of the empirical and population gradients by the difference of the potentials themselves and an error term of order $\mathcal{O}_{\mathbb{P}}(n^{-\frac{1}{2}})$. This estimate will be instrumental for our subsequent arguments towards the CLT of Theorem 3.1, specifically for Lemma 5.3.

Lemma 4.6. *We have*

$$\|\nabla \varphi_n - \nabla \varphi_{\varepsilon}\|_{\infty} = \mathcal{O}_{\mathbb{P}} \left(n^{-\frac{1}{2}} + \|(\varphi_n, \psi_n) - (\varphi_{\varepsilon}, \psi_{\varepsilon})\|_{\oplus} \right).$$

As a consequence, the norm $\|\cdot\|_{\oplus, 1}$ on $\mathcal{B}_{\oplus}^{0,1} = (\mathcal{C}^{0,1}(\mathcal{X}) \times \mathcal{C}^{0,1}(\mathcal{Y})) / \sim_{\oplus}$ satisfies

$$\|(\varphi_n, \psi_n) - (\varphi_{\varepsilon}, \psi_{\varepsilon})\|_{\oplus, 1} = \mathcal{O}_{\mathbb{P}} \left(n^{-\frac{1}{2}} + \|(\varphi_n, \psi_n) - (\varphi_{\varepsilon}, \psi_{\varepsilon})\|_{\oplus} \right).$$

Proof. Recall (6) and (13), and in particular the random sets $\hat{\mathcal{S}}_x = \hat{\mathcal{S}}_{x,n}$. Decompose

$$\begin{aligned} \nabla \varphi_\varepsilon(x) - \nabla \varphi_n(x) &= \frac{\int_{\mathcal{S}_x} y dQ(y)}{Q(\mathcal{S}_x)} - \frac{\int_{\hat{\mathcal{S}}_x} y dQ_n(y)}{Q_n(\hat{\mathcal{S}}_x)} \\ &= \underbrace{\frac{\int_{\mathcal{S}_x} y dQ(y) - \int_{\hat{\mathcal{S}}_x} y dQ_n(y)}{Q(\mathcal{S}_x)}}_{A_n(x)} + \underbrace{\frac{\int_{\hat{\mathcal{S}}_x} y dQ_n(y)}{Q_n(\hat{\mathcal{S}}_x)Q(\mathcal{S}_x)} (Q_n(\hat{\mathcal{S}}_x) - Q(\mathcal{S}_x))}_{B_n(x)}. \end{aligned} \quad (21)$$

We first focus on $A_n(x)$. In view of the uniform bound (9), it is enough to estimate the real-valued function

$$x \mapsto D_n(x) := \int_{\mathcal{S}_x} y_i dQ(y) - \int_{\hat{\mathcal{S}}_x} y_i dQ_n(y), \quad (22)$$

where y_i is the i -th component of y . Recalling the definition of $\mathcal{S}_x(\cdot)$ from (16), we check

$$\mathcal{S}_x(-\|(\varphi_n, \psi_n) - (\varphi_\varepsilon, \psi_\varepsilon)\|_\oplus) \subset \hat{\mathcal{S}}_x \subset \mathcal{S}_x(\|(\varphi_n, \psi_n) - (\varphi_\varepsilon, \psi_\varepsilon)\|_\oplus).$$

Moreover, we trivially have $\mathcal{S}_x(-\|(\varphi_n, \psi_n) - (\varphi_\varepsilon, \psi_\varepsilon)\|_\oplus) \subset \mathcal{S}_x \subset \mathcal{S}_x(\|(\varphi_n, \psi_n) - (\varphi_\varepsilon, \psi_\varepsilon)\|_\oplus)$. We can thus apply Proposition 4.4 with

$$g(y) = y_i, \quad \Lambda = \emptyset, \quad V_{n,\lambda,x} = \mathbf{1}_{\mathcal{S}_x}, \quad W_{n,\lambda,x} = \mathbf{1}_{\hat{\mathcal{S}}_x}, \quad \beta_n = -\alpha_n = \|(\varphi_n, \psi_n) - (\varphi_\varepsilon, \psi_\varepsilon)\|_\oplus$$

to conclude that $\sup_x |A_n(x)| = \mathcal{O}_{\mathbb{P}}(n^{-\frac{1}{2}} + \|(\varphi_n, \psi_n) - (\varphi_\varepsilon, \psi_\varepsilon)\|_\oplus)$.

It remains to bound the second term, $B_n(x)$, of (21). Noting the inequality

$$|B_n(x)| \leq \|\mathcal{Y}\|_\infty \frac{|Q_n(\hat{\mathcal{S}}_x) - Q(\mathcal{S}_x)|}{Q(\mathcal{S}_x)}$$

and recalling from (9) the uniform lower bound for the denominator, it suffices to estimate $Q_n(\hat{\mathcal{S}}_x) - Q(\mathcal{S}_x)$. We apply Proposition 4.4 (with the same data as above except that now $g(y) \equiv 1$) to get $\sup_x |Q_n(\hat{\mathcal{S}}_x) - Q(\mathcal{S}_x)| = \mathcal{O}_{\mathbb{P}}(n^{-\frac{1}{2}} + \|(\varphi_n, \psi_n) - (\varphi_\varepsilon, \psi_\varepsilon)\|_\oplus)$. $\square$

5 Proof of the CLT for potentials

This section is devoted to the proof of Theorem 3.1. As outlined in Section 3.1, we follow the procedure for Z-estimation problems in empirical process theory. Define the nonlinear operator

$$\mathcal{B}_\oplus \ni \begin{pmatrix} f \\ g \end{pmatrix} \mapsto \Phi \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \int (\langle \cdot, y \rangle - f(\cdot) - g(y))_+ dQ(y) \\ \int (\langle x, \cdot \rangle - f(x) - g(\cdot))_+ dP(x) \end{pmatrix} \in \mathcal{C}(\mathcal{X}) \times \mathcal{C}(\mathcal{Y})$$

and its empirical version

$$\mathcal{B}_\oplus \ni \begin{pmatrix} f \\ g \end{pmatrix} \mapsto \Phi_n \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \int (\langle \cdot, y \rangle - f(\cdot) - g(y))_+ dQ_n(y) \\ \int (\langle x, \cdot \rangle - f(x) - g(\cdot))_+ dP_n(x) \end{pmatrix} \in \mathcal{C}(\mathcal{X}) \times \mathcal{C}(\mathcal{Y}).$$

We denote by $[\Phi]_{\oplus}$ the composition of Φ with the quotient map $[\cdot]_{\oplus}$ of $\sim_{\oplus}$, and similarly for $[\Phi_n]_{\oplus}$. Below, we first show that Φ is differentiable at the population potentials, with invertible derivative. Then, we show that

$$\sqrt{n} \left(\Phi_n \begin{pmatrix} \varphi_{\varepsilon} \\ \psi_{\varepsilon} \end{pmatrix} - \Phi \begin{pmatrix} \varphi_{\varepsilon} \\ \psi_{\varepsilon} \end{pmatrix} \right)$$

converges to a tight Gaussian element of $\mathcal{C}(\mathcal{X}) \times \mathcal{C}(\mathcal{Y})$. Finally, we show that the remainder

$$[\Phi_n - \Phi] \begin{pmatrix} \varphi_{\varepsilon} - \varphi_n \\ \psi_{\varepsilon} - \psi_n \end{pmatrix} = o_{\mathbb{P}} \left(n^{-\frac{1}{2}} + \left\| \begin{pmatrix} \varphi_{\varepsilon} - \varphi_n \\ \psi_{\varepsilon} - \psi_n \end{pmatrix} \right\|_{\oplus} \right)$$

tends to zero in probability.

