On the Guyon–Lekeufack Volatility Model*

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Abstract

Guyon and Lekeufack recently proposed a path-dependent volatility model and documented its excellent performance in fitting market data and capturing stylized facts. The instantaneous volatility is modeled as a linear combination of two processes, one is an integral of weighted past price returns and the other is the square-root of an integral of weighted past squared volatility. Each of the weightings is built using two exponential kernels reflecting long and short memory. Mathematically, the model is a coupled system of four stochastic differential equations. Our main result is the wellposedness of this system: the model has a unique strong (non-explosive) solution for all parameter values. We also study the positivity of the resulting volatility process and the martingale property of the associated exponential price process.

Keywords Path-dependent volatility model, SDE, wellposedness, explosion
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1 Introduction

Path-dependent volatility models (PDV) are stochastic models for security prices where the instantaneous volatility is a function of the price path. Starting with [5, 2, 11, 1, 3], such models emphasize that prices have a feedback on volatility (e.g., the leverage effect) rather than the volatility being an exogenous factor driving the price. In their recent paper [4], Guyon and Lekeufack empirically study the volatility of the S&P 500 index (and other indexes) and conclude that the majority of the variation can be explained by past index returns. Indeed, the relevant statistics are 1. weighted sum of past daily returns and 2. square-root of weighted sum of past daily squared returns (i.e., squared volatility). More specifically, long and short memory are both found to be important, hence the authors recommend using two decay kernels with different time scales. This leads to four processes feeding into the volatility: weighted sum of past returns at two timescales (indexed as (1,0) and (1,1) below) and weighted sum of past squared returns at two (different) timescales (indexed as (2,0) and (2,1) below).

For practical purposes, [4] finds that exponential kernels provide a tractable model with good fit. This leads the authors to propose a Markovian model with nine parameters, called

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the Markovian 4-factor PDV model. They convincingly argue that this model captures the important stylized facts of volatility, produces realistic price and volatility paths, and can jointly fit S&P 500 and VIX smiles. Specifically, the volatility process of the 4-factor PDV model is given as

$$\sigma_t = \beta_0 + \beta_1 R_{1,t} + \beta_2 \sqrt{R_{2,t}}\,.$$

Here $R_{1,t}$ is the convex combination $(1-\theta_1)R_{1,0,t} + \theta_1 R_{1,1,t}$ of the past returns weighted with different decay rates $\lambda_{1,j}$; i.e., $R_{1,j,t}$ is an Ornstein–Uhlenbeck process $dR_{1,j,t} = \lambda_{1,j}\sigma_t dW_t - \lambda_{1,j}R_{1,j,t}dt$ for $j \in \{0,1\}$. Moreover, $R_{2,t}$ is a convex combination $(1-\theta_2)R_{2,0,t} + \theta_2 R_{2,1,t}$ of the past squared volatility weighted with different decay rates $\lambda_{2,j}$; i.e., $R_{2,j,t}$ is an exponential moving average $dR_{2,j,t} = \lambda_{2,j}\left(\sigma_t^2 - R_{2,j,t}\right)dt$ for $j \in \{0,1\}$.

Altogether, this leads to a coupled SDE system for the four processes $(R_{i,j,t})$, stated as (4-PDV) below. Due to the square and square-root terms in the dynamics, its wellposedness is not obvious. Most importantly, it is not clear if the system explodes in finite time (the numerical simulations in [4] truncate the volatility at a fixed upper bound). The purpose of this paper is to provide existence and uniqueness for (4-PDV). Our results extend to other models where the relationship between $(R_{1,t}, R_{2,t})$ and σ_t has a more general form satisfying certain regularity and growth properties (see Remarks 2.5, 3.3 and 4.4).

The main results are summarized in the subsequent section. There, we first discuss a simpler model which uses only one timescale for each process $R_{i,t}$, corresponding to the special case $\theta_i \in \{0,1\}$. While [4] details that this 2-factor model does not provide a good fit in practice, the authors find it useful to gain intuition about the more complicated 4-factor model. Following their didactic lead, we first prove our results for the 2-factor model in Section 2. In this case, the equations are simpler and the algebraic expressions clearly motivate our strategy of proof. Guided by those insights, the 4-factor model can be treated using a similar strategy (detailed in Section 3), though the expressions are more convoluted. Section 4 concludes by studying the martingale property of the exponential local martingale (price) process associated with the volatility models, a problem posed to us by an anonymous referee.

1.1 Main Results

The 2-factor model of [4] is specified by an SDE driven by a standard Brownian motion W,

$$\begin{aligned} \sigma_t &= \beta_0 + \beta_1 R_{1,t} + \beta_2 \sqrt{R_{2,t}} \\ dR_{1,t} &= \lambda_1 \sigma_t dW_t - \lambda_1 R_{1,t} dt \\ dR_{2,t} &= \left(\lambda_2 \sigma_t^2 - \lambda_2 R_{2,t}\right) dt \end{aligned} \tag{2-PDV}$$

with parameters

$$\beta_0, \beta_2, \lambda_1, \lambda_2 \geq 0 \quad \text{and} \quad \beta_1 \leq 0$$

and initial values $R_{1,0} \in \mathbb{R}$ and $R_{2,0} \in (0,\infty)$. The above is an autonomous SDE for the processes $(R_{1,t}, R_{2,t})$, with σ_t merely acting as an abbreviation. On the other hand, if σ_t is given, the equations for $R_{1,t}$ and $R_{2,t}$ in (2-PDV) are straightforward: $R_{1,t}$ is an Ornstein–Uhlenbeck process driven by the log-returns $\sigma_t dW_t$,

$$R_{1,t} = R_{1,0}e^{-\lambda_1 t} + \lambda_1 \int_0^t e^{-\lambda_2(t-s)}\sigma_s dW_s,$$

and $R_{2,t}$ is an exponential moving average of σ_t^2,

$$R_{2,t} = R_{2,0}e^{-\lambda_2 t} + \lambda_2 \int_0^t \sigma_s^2 e^{-\lambda_2(t-s)} ds > 0. \tag{1}$$

In particular, the expression $\sqrt{R_{2,t}}$ in (2-PDV) is well-defined.

Theorem 1.1. *The 2-factor model* (2-PDV) *has a unique strong solution.*

The proof is detailed in Section 2. There, we first observe that strong existence and uniqueness readily hold up to a possible explosion time, and then proceed to show the absence of explosions in finite time. Theorem 1.1 extends to certain more general models (Remark 2.5). Table 1 reports the parameters used in [4]. In Theorem 2.6, we provide the condition $\lambda_2 < 2\lambda_1$ ensuring $\sigma_t > 0$; that condition is satisfied by the values in Table 1.

β_0	β_1	β_2	λ_1	λ_2
0.08	-0.08	0.5	62	40

Table 1: Example parameters for the 2-factor model from [4, Table 7]

Next, we move on to the 4-factor model of [4]. It is specified by the SDE

$$\begin{aligned} \sigma_t &= \beta_0 + \beta_1 R_{1,t} + \beta_2 \sqrt{R_{2,t}} \\ R_{1,t} &= (1-\theta_1)R_{1,0,t} + \theta_1 R_{1,1,t} \\ R_{2,t} &= (1-\theta_2)R_{2,0,t} + \theta_2 R_{2,1,t} \\ dR_{1,j,t} &= \lambda_{1,j}\sigma_t dW_t - \lambda_{1,j} R_{1,j,t} dt, \quad j \in \{0,1\} \\ dR_{2,j,t} &= \lambda_{2,j}\left(\sigma_t^2 - R_{2,j,t}\right) dt, \quad j \in \{0,1\} \end{aligned} \tag{4-PDV}$$

with parameters

$$\beta_0, \beta_2, \lambda_{1,j}, \lambda_{2,j} \geq 0, \qquad \beta_1 \leq 0, \qquad \theta_1, \theta_2 \in [0,1]$$

and initial values $R_{1,j,0} \in \mathbb{R}$ and $R_{2,j,0} > 0$, $j \in \{0,1\}$. The above is an autonomous SDE for the four processes $(R_{1,j,t}, R_{2,j,t})_{j\in\{0,1\}}$. We note that (4-PDV) generalizes the 2-factor model (2-PDV); the latter is recovered when $\theta_1, \theta_2 \in \{0,1\}$. Whereas for $\theta_1, \theta_2 \in (0,1)$, the difference with (2-PDV) is that $R_{1,t}, R_{2,t}$ are proper convex combinations of processes with different time scales. Once again, $R_{1,j,t}$ and $R_{2,j,t}$ have straightforward expressions once σ_t is given, and $R_{2,j,t} > 0$ as in (1). Our main result reads as follows.

Theorem 1.2. *The 4-factor model* (4-PDV) *has a unique strong solution.*

The proof is stated in Section 3. Again, strong existence and uniqueness readily hold up to a possible explosion time, and we prove absence of explosions in finite time. Theorem 1.2 extends to certain more general models (Remark 3.3). The parameters used in [4] are reproduced in Table 2. While in the 2-factor model, σ_t remains strictly positive for a certain parameter range (Theorem 2.6), that property can fail in the 4-factor model (Proposition 3.4).