5.1 Differentiability and invertibility of the optimality conditions

Our first result establishes the Fréchet differentiability of Φ at the point $(\varphi_{\varepsilon}, \psi_{\varepsilon})$. We recall that Φ is Fréchet differentiable at (f_*, g_*) with derivative $D\Phi_{(f_*, g_*)}$ if

$$\lim_{\|(f, g)\|_{\oplus} \rightarrow 0} \frac{\|\Phi(f_* + f, g_* + g) - \Phi(f_*, g_*) - D\Phi_{(f_*, g_*)}(f, g)\|_{\mathcal{C}(\mathcal{X}) \times \mathcal{C}(\mathcal{Y})}}{\|(f, g)\|_{\oplus}} = 0.$$

Lemma 5.1. *The function $\Phi : \mathcal{B}_{\oplus} \rightarrow \mathcal{C}(\mathcal{X}) \times \mathcal{C}(\mathcal{Y})$ is Fréchet differentiable at $(\varphi_{\varepsilon}, \psi_{\varepsilon})$ with derivative*

$$D\Phi_{(\varphi_{\varepsilon}, \psi_{\varepsilon})} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} D\Phi_1(f, g) \\ D\Phi_2(f, g) \end{pmatrix},$$

where

$$\begin{aligned} D\Phi_1(f, g)(x) &= -f(x)Q(\mathcal{S}_x) - \int_{\mathcal{S}_x} g(y)dQ(y) \quad \text{for } x \in \mathcal{X}, \\ D\Phi_2(f, g)(y) &= -g(y)P(\mathcal{T}_y) - \int_{\mathcal{T}_y} f(x)dP(x) \quad \text{for } y \in \mathcal{Y}. \end{aligned}$$

Proof. Given $(h_1, h_2) \in \mathcal{B}_{\oplus}$, set

$$\begin{aligned} A(h_1, h_2)(x) &:= \int (\langle x, y \rangle - (\varphi_{\varepsilon} + h_1)(x) - (\psi_{\varepsilon} + h_2)(y))_+ dQ(y) \\ &\quad - \int (\langle x, y \rangle - \varphi_{\varepsilon}(x) - \psi_{\varepsilon}(y))_+ dQ(y). \end{aligned}$$

With the expression for $D\Phi_1$ stated in the lemma, our goal is to show that

$$\lim_{\|(h_1, h_2)\|_{\oplus} \rightarrow 0} \frac{\|A(h_1, h_2) - D\Phi_1(h_1, h_2)\|_{\infty}}{\|(h_1, h_2)\|_{\oplus}} = 0. \quad (23)$$

For any $a, b \in \mathbb{R}$, applying the fundamental theorem of calculus to the absolutely continuous function $\phi(t) = ((1-t)b + ta)_+$ yields

$$(a)_+ - (b)_+ = \phi(1) - \phi(0) = (a - b)\mathcal{L}_1(t \in [0, 1] : (1-t)b + ta \geq 0). \quad (24)$$

Set

$$\mathcal{J}_x = \{(t, y) \in [0, 1] \times \mathcal{Y} : \langle x, y \rangle - \varphi_\varepsilon(x) - \psi_\varepsilon(y) - th_1(x) - th_2(y) \geq 0\},$$

then the above yields

$$A(h_1, h_2)(x) = -h_1(x)(\mathcal{L}_1 \otimes Q)(\mathcal{J}_x) - \int_{\mathcal{J}_x} h_2(y)d(\mathcal{L}_1 \otimes Q)(t, y).$$

Inserting this expression into (23), we see that a sufficient condition for (23) is

$$\sup_{\|g\|_\infty \leq 1} \left\| \int_{\mathcal{J}_{(\cdot)}} g(y)d(\mathcal{L}_1 \otimes Q)(t, y) - \int_{\mathcal{S}_{(\cdot)}} g(y)dQ(y) \right\|_\infty \rightarrow 0 \quad \text{as } \|(h_1, h_2)\|_\oplus \rightarrow 0. \quad (25)$$

Fix $x \in \mathcal{X}$ and $g \in \mathcal{C}(\mathcal{Y})$ with $\|g\|_\infty \leq 1$. Decomposing $g = (g)_+ - (g)_-$ and using the triangle inequality, we may assume without loss of generality that $g \geq 0$. Recall the notation $\mathcal{S}_x(\beta) = \{y : \varphi_\varepsilon(x) + \psi_\varepsilon(y) \leq \langle x, y \rangle + \beta\}$ from (16). We observe that

$$\begin{aligned} \mathcal{J}_x &\subset \{(t, y) \in [0, 1] \times \mathcal{Y} : \langle x, y \rangle - \varphi_\varepsilon(x) - \psi_\varepsilon(y) \geq -\|(h_1, h_2)\|_\oplus\} \\ &= [0, 1] \times \mathcal{S}_x(\|(h_1, h_2)\|_\oplus) \end{aligned}$$

and similarly $[0, 1] \times \mathcal{S}_x(-\|(h_1, h_2)\|_\oplus) \subset \mathcal{J}_x$. As $g \geq 0$, it follows that

$$\int_{\mathcal{S}_x(-\|(h_1, h_2)\|_\oplus)} g(y)dQ(y) \leq \int_{\mathcal{J}_x} g(y)d(\mathcal{L}_1 \otimes Q)(t, y) \leq \int_{\mathcal{S}_x(\|(h_1, h_2)\|_\oplus)} g(y)dQ(y).$$

By Proposition 4.1, the upper and lower bounds both converge to $\int_{\mathcal{S}_x} g(y)dQ(y)$, uniformly in x and g , as $\|(h_1, h_2)\|_\oplus \rightarrow 0$. This completes the proof of (25) and hence of the lemma. $\square$

Lemma 5.1 implies that

$$\left\| \Phi \begin{pmatrix} \varphi_\varepsilon \\ \psi_\varepsilon \end{pmatrix} - \Phi \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} - \mathbb{D}\Phi_{(\varphi_\varepsilon, \psi_\varepsilon)} \begin{pmatrix} \varphi_\varepsilon - \varphi_n \\ \psi_\varepsilon - \psi_n \end{pmatrix} \right\|_{\mathcal{C}(\mathcal{X}) \times \mathcal{C}(\mathcal{Y})} = o_{\mathbb{P}} \left(\left\| \begin{pmatrix} \varphi_\varepsilon - \varphi_n \\ \psi_\varepsilon - \psi_n \end{pmatrix} \right\|_\oplus \right).$$

Using also the uniform bounds (9), it follows that

$$\left\| \tilde{\Phi} \begin{pmatrix} \varphi_\varepsilon \\ \psi_\varepsilon \end{pmatrix} - \tilde{\Phi} \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} + \mathbb{L} \begin{pmatrix} \varphi_\varepsilon - \varphi_n \\ \psi_\varepsilon - \psi_n \end{pmatrix} \right\|_{\mathcal{C}(\mathcal{X}) \times \mathcal{C}(\mathcal{Y})} = o_{\mathbb{P}} \left(\left\| \begin{pmatrix} \varphi_\varepsilon - \varphi_n \\ \psi_\varepsilon - \psi_n \end{pmatrix} \right\|_\oplus \right), \quad (26)$$

where

$$\tilde{\Phi} \begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} \frac{1}{Q(\mathcal{S}_{(\cdot)})} \int (\langle \cdot, y \rangle - f(\cdot) - g(y))_+ dQ(y) \\ \frac{1}{P(\mathcal{T}_{(\cdot)})} \int (\langle x, \cdot \rangle - f(x) - g(\cdot))_+ dP(x) \end{pmatrix}$$

and $\mathbb{L} := \mathbb{I} + \mathbb{A}$ with $\mathbb{I}$ denoting the identity operator and

$$\mathbb{A} \begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} \mathbb{A}_1(g) \\ \mathbb{A}_2(f) \end{pmatrix} = \begin{pmatrix} \frac{1}{Q(\mathcal{S}_{(\cdot)})} \int_{\mathcal{S}_{(\cdot)}} g(y)dQ(y) \\ \frac{1}{P(\mathcal{T}_{(\cdot)})} \int_{\mathcal{T}_{(\cdot)}} f(x)dP(x) \end{pmatrix}.$$

In the preceding displays, it is tacitly understood that the left-hand side takes two arguments (x', y') ; on the right-hand side, x' is inserted into $\mathcal{S}_{(\cdot)}$ and $f(\cdot)$, whereas y' is inserted into $\mathcal{T}_{(\cdot)}$ and $g(\cdot)$.

We denote by $[\mathbb{L}]_\oplus$ the composition of $\mathbb{L}$ with the quotient map $[\cdot]_\oplus$.

Lemma 5.2. *The operator $[\mathbb{L}]_{\oplus} : \mathcal{B}_{\oplus} \rightarrow \mathcal{B}_{\oplus}$ is a bounded bijection.*

Proof. As $\mathcal{B}_{\oplus}$ is a Banach space, it suffices to show that the bounded linear operator $[\mathbb{L}]_{\oplus}$ is invertible. Indeed, the operator $\mathbb{A}$ is compact by [17, Lemma 4.2]. Hence, by the Fredholm alternative, $[\mathbb{L}]_{\oplus} = [\mathbb{I} + \mathbb{A}]_{\oplus}$ is invertible if and only if $\mathbb{L}(f, g) \sim_{\oplus} 0$ implies $(f, g) \sim_{\oplus} 0$. In view of the definition of $\mathbb{L}$, the latter is in turn equivalent to $\mathbb{A}(f, g) \sim_{\oplus} -(f, g)$ implying $(f, g) \sim_{\oplus} 0$, and that holds by [17, Lemma 4.3]. $\square$

Taking equivalence classes in (26), we have

$$\left\| [\tilde{\Phi}]_{\oplus} \begin{pmatrix} \varphi_{\varepsilon} \\ \psi_{\varepsilon} \end{pmatrix} - [\tilde{\Phi}]_{\oplus} \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} + [\mathbb{L}]_{\oplus} \begin{pmatrix} \varphi_{\varepsilon} - \varphi_n \\ \psi_{\varepsilon} - \psi_n \end{pmatrix} \right\|_{\oplus} = o_{\mathbb{P}} \left(\left\| \begin{pmatrix} \varphi_{\varepsilon} - \varphi_n \\ \psi_{\varepsilon} - \psi_n \end{pmatrix} \right\|_{\oplus} \right).$$

Setting also

$$\tilde{\Phi}_n \begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} \frac{1}{Q(\mathcal{S}_{(\cdot)})} \int (\langle \cdot, y \rangle - f(\cdot) - g(y))_+ dQ_n(y) \\ \frac{1}{P(\mathcal{T}_{(\cdot)})} \int (\langle x, \cdot \rangle - f(x) - g(\cdot))_+ dP_n(x) \end{pmatrix},$$

the first-order conditions (5) and (12) imply

$$[\tilde{\Phi}]_{\oplus} \begin{pmatrix} \varphi_{\varepsilon} \\ \psi_{\varepsilon} \end{pmatrix} = \begin{pmatrix} \frac{\varepsilon}{Q(\mathcal{S}_{(\cdot)})} \\ \frac{\varepsilon}{P(\mathcal{T}_{(\cdot)})} \end{pmatrix} = [\tilde{\Phi}_n]_{\oplus} \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix}$$

and we conclude that

$$\left\| [\tilde{\Phi}_n]_{\oplus} \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} - [\tilde{\Phi}]_{\oplus} \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} + [\mathbb{L}]_{\oplus} \begin{pmatrix} \varphi_{\varepsilon} - \varphi_n \\ \psi_{\varepsilon} - \psi_n \end{pmatrix} \right\|_{\oplus} = o_{\mathbb{P}} \left(\left\| \begin{pmatrix} \varphi_{\varepsilon} - \varphi_n \\ \psi_{\varepsilon} - \psi_n \end{pmatrix} \right\|_{\oplus} \right). \quad (27)$$

5.2 The remainder term

Defining the “remainder” term

$$\Delta_n := [\tilde{\Phi}_n]_{\oplus} \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} - [\tilde{\Phi}]_{\oplus} \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} - \left([\tilde{\Phi}_n]_{\oplus} \begin{pmatrix} \varphi_{\varepsilon} \\ \psi_{\varepsilon} \end{pmatrix} - [\tilde{\Phi}]_{\oplus} \begin{pmatrix} \varphi_{\varepsilon} \\ \psi_{\varepsilon} \end{pmatrix} \right),$$

we have from (27) that

$$\begin{aligned} \left\| [\tilde{\Phi}_n]_{\oplus} \begin{pmatrix} \varphi_{\varepsilon} \\ \psi_{\varepsilon} \end{pmatrix} - [\tilde{\Phi}]_{\oplus} \begin{pmatrix} \varphi_{\varepsilon} \\ \psi_{\varepsilon} \end{pmatrix} + [\mathbb{L}]_{\oplus} \begin{pmatrix} \varphi_{\varepsilon} - \varphi_n \\ \psi_{\varepsilon} - \psi_n \end{pmatrix} \right\|_{\oplus} \\ = \mathcal{O}_{\mathbb{P}}(\|\Delta_n\|_{\oplus}) + o_{\mathbb{P}} \left(\left\| \begin{pmatrix} \varphi_{\varepsilon} - \varphi_n \\ \psi_{\varepsilon} - \psi_n \end{pmatrix} \right\|_{\oplus} \right). \end{aligned}$$

As $[\mathbb{L}]_{\oplus}$ is continuously invertible by Lemma 5.2, this implies

$$\begin{aligned} \left\| [\mathbb{L}]_{\oplus}^{-1} \left([\tilde{\Phi}_n]_{\oplus} \begin{pmatrix} \varphi_{\varepsilon} \\ \psi_{\varepsilon} \end{pmatrix} - [\tilde{\Phi}]_{\oplus} \begin{pmatrix} \varphi_{\varepsilon} \\ \psi_{\varepsilon} \end{pmatrix} \right) + \begin{pmatrix} \varphi_{\varepsilon} - \varphi_n \\ \psi_{\varepsilon} - \psi_n \end{pmatrix} \right\|_{\oplus} \\ = \mathcal{O}_{\mathbb{P}}(\|\Delta_n\|_{\oplus}) + o_{\mathbb{P}} \left(\left\| \begin{pmatrix} \varphi_{\varepsilon} - \varphi_n \\ \psi_{\varepsilon} - \psi_n \end{pmatrix} \right\|_{\oplus} \right). \quad (28) \end{aligned}$$

The key step of the proof is the following estimate for $\|\Delta_n\|_{\oplus}$, which will be obtained on the strength of Proposition 4.4.