β_0	β_1	β_2	$\lambda_{1,0}$	$\lambda_{1,1}$	$\lambda_{2,0}$	$\lambda_{2,1}$	θ_1	θ_2
0.04	-0.13	0.65	55	10	20	3	0.25	0.5

Table 2: Example parameters for the 4-factor model from [4, Table 8]

Finally, we study the martingale property of the resulting price process $(X_t)_{t\geq 0}$, a problem posed by an anonymous referee. We provide a positive result for the 2-factor model; the problem remains open for the 4-factor model. For the sake of generality, we allow processes $(\sigma_t)_{t\geq 0}$ that can become negative (even if those may be undesirable in practice), but stop them at some level for technical reasons; that gives rise to the volatility process $(\nu_t)_{t\geq 0}$ in the theorem below. If the process $(\sigma_t)_{t\geq 0}$ is nonnegative, as is guaranteed for the parameters mentioned in Theorem 2.6, then clearly $\nu_t = \sigma_t$.

Theorem 1.3. *Let $(\sigma_t)_{t\geq 0}$ be given by* (2-PDV) *and $\nu_t := \sigma_{t\wedge\tau}$ where $\tau = \inf\{t \geq 0 : \sigma_t < -C\}$ for some $C \geq 0$. The exponential local martingale $(X_t)_{t\geq 0}$ given by*

$$dX_t = \nu_t X_t dW_t, \quad X_0 = x_0 > 0$$

is a true martingale.

The proof is reported in Section 4. Following an idea in [10] and [7], we characterize the martingale property of $(X_t)_{t\geq 0}$ as the non-explosiveness of $(\nu_t)_{t\geq 0}$ under a changed measure, and then prove the latter by an estimate for the associated stochastic differential equation. This line of argument may extend to the 4-factor model, but the present argument for non-explosiveness in the 2-factor case does not apply in the 4-factor case (see Remark 4.5).

2 Analysis of the 2-Factor Model (2-PDV)

We first show, using fairly standard arguments, that (2-PDV) has a unique strong solution up to a possible explosion time. Then, we prove the absence of explosions. In Section 2.2, we study the positivity of σ_t.

2.1 Wellposedness and Absence of Explosions

To detail the aforementioned arguments, we introduce a more concise notation for (2-PDV): writing $R_t := (R_{1,t}, R_{2,t})$, we can rewrite (2-PDV) as

$$dR_t = b(R_t)dt + \nu(R_t)dW_t, \qquad R_0 = (R_{1,0}, R_{2,0})$$

$$\nu(x, y) = \begin{pmatrix} \lambda_1(\beta_0 + \beta_1 x + \beta_2\sqrt{y}) \\ 0 \end{pmatrix}$$

$$b(x, y) = \begin{pmatrix} -\lambda_1 x \\ \lambda_2\left(\beta_0^2 + \beta_1^2 x^2 + (\beta_2^2 - 1)y + 2\beta_0\beta_1 x + 2\beta_0\beta_2\sqrt{y} + 2\beta_1\beta_2 x\sqrt{y}\right) \end{pmatrix}.$$

As the coefficients $\nu(x, y)$ and $b(x, y)$ are continuous in their domains and the initial condition is deterministic, the general existence result of [6, Theorem IV.2.3] (applied with $\sqrt{y} := 0$ for $y < 0$) shows the following.

Lemma 2.1. *The SDE* (2-PDV) *has a weak solution up to a possible explosion time.*

Next, we establish pathwise uniqueness. The usual local Lipschitz condition (e.g., [6, Theorem IV.3.1]) fails because of the term $\sqrt{y}$ in the coefficients. However, as this failure only occurs at the boundary of the relevant domain, a modification of the usual proof applies.

Lemma 2.2. *The SDE* (2-PDV) *satisfies pathwise uniqueness.*

Proof. Following the proof of [6, Theorem IV.3.1], we consider two solutions (R, W) and (R', W) of (2-PDV) on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Brownian motion W, and with the same initial values $R_0 = R'_0 = (R_{1,0}, R_{2,0})$.