Lemma 5.3. *We have*

$$\|\Delta_n\|_{\oplus} = o_{\mathbb{P}} \left(\|(\varphi_n, \psi_n) - (\varphi_{\varepsilon}, \psi_{\varepsilon})\|_{\oplus} + n^{-\frac{1}{2}} \right).$$

Proof. We use the shorthands

$$\xi_{\varepsilon}(x, y) = \langle x, y \rangle - \varphi_{\varepsilon}(x) - \psi_{\varepsilon}(y), \quad \xi_n(x, y) = \langle x, y \rangle - \varphi_n(x) - \psi_n(y). \quad (29)$$

Expanding the definition of Δ_n , using the continuity of the quotient map, and the uniform bound (9), we need to show that

$$\begin{aligned} \left\| \int (\xi_n(\cdot, y))_+ - (\xi_{\varepsilon}(\cdot, y))_+ d(Q_n - Q)(y) \right\|_{\infty} &\leq o_{\mathbb{P}} \left(\|(\varphi_n, \psi_n) - (\varphi_{\varepsilon}, \psi_{\varepsilon})\|_{\oplus} + n^{-\frac{1}{2}} \right), \\ \left\| \int (\xi_n(x, \cdot))_+ - (\xi_{\varepsilon}(x, \cdot))_+ d(P_n - P)(x) \right\|_{\infty} &\leq o_{\mathbb{P}} \left(\|(\varphi_n, \psi_n) - (\varphi_{\varepsilon}, \psi_{\varepsilon})\|_{\oplus} + n^{-\frac{1}{2}} \right). \end{aligned}$$

We prove the first statement, the second is analogous. Thanks to Lemma 4.6, it suffices to show

$$\left\| \int (\xi_n(\cdot, y))_+ - (\xi_{\varepsilon}(\cdot, y))_+ d(Q_n - Q)(y) \right\|_{\infty} \leq o_{\mathbb{P}} (\|(\varphi_n, \psi_n) - (\varphi_{\varepsilon}, \psi_{\varepsilon})\|_{\oplus, 1}). \quad (30)$$

Using the fundamental theorem of calculus as in (24) yields

$$\begin{aligned} \delta_n(x, y) &:= (\xi_n(x, y))_+ - (\xi_{\varepsilon}(x, y))_+ \\ &= (\xi_n(x, y) - \xi_{\varepsilon}(x, y)) \cdot \mathcal{L}_1(\lambda \in [0, 1] : \lambda \xi_n(x, y) + (1 - \lambda) \xi_{\varepsilon}(x, y) \geq 0). \end{aligned}$$

Hence, for every $x \in \mathcal{X}$,

$$\begin{aligned} &\int \delta_n(x, y) d(Q_n - Q)(y) \\ &= \int (\xi_n(x, y) - \xi_{\varepsilon}(x, y)) \cdot \mathcal{L}_1(\lambda \in [0, 1] : \lambda \xi_n(x, y) + (1 - \lambda) \xi_{\varepsilon}(x, y) \geq 0) d(Q_n - Q)(y). \end{aligned}$$

Defining the interpolated sections $\mathcal{S}_{\lambda, x} = \{y : \lambda \xi_n(x, y) + (1 - \lambda) \xi_{\varepsilon}(x, y) \geq 0\}$ (which are random and depend on n , a fact suppressed in the notation), we can bound this as

$$\begin{aligned} \left| \int \delta_n(x, y) d(Q_n - Q)(y) \right| &\leq \|\xi_n - \xi_{\varepsilon}\|_{0,1} \sup_{\lambda \in [0,1]} \left| \int_{\mathcal{S}_{\lambda, x}} \frac{\xi_n(x, y) - \xi_{\varepsilon}(x, y)}{\|\xi_n - \xi_{\varepsilon}\|_{0,1}} d(Q_n - Q)(y) \right| \\ &\leq \|\xi_n - \xi_{\varepsilon}\|_{0,1} \sup_{\|g\|_{0,1} \leq 1, \lambda \in [0,1], x \in \mathcal{X}} \left| \int_{\mathcal{S}_{\lambda, x}} g(y) d(Q_n - Q)(y) \right|. \end{aligned}$$

Comparing with (30) and noting that $\|\xi_n - \xi_{\varepsilon}\|_{0,1} \leq \|(\varphi_n, \psi_n) - (\varphi_{\varepsilon}, \psi_{\varepsilon})\|_{\oplus, 1}$, it then suffices to prove

$$\sup_{\|g\|_{0,1} \leq 1, \lambda \in [0,1], x \in \mathcal{X}} \left| \int_{\mathcal{S}_{\lambda, x}} g(y) d(Q_n - Q)(y) \right| = o_{\mathbb{P}}(1). \quad (31)$$

Step 1. To that end, we first show that for any fixed bounded and measurable $g : \mathcal{Y} \rightarrow \mathbb{R}$,

$$\sup_{\lambda \in [0,1], x \in \mathcal{X}} \left| \int_{\mathcal{S}_{\lambda,x}} g(y) d(Q_n - Q)(y) \right| = o_{\mathbb{P}}(1). \quad (32)$$

Note that $\xi_n(x, y) \wedge \xi_\varepsilon(x, y) \geq 0$ implies $y \in \mathcal{S}_{\lambda,x}$ for all $\lambda \in [0, 1]$. Moreover, if $y \in \mathcal{S}_{\lambda,x}$ for some $\lambda \in [0, 1]$, we must have $\xi_n(x, y) \vee \xi_\varepsilon(x, y) \geq 0$. Recalling $\mathcal{S}_x(\cdot)$ from (16), this yields

$$\mathcal{S}_x(-\|(\varphi_n, \psi_n) - (\varphi_\varepsilon, \psi_\varepsilon)\|_\oplus) \subset \mathcal{S}_{\lambda,x} \subset \mathcal{S}_x(\|(\varphi_n, \psi_n) - (\varphi_\varepsilon, \psi_\varepsilon)\|_\oplus)$$

for all $\lambda \in [0, 1]$. We can now apply Proposition 4.4 with

$$\Lambda = [0, 1], \quad V_{n,\lambda,x} = W_{n,\lambda,x} = \mathbf{1}_{\mathcal{S}_{\lambda,x}}, \quad \beta_n = -\alpha_n = \|(\varphi_n, \psi_n) - (\varphi_\varepsilon, \psi_\varepsilon)\|_\oplus$$

to obtain

$$\sup_{\lambda \in [0,1], x \in \mathcal{X}} \left| \int_{\mathcal{S}_{\lambda,x}} g(y) d(Q_n - Q)(y) \right| = \mathcal{O}_{\mathbb{P}} \left(n^{-\frac{1}{2}} + \|(\varphi_n, \psi_n) - (\varphi_\varepsilon, \psi_\varepsilon)\|_\oplus \right) = o_{\mathbb{P}}(1),$$

where the last equality is due to Lemma 2.4. This completes the proof of (32).

Step 2. It remains to infer the uniform convergence (31) from (32). This follows easily from the compactness of $\overline{\mathbb{B}}_{\|\cdot\|_{0,1}} := \{\|g\|_{0,1} \leq 1\}$ in $\mathcal{C}(\mathcal{Y})$. Indeed, fix $\alpha > 0$. By the Arzelà–Ascoli theorem, there exist $g_1, \dots, g_{N_\alpha} \in \overline{\mathbb{B}}_{\|\cdot\|_{0,1}}$ such that

$$\sup_{g \in \overline{\mathbb{B}}_{\|\cdot\|_{0,1}}} \inf_{k=1, \dots, N_\alpha} \|g - g_k\|_\infty \leq \frac{\alpha}{4}.$$

We can then write

$$\begin{aligned} & \left| \int_{\mathcal{S}_{\lambda,x}} g(y) d(Q_n - Q)(y) \right| \\ & \leq \inf_k \left(\left| \int_{\mathcal{S}_{\lambda,x}} (g(y) - g_k(y)) d(Q_n - Q)(y) \right| + \left| \int_{\mathcal{S}_{\lambda,x}} g_k(y) d(Q_n - Q)(y) \right| \right) \\ & \leq \frac{\alpha}{2} + \sum_{k=1}^{N_\alpha} \left| \int_{\mathcal{S}_{\lambda,x}} g_k(y) d(Q_n - Q)(y) \right| \end{aligned}$$

and, using also (32), conclude that

$$\begin{aligned} & \mathbb{P} \left(\left| \sup_{\|g\|_{0,1} \leq 1, \lambda \in [0,1], x \in \mathcal{X}} \int_{\mathcal{S}_{\lambda,x}} g(y) d(Q_n - Q)(y) \right| \geq \alpha \right) \\ & \leq \mathbb{P} \left(\sum_{k=1}^{N_\alpha} \left| \sup_{\lambda \in [0,1], x \in \mathcal{X}} \int_{\mathcal{S}_{\lambda,x}} g_k(y) d(Q_n - Q)(y) \right| \geq \frac{\alpha}{2} \right) \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, completing the proof. $\square$

5.3 End of the proof of Theorem 3.1

Returning to the proof of Theorem 3.1, note that combining (28) with Lemma 5.3 yields

$$\begin{aligned} \left\| [\mathbb{L}]_{\oplus}^{-1} \left([\tilde{\Phi}_n]_{\oplus} \begin{pmatrix} \varphi_{\varepsilon} \\ \psi_{\varepsilon} \end{pmatrix} - [\tilde{\Phi}]_{\oplus} \begin{pmatrix} \varphi_{\varepsilon} \\ \psi_{\varepsilon} \end{pmatrix} \right) + \begin{pmatrix} \varphi_{\varepsilon} - \varphi_n \\ \psi_{\varepsilon} - \psi_n \end{pmatrix} \right\|_{\oplus} \\ = o_{\mathbb{P}} \left(\left\| \begin{pmatrix} \varphi_{\varepsilon} - \varphi_n \\ \psi_{\varepsilon} - \psi_n \end{pmatrix} \right\|_{\oplus} + n^{-\frac{1}{2}} \right). \end{aligned} \quad (33)$$

We can now complete the proof using standard arguments. For $(X, Y) \sim P \otimes Q$, consider the centered random processes

$$\begin{aligned} \mathcal{X} \ni x \mapsto \mathbb{U}_Q(x) &:= (\langle x, Y \rangle - \varphi_{\varepsilon}(x) - \psi_{\varepsilon}(Y))_+ - \int (\langle x, y \rangle - \varphi_{\varepsilon}(x) - \psi_{\varepsilon}(y))_+ dQ(y), \\ \mathcal{Y} \ni y \mapsto \mathbb{U}_P(y) &:= (\langle X, y \rangle - \varphi_{\varepsilon}(X) - \psi_{\varepsilon}(y))_+ - \int (\langle x, y \rangle - \varphi_{\varepsilon}(x) - \psi_{\varepsilon}(y))_+ dP(x), \end{aligned}$$

As the potentials are Lipschitz by Lemma 2.2, these processes have Lipschitz sample paths. Hence, by [22, Theorem 3.5], they satisfy the central limit theorem in $\mathcal{C}(\mathcal{X})$ and $\mathcal{C}(\mathcal{Y})$, respectively. As a consequence,

$$\sqrt{n} \left(\tilde{\Phi}_n \begin{pmatrix} \varphi_{\varepsilon} \\ \psi_{\varepsilon} \end{pmatrix} - \tilde{\Phi} \begin{pmatrix} \varphi_{\varepsilon} \\ \psi_{\varepsilon} \end{pmatrix} \right) = \sqrt{n} \left(\frac{\frac{1}{Q(\mathcal{S}_{(\cdot)})} \int (\langle \cdot, y \rangle - \varphi_{\varepsilon}(\cdot) - \psi_{\varepsilon}(y))_+ d(Q_n - Q)(y)}{\frac{1}{P(\mathcal{T}_{(\cdot)})} \int (\langle x, \cdot \rangle - \varphi_{\varepsilon}(x) - \psi_{\varepsilon}(\cdot))_+ d(P_n - P)(x)} \right)$$

converges weakly in $\mathcal{C}(\mathcal{X}) \times \mathcal{C}(\mathcal{Y})$ to $(\frac{\mathbf{G}_Q}{Q(\mathcal{S}_{(\cdot)})}, \frac{\mathbf{G}_P}{P(\mathcal{T}_{(\cdot)})})$, where $(\mathbf{G}_Q, \mathbf{G}_P) \in \mathcal{C}(\mathcal{X}) \times \mathcal{C}(\mathcal{Y})$ are independent, Gaussian, centered, and tight, with covariances as stated in Theorem 3.1. By the continuity and linearity of the quotient map $[\cdot]_{\oplus}$, it follows that

$$\sqrt{n} \left([\tilde{\Phi}_n]_{\oplus} \begin{pmatrix} \varphi_{\varepsilon} \\ \psi_{\varepsilon} \end{pmatrix} - [\tilde{\Phi}]_{\oplus} \begin{pmatrix} \varphi_{\varepsilon} \\ \psi_{\varepsilon} \end{pmatrix} \right) \overset{\mathcal{B}_{\oplus}}{\rightsquigarrow} \left[\begin{pmatrix} \frac{\mathbf{G}_Q}{Q(\mathcal{S}_{(\cdot)})} \\ \frac{\mathbf{G}_P}{P(\mathcal{T}_{(\cdot)})} \end{pmatrix} \right]_{\oplus}. \quad (34)$$

Combining (34) with (33) and the continuous mapping theorem, we conclude that

$$\sqrt{n} \begin{pmatrix} \varphi_{\varepsilon} - \varphi_n \\ \psi_{\varepsilon} - \psi_n \end{pmatrix} \overset{\mathcal{B}_{\oplus}}{\rightsquigarrow} [\mathbb{L}]_{\oplus}^{-1} \left[\begin{pmatrix} \frac{\mathbf{G}_Q}{Q(\mathcal{S}_{(\cdot)})} \\ \frac{\mathbf{G}_P}{P(\mathcal{T}_{(\cdot)})} \end{pmatrix} \right]_{\oplus},$$

which was the claim of Theorem 3.1. $\square$

6 Proofs of the CLTs for the optimal costs and couplings

On the strength of the central limit theorem for the potentials (Theorem 3.1) and the statistical gradient estimate (Lemma 4.6), we can now easily derive the central limit theorem for the optimal costs.