Given $N, \varepsilon > 0$, there exists $K_{\varepsilon,N} > 0$ such that

$$\left\|\nu(x,y) - \nu(x',y')\right\|^2 + \left\|b(x,y) - b(x',y')\right\|^2 \leq K_{\varepsilon,N} \left\|(x,y) - (x',y')\right\|^2$$

for all $(x,y), (x',y') \in \mathbb{R}^2$ with $\|(x,y)\|, \|(x',y')\| \leq N$ and $y, y' \geq \varepsilon$. Define

$$S_N := \inf\{t \geq 0 : \ \|R_t\| \geq N\}, \qquad T_\varepsilon := \inf\{t \geq 0 : \ |R_{2,t}| \leq \varepsilon\}$$

and similarly S'_N, T'_ε for R' instead of R. Set $S_{\varepsilon,N} := S_N \wedge S'_N \wedge T_\varepsilon \wedge T'_\varepsilon$ and note that

$$R_{t\wedge S_{\varepsilon,N}} - R'_{t\wedge S_{\varepsilon,N}} = \int_0^{t\wedge S_{\varepsilon,N}} \big(\nu(R_s) - \nu(R'_s)\big)\, dW_s + \int_0^{t\wedge S_{\varepsilon,N}} \big(b(R_s) - b(R'_s)\big)\, ds.$$

Fix $T \in (0, \infty)$. For $t \leq T$, Itô's isometry and Hölder's inequality yield

$$\begin{aligned}
&\mathbb{E}\left(\|R_{t\wedge S_{\varepsilon,N}} - R'_{t\wedge S_{\varepsilon,N}}\|^2\right) \\
&\leq 2\mathbb{E}\left(\left\|\int_0^{t\wedge S_{\varepsilon,N}} \big(\nu(R_s) - \nu(R'_s)\big)\, dW_s\right\|^2\right) + 2\mathbb{E}\left(\left\|\int_0^{t\wedge S_{\varepsilon,N}} \big(b(R_s) - b(R'_s)\big)\, ds\right\|^2\right) \\
&\leq 2\mathbb{E}\left(\int_0^t \left\|\nu(R_{s\wedge S_{\varepsilon,N}}) - \nu(R'_{s\wedge S_{\varepsilon,N}})\right\|^2 ds\right) + 2T\mathbb{E}\left(\int_0^t \left\|b(R_{s\wedge S_{\varepsilon,N}}) - b(R'_{s\wedge S_{\varepsilon,N}})\right\|^2 ds\right) \\
&\leq 2(1+T)K_{\varepsilon,N}\mathbb{E}\left(\int_0^t \left\|R_{s\wedge S_{\varepsilon,N}} - R'_{s\wedge S_{\varepsilon,N}}\right\|^2 ds\right),
\end{aligned}$$

and then Grönwall's inequality shows

$$\mathbb{E}\left(\|R_{t\wedge S_{\varepsilon,N}} - R'_{t\wedge S_{\varepsilon,N}}\|^2\right) = 0.$$

As $T > 0$ was arbitrary, this holds for all $t \geq 0$.

Next, we let $\varepsilon \to 0$. In view of the positivity (1), we have $T_\varepsilon, T'_\varepsilon \to \infty$ and conclude that

$$R_{t\wedge S_N \wedge S'_N} = R'_{t\wedge S_N\wedge S'_N} \quad \forall t \geq 0.$$

Together with the continuity of the paths, it follows that $S_N = S'_N$, and since this holds for all $N > 0$, we have shown that $R = R'$ up to a possible (common) time of explosion. □

As weak existence together with pathwise uniqueness implies strong existence [6, Theorem IV.1.1], the results so far establish the strong wellposedness of (2-PDV) up to explosion.

Corollary 2.3. *The SDE* (2-PDV) *satisfies strong existence and uniqueness up to a possible explosion time.*

Turning to the main contribution of this section, we now show the absence of explosions.

Lemma 2.4. *A solution* $(R_{1,t}, R_{2,t})$ *of* (2-PDV) *cannot explode in finite time. Moreover,* $\sup_{t\le T}\mathbb{E}(R_{1,t}^2 + R_{2,t}) < \infty$ *for any* $T \in [0,\infty)$.

Proof. Fix $M > 0$ and define the stopping times

$$T_M^1 := \inf\left\{t \ge 0 : \, R_{1,t}^2 \ge M^2\right\} \qquad T_M^2 := \inf\left\{t \ge 0 : \, R_{2,t} \ge M^2\right\}$$
$$T_M := T_M^1 \wedge T_M^2.$$

Fix also $t \ge 0$, and note that

$$M^2\mathbb{P}(T_M \le t) = \mathbb{E}(M^2\mathbf{1}_{T_M\le t}) \le \mathbb{E}(\max(R_{1,T_M\wedge t}^2, R_{2,T_M\wedge t})) \le \mathbb{E}(R_{1,T_M\wedge t}^2 + R_{2,T_M\wedge t}).$$

In the main part of the proof below, we show that

$$\mathbb{E}(R_{1,T_M\wedge t}^2 + R_{2,T_M\wedge t}) \le c(t) \tag{2}$$

with $c(t) < \infty$ independent of M. It will then follow that $\lim_{M\to\infty}\mathbb{P}(T_M \le t) = 0$, showing that R_1 and R_2 have bounded paths on any compact time interval and hence completing the proof.