Proof of Theorem 3.2. The optimality of the population potentials $(f_\varepsilon, g_\varepsilon)$ for the population dual problem (1) yields

$$\begin{aligned} \text{QOT}_\varepsilon(P_n, Q_n) - \text{QOT}_\varepsilon(P, Q) &\geq \int f_\varepsilon(x) d(P_n - P)(x) + \int g_\varepsilon(y) d(Q_n - Q)(y) \\ &\quad - \frac{1}{2\varepsilon} \int \left(f_\varepsilon(x) + g_\varepsilon(y) - \frac{\|x - y\|^2}{2} \right)_+^2 d(P_n \otimes Q_n - P \otimes Q)(x, y) \end{aligned} \quad (35)$$

whereas the optimality of the empirical potentials (f_n, g_n) for the empirical dual yields

$$\begin{aligned} \text{QOT}_\varepsilon(P_n, Q_n) - \text{QOT}_\varepsilon(P, Q) &\leq \int f_n(x) d(P_n - P)(x) + \int g_n(y) d(Q_n - Q)(y) \\ &\quad - \frac{1}{2\varepsilon} \int \left(f_n(x) + g_n(y) - \frac{\|x - y\|^2}{2} \right)_+^2 d(P_n \otimes Q_n - P \otimes Q)(x, y). \end{aligned} \quad (36)$$

Note that the right-hand side of (35) gives the limit described in Theorem 3.2. Hence, it suffices to show that the difference between the right-hand sides of (35) and (36) behaves as $o_{\mathbb{P}}(n^{-\frac{1}{2}})$; that is,

$$\int (\tau_\varepsilon - \tau_n) d(P_n \otimes Q_n - P \otimes Q) = o_{\mathbb{P}}(n^{-\frac{1}{2}}) \quad (37)$$

for

$$\begin{aligned} \tau_\varepsilon(x, y) &:= f_\varepsilon(x) + g_\varepsilon(y) - \frac{1}{2\varepsilon} \left(f_\varepsilon(x) + g_\varepsilon(y) - \frac{\|x - y\|^2}{2} \right)_+^2, \\ \tau_n(x, y) &:= f_n(x) + g_n(y) - \frac{1}{2\varepsilon} \left(f_n(x) + g_n(y) - \frac{\|x - y\|^2}{2} \right)_+^2. \end{aligned}$$

To see this, we first write

$$\begin{aligned} \left| \int (\tau_\varepsilon - \tau_n) d(P_n \otimes Q_n - P \otimes Q) \right| &\leq \|\tau_\varepsilon - \tau_n\|_{0,1} \sup_{\|h\|_{0,1} \leq 1} \int h d(P_n \otimes Q_n - P \otimes Q) \\ &= o_{\mathbb{P}}(\|\tau_\varepsilon - \tau_n\|_{0,1}), \end{aligned} \quad (38)$$

where the second estimate holds because the unit ball in $\mathcal{C}^{0,1}(\mathcal{X} \times \mathcal{Y})$ is a Glivenko–Cantelli class. Noting that the function $(\cdot)_+^2/2$ occurring in the definitions of τ_ε and τ_n has Lipschitz-continuous derivative $(\cdot)_+$, our gradient estimate in Lemma 4.6 yields

$$o_{\mathbb{P}}(\|\tau_\varepsilon - \tau_n\|_{0,1}) = o_{\mathbb{P}}\left(n^{-\frac{1}{2}} + \|(f_n, g_n) - (f_\varepsilon, g_\varepsilon)\|_\oplus\right).$$

As we already know from Theorem 3.1 that $\|(f_n, g_n) - (f_\varepsilon, g_\varepsilon)\|_\oplus = \mathcal{O}_{\mathbb{P}}(n^{-\frac{1}{2}})$, this completes the proof of (37). $\square$

Lastly, we prove the central limit theorem for the optimal couplings.

Proof of Theorem 3.3. Note that $\int \eta d(\pi_n - \pi) = \int \bar{\eta} d(\pi_n - \pi)$ and define

$$\mathcal{E}_n := \int \bar{\eta}(x, y) (\xi_n(x, y))_+ d(P_n \otimes Q_n)(x, y) - \int \bar{\eta}(x, y) (\xi_\varepsilon(x, y))_+ d(P \otimes Q)(x, y).$$

Recalling (10), we need to show $\sqrt{n}\mathcal{E}_n/\varepsilon \xrightarrow{w} N(0, \sigma^2(\eta)/\varepsilon^2)$. To that end, we decompose

$$\mathcal{E}_n = A_n + B_n + C_n$$

where

$$\begin{aligned} A_n &:= \int \bar{\eta}(x, y) \{(\xi_n(x, y))_+ - (\xi_\varepsilon(x, y))_+\} d(P_n \otimes Q_n - P \otimes Q)(x, y), \\ B_n &:= \int \bar{\eta}(x, y) \{(\xi_n(x, y))_+ - (\xi_\varepsilon(x, y))_+\} d(P \otimes Q)(x, y), \\ C_n &:= \int \bar{\eta}(x, y) (\xi_\varepsilon(x, y))_+ d(P_n \otimes Q_n - P \otimes Q)(x, y). \end{aligned}$$

Step 1. We first address the key part of the proof, which is to show

$$A_n = o_{\mathbb{P}}\left(n^{-\frac{1}{2}}\right). \quad (39)$$

Using the fundamental theorem of calculus as in (24) yields

$$(\xi_n(x, y))_+ - (\xi_\varepsilon(x, y))_+ = (\xi_n(x, y) - \xi_\varepsilon(x, y))V_n(x, y)$$

where $V_n(x, y) := \mathcal{L}_1(\{\lambda \in [0, 1] : \lambda \xi_n(x, y) + (1 - \lambda)\xi_\varepsilon(x, y) \geq 0\})$. We can therefore write

$$\begin{aligned} A_n &= \|\xi_n - \xi_\varepsilon\|_{0,1} \int \bar{\eta}(x, y) \frac{(\xi_n - \xi_\varepsilon)(x, y)}{\|\xi_n - \xi_\varepsilon\|_{0,1}} V_n(x, y) d(P_n \otimes Q_n - P \otimes Q)(x, y) \\ &\leq \|\xi_n - \xi_\varepsilon\|_{0,1} \sup_{\|h\|_{0,1} \leq 1} \int \bar{\eta}(x, y) h(x, y) V_n(x, y) d(P_n \otimes Q_n - P \otimes Q)(x, y). \end{aligned}$$

An analogous lower bound holds with an infimum; we only discuss the supremum. Once again, our gradient estimate in Lemma 4.6 and Theorem 3.1 yield

$$o_{\mathbb{P}}(\|\xi_n - \xi_\varepsilon\|_{0,1}) = o_{\mathbb{P}}\left(n^{-\frac{1}{2}} + \|(\varphi_n, \psi_n) - (\varphi_\varepsilon, \psi_\varepsilon)\|_{\oplus}\right) = \mathcal{O}_{\mathbb{P}}\left(n^{-\frac{1}{2}}\right).$$

Thus, it remains to show that

$$\sup_{\|h\|_{0,1} \leq 1} \int \bar{\eta}(x, y) h(x, y) V_n(x, y) d(P_n \otimes Q_n - P \otimes Q)(x, y) = o_{\mathbb{P}}(1). \quad (40)$$