To show (2), we first apply Itô's formula to obtain

$$R_{1,t\wedge T_M}^2 = R_{1,0}^2 + \int_0^{t\wedge T_M} 2\lambda_1\sigma_s R_{1,s}dW_s + \int_0^{t\wedge T_M}(\lambda_1^2\sigma_s^2 - 2\lambda_1 R_{1,s}^2)ds.$$

As $\sigma_s R_{1,s}$ is uniformly bounded up to the stopping time $t\wedge T_M$, it follows that

$$\mathbb{E}(R_{1,T_M\wedge t}^2) = R_{1,0}^2 + \mathbb{E}\left(\int_0^{t\wedge T_M}(\lambda_1^2\sigma_s^2 - 2\lambda_1 R_{1,s}^2)ds\right)$$

and thus, by Fubini's theorem,

$$\mathbb{E}(R_{1,T_M\wedge t}^2) = R_{1,0}^2 + \int_0^t \mathbb{E}\left((\lambda_1^2\sigma_s^2 - 2\lambda_1 R_{1,s}^2)\mathbf{1}_{s\le t\wedge T_M}\right)ds. \tag{3}$$

Next, we insert the definition of σ_s^2 to get

$$\mathbb{E}(R_{1,T_M\wedge t}^2) = R_{1,0}^2 + \int_0^t \mathbb{E}\Big(\Big\{\lambda_1^2(\beta_0 + \beta_2\sqrt{R_{2,s}})^2 + 2\lambda_1^2\beta_1 R_{1,s}(\beta_0 + \beta_2\sqrt{R_{2,s}}) + (\lambda_1^2\beta_1^2 - 2\lambda_1)R_{1,s}^2\Big\}\mathbf{1}_{s\le t\wedge T_M}\Big)\,ds.$$

Using the elementary inequalities

$$2ab \le a^2 + b^2 \qquad \text{and} \qquad (a+b)^2 \le 2a^2 + 2b^2$$

we deduce

$$\mathbb{E}(R^2_{1,T_M\wedge t}) \leq R^2_{1,0} + \int_0^t \mathbb{E}\Big(\Big\{2\lambda_1^2\beta_0^2 + 2\lambda_1^2\beta_2^2 R_{2,s} + \lambda_1^2\beta_0^2 + \lambda_1^2\beta_1^2 R^2_{1,s} + \lambda_1^2\beta_1^2 R^2_{1,s} + \lambda_1^2\beta_2^2 R_{2,s} + (\lambda_1^2\beta_1^2 - 2\lambda_1) R^2_{1,s}\Big\}\mathbf{1}_{s\leq t\wedge T_M}\Big)\, ds$$

and therefore

$$\begin{aligned}\mathbb{E}(R^2_{1,T_M\wedge t}) &\leq R^2_{1,0} + \int_0^t \mathbb{E}\left(\left(3\lambda_1^2\beta_0^2 + 3\lambda_1^2\beta_2^2 R_{2,s} + (3\lambda_1^2\beta_1^2 - 2\lambda_1)R^2_{1,s}\right)\mathbf{1}_{s\leq t\wedge T_M}\right) ds \\ &\leq R^2_{1,0} + 3\lambda_1^2\beta_0^2 t + \max\left\{3\lambda_1^2\beta_2^2, (3\lambda_1^2\beta_1^2 - 2\lambda_1)\right\}\int_0^t \mathbb{E}\left(R^2_{1,s\wedge T_M} + R_{2,s\wedge T_M}\right) ds \\ &=: c_{1,1} + c_{1,2}t + c_{1,3}\int_0^t \mathbb{E}\left(R^2_{1,s\wedge T_M} + R_{2,s\wedge T_M}\right) ds \end{aligned} \tag{4}$$

where the constants $c_{1,1}, c_{1,2}, c_{1,3} > 0$ are independent of M and t.

Our next goal is a similar bound for R_2 instead of R_1^2. From the SDE for R_2,

$$\begin{aligned}&\mathbb{E}(R_{2,T_M\wedge t}) \\ &= R_{2,0} + \lambda_2\mathbb{E}\left(\int_0^{t\wedge T_M}\left(\sigma_s^2 - R_{2,s}\right) ds\right) \\ &