We first show, for any bounded measurable $h : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$, that

$$\int \bar{\eta}(x, y) h(x, y) V_n(x, y) d(P_n \otimes Q_n - P \otimes Q)(x, y) = o_{\mathbb{P}}(1). \quad (41)$$

Let $\delta_n = \|\xi_n - \xi_\varepsilon\|_\infty$ and recall from (20) the notation $\mathcal{M}(\cdot)$. On the set $\mathcal{M}(-\delta_n) = \{\xi_\varepsilon \geq \delta_n\}$, we have both $\xi_\varepsilon \geq 0$ and $\xi_n \geq 0$, so that $V_n = 1$. In view of $0 \leq V_n \leq 1$, it follows

that $\mathbf{1}_{\mathcal{M}(-\delta_n)} \leq V_n$. On the set $\{V_n > 0\}$, we have either $\xi_\varepsilon \geq 0$ or $\xi_n \geq 0$, and hence $\xi_\varepsilon + \|\xi_n - \xi_\varepsilon\|_\infty \geq 0$. Thus, $\{V_n > 0\} \subset \{\xi_\varepsilon \geq -\delta_n\} = \mathcal{M}(\delta_n)$, showing that $V_n \leq \mathbf{1}_{\mathcal{M}(\delta_n)}$. In summary,

$$\mathbf{1}_{\mathcal{M}(-\delta_n)} \leq V_n \leq \mathcal{M}(\delta_n).$$

Recalling that $\delta_n = o_{\mathbb{P}}(1)$ by Theorem 3.1, we can thus apply Proposition 4.5 with

$$g = \bar{\eta}h, \quad \Lambda = \emptyset, \quad V_{n,\lambda} = W_{n,\lambda} = V_n, \quad \beta_n = -\alpha_n = \delta_n$$

to conclude that

$$\int \bar{\eta}(x, y) h(x, y) V_n(x, y) d(P_n \otimes Q_n - P \otimes Q)(x, y) = \mathcal{O}_{\mathbb{P}}\left(n^{-\frac{1}{2}} + \delta_n\right) = o_{\mathbb{P}}(1),$$

which is (41). Finally, (40) follows from (41) and the compactness of $\{\|h\|_{0,1} \leq 1\}$, exactly as in Step 2 in the proof of Lemma 5.3. This completes the proof of (39).

Step 2. It remains to show that $\sqrt{n}(B_n + C_n)/\varepsilon \xrightarrow{w} N(0, \sigma^2(\eta)/\varepsilon^2)$. The real-valued function

$$\mathcal{B}_{\oplus} \ni (f, g) \mapsto \int \bar{\eta}(x, y) \left(f(x) + g(y) - \frac{\|x - y\|^2}{2} \right)_+ d(P \otimes Q)(x, y)$$

is Fréchet differentiable at the population potentials $(f_\varepsilon, g_\varepsilon)$ with derivative

$$\mathcal{B}_{\oplus} \ni (f, g) \mapsto \int_{\xi_\varepsilon \geq 0} \bar{\eta}(x, y) (f(x) + g(y)) d(P \otimes Q)(x, y).$$

The proof of this fact uses the continuity of $(f_\varepsilon, g_\varepsilon)$ and that $(P \otimes Q)(\partial\{\xi_\varepsilon \geq 0\}) = 0$ as a consequence of Fubini's theorem and $Q(\partial\mathcal{S}_x) = 0$; we omit the details as the argument is similar to (but simpler than) the proof of Lemma 5.1. As a consequence,

$$B_n = \int_{\xi_\varepsilon \geq 0} \bar{\eta}(x, y) (\xi_n(x, y) - \xi_\varepsilon(x, y)) d(P \otimes Q)(x, y) + o_{\mathbb{P}}(\|\xi_n - \xi_\varepsilon\|_\infty),$$

and now Theorem 3.1 yields that $B_n = B'_n + o_{\mathbb{P}}(n^{-\frac{1}{2}})$ for

$$B'_n = \frac{1}{n^2} \sum_{i,j=1}^n \int_{\xi_\varepsilon \geq 0} \left\{ \oplus \left(\mathbb{L}^{-1} \left[\left(\frac{(\xi_\varepsilon(\cdot, Y_j))_+ - \int (\xi_\varepsilon(\cdot, y'))_+ dQ(y')}{Q(\mathcal{S}_{(\cdot)})} \right) \right] \right) \bar{\eta} \right\} d(P \otimes Q),$$

where we recall that X_i, Y_j denote the samples defining P_n and Q_n . We note that both B'_n and

$$C_n = \frac{1}{n^2} \sum_{i,j=1}^n \bar{\eta}(X_i, Y_j) (\xi_\varepsilon(X_i, Y_j))_+ - \int \bar{\eta}(\xi_\varepsilon)_+ d(P \otimes Q)$$

are centered (w.r.t. $\mathbb{P}$) and have finite variance. In summary,

$$\int \bar{\eta} d(\pi_n - \pi) = B'_n + C_n + o_{\mathbb{P}}(n^{-\frac{1}{2}})$$

is a U -statistic with finite variance up to $o_{\mathbb{P}}(n^{-\frac{1}{2}})$ terms, so that the central limit theorem for U -statistics (cf. [29, Theorem 12.6]) gives the desired result. $\square$

A Omitted Proofs

This section collects the proofs for the statements in Section 2, which are either known or follow easily from known results.

Proof of Lemma 2.2. Item (i), the first half of (ii), item (iv), and the first part of (v) up to (10), can all be found in [27]. The convexity of φ_ε and ψ_ε and the formula (6) are shown in [32] and [21] (or see the proof of Lemma 2.3 below).

For any $x \in \mathcal{X}$, the set $\mathcal{S}_x = \{y \in \mathcal{Y} : \varphi_\varepsilon(x) + \psi_\varepsilon(y) - \langle x, y \rangle \leq 0\}$ is convex as a sublevel set of the convex function $y \mapsto \varphi_\varepsilon(x) + \psi_\varepsilon(y) - \langle x, y \rangle$. Clearly

$$\partial \mathcal{S}_x \subset \{y \in \mathcal{Y} : \varphi_\varepsilon(x) + \psi_\varepsilon(y) - \langle x, y \rangle = 0\}.$$

For the reverse inclusion, it suffices to observe that a point y with $\varphi_\varepsilon(x) + \psi_\varepsilon(y) - \langle x, y \rangle = 0$ cannot be a minimum of the convex function $y \mapsto \varphi_\varepsilon(x) + \psi_\varepsilon(y) - \langle x, y \rangle$, as otherwise it would follow that this function is nonnegative, contradicting (5).

As $\partial \mathcal{S}_x$ is the boundary of a convex set and $Q \ll \mathcal{L}_d$, we have that $Q(\partial \mathcal{S}_x) = 0$ and hence $Q(\{y \in \mathcal{Y} : \varphi_\varepsilon(x) + \psi_\varepsilon(y) - \langle x, y \rangle = 0\}) = 0$. By the continuity of $\varphi_\varepsilon(x) + \psi_\varepsilon(y) - \langle x, y \rangle$, it follows that $\mathbf{1}_{\mathcal{S}_{x_n}} \rightarrow \mathbf{1}_{\mathcal{S}_x}$ Q -a.s. for $x_n \rightarrow x$. In particular, $x \mapsto Q(\mathcal{S}_x)$ is continuous and, in view of its formula (6), $\nabla \varphi_\varepsilon$ is continuous.

As before, $Q(\mathcal{S}_x) > 0$ by (5), and hence continuity of $x \mapsto Q(\mathcal{S}_x)$ also implies the uniform bound (9). (A different proof of (9) can be found in [2].) The expression for $\text{spt } \pi_\varepsilon$ in (iv) follows from (10) and the aforementioned fact that $Q(\partial \mathcal{S}_x) = 0$. $\square$

Proof of Lemma 2.3. The existence of the potentials and their continuous extension to $\mathbb{R}^d$ satisfying (12) can again be found in [27]. We detail the proof of the convexity and differentiability properties as [32] and [21] do not state them in the discrete setting; however, we remark that the proof is similar.

We first verify that φ_n is convex. Recall from (12) that

$$\varepsilon = \int (\langle x, y \rangle - \varphi_n(x) - \psi_n(y))_+ dQ_n(y) \quad \text{for all } x \in \mathcal{X}. \quad (42)$$

For every $x, x' \in \mathcal{X}$ and every $\lambda \in (0, 1)$, convexity of $(\cdot)_+$ yields

$$\begin{aligned} & \int (\langle \lambda x + (1 - \lambda)x', y \rangle - \lambda \varphi_n(x) - (1 - \lambda) \varphi_n(x') - \psi_n(y))_+ dQ_n(y) \\ & \leq \lambda \int (\langle x, y \rangle - \varphi_n(x) - \psi_n(y))_+ dQ_n(y) + (1 - \lambda) \int (\langle x', y \rangle - \varphi_n(x') - \psi_n(y))_+ dQ_n(y) \end{aligned}$$

and the right-hand side equals ε by (42). On the other hand, (42) at the point $\lambda x + (1 - \lambda)x'$ states that

$$\varepsilon = \int (\langle \lambda x + (1 - \lambda)x', y \rangle - \varphi_n(\lambda x + (1 - \lambda)x') - \psi_n(y))_+ dQ_n(y).$$

Together, we have

$$\begin{aligned} & \int (\langle \lambda x + (1 - \lambda)x', y \rangle - \lambda \varphi_n(x) - (1 - \lambda) \varphi_n(x') - \psi_n(y))_+ dQ_n(y) \\ & \leq \int (\langle \lambda x + (1 - \lambda)x', y \rangle - \varphi_n(\lambda x + (1 - \lambda)x') - \psi_n(y))_+ dQ_n(y), \end{aligned}$$

and moreover the right-hand side equals $\varepsilon > 0$. It follows that the non-increasing function

$$t \mapsto \int (\langle \lambda x + (1 - \lambda)x', y \rangle - t - \psi_n(y))_+ dQ_n(y)$$

is strictly decreasing for t in a neighborhood of $\varphi_n(\lambda x + (1 - \lambda)x')$, so that we can conclude

$$\varphi_n(\lambda x + (1 - \lambda)x') \leq \lambda \varphi_n(x) + (1 - \lambda) \varphi_n(x').$$

In other words, φ_n is convex.

Next, we observe that (42) and the inequality $(a)_+ - (b)_+ \leq \mathbb{I}_{\{a \geq 0\}}(a - b)$, applied with $a = (\langle x, y \rangle - \varphi_n(x) - \psi_n(y))_+$ and $b = (\langle x', y \rangle - \varphi_n(x') - \psi_n(y))_+$, imply that for $x, x' \in \mathcal{X}$,

$$\begin{aligned} 0 &= \int (\langle x, y \rangle - \varphi_n(x) - \psi_n(y))_+ dQ_n(y) - \int (\langle x', y \rangle - \varphi_n(x') - \psi_n(y))_+ dQ_n(y) \\ &\leq \int_{\hat{\mathcal{S}}_x} \langle x - x', y \rangle dQ_n(y) - Q_n(\hat{\mathcal{S}}_x)(\varphi_n(x) - \varphi_n(x')). \end{aligned}$$

Note that $Q_n(\hat{\mathcal{S}}_x)$ cannot vanish, by (42). Rearranging then gives

$$\varphi_n(x') - \varphi_n(x) \geq \left\langle x' - x, \frac{\int_{\hat{\mathcal{S}}_x} y dQ_n(y)}{Q_n(\hat{\mathcal{S}}_x)} \right\rangle,$$

showing that $\frac{\int_{\hat{\mathcal{S}}_x} y dQ_n(y)}{Q_n(\hat{\mathcal{S}}_x)}$ is a subgradient of φ_n at x . On the other hand, the convex function φ_n is $\mathcal{L}_d$ -a.e. differentiable by Rademacher's theorem. Together, we conclude (13). This formula also shows that $\|\nabla \varphi_n(x)\| \leq \|\mathcal{Y}\|_\infty$ a.e., and hence that φ_n is $\|\mathcal{Y}\|_\infty$ -Lipschitz. $\square$

Proof of Lemma 2.4. Almost surely, P_n and Q_n are probability measures supported in $\mathcal{X}$ and $\mathcal{Y}$ which converge weakly to P and Q , respectively. We fix such a realization and show that any potentials (φ_n, ψ_n) associated with P_n and Q_n converge to $(\varphi_\infty, \psi_\infty)$ in $\mathcal{B}_\oplus$. As we deal with equivalence classes, we may choose representatives of (φ_n, ψ_n) with $\varphi_n(x_0) = 0$ for some fixed $x_0 \in \mathcal{X}$. We recall from Lemma 2.3 that (φ_n, ψ_n) are Lipschitz with a constant depending only on $\mathcal{X}$ and $\mathcal{Y}$. Observe that given any $y \in \mathcal{Y}$, (12) implies that $\langle x, y \rangle - \varphi_n(x) - \psi_n(y) > 0$ for some $x \in \mathcal{X}$. Using $\varphi_n(x_0) = 0$ and boundedness of the supports, we deduce a uniform upper bound $\psi_n(y) \leq \varphi_n(x) - \langle x, y \rangle \leq C$, which by the Lipschitz continuity also implies a uniform lower bound. Moreover, $\langle x, y \rangle - \varphi_n(x) - \psi_n(y) > 0$ now yields another upper bound $\varphi_n(x) \leq C$, which again also implies a lower bound. We conclude that (φ_n, ψ_n) are uniformly bounded and equicontinuous, so that by the Arzelà–Ascoli theorem, a subsequence $\{(\varphi_{n_k}, \psi_{n_k})\}_k$ converges uniformly to a limit $(\varphi_\infty, \psi_\infty)$. Note that $(x, y) \mapsto \langle x, y \rangle - \varphi_n(x) - \psi_n(y)$ is uniformly Lipschitz with some constant L and hence

$$\left| \int (\langle x, y \rangle - \varphi_{n_k}(x) - \psi_{n_k}(y))_+ d(Q_{n_k} - Q)(y) \right| \leq L \cdot \sup_{\|f\|_{0,1} \leq 1} \int f d(Q_{n_k} - Q) \rightarrow 0.$$

Using this observation and the empirical first-order condition (12), passing to the limit shows that $(\varphi_\infty, \psi_\infty)$ satisfies the population first-order condition (5). Thus $(\varphi_\infty, \psi_\infty)$ are population potentials and the claim follows by their uniqueness. $\square$

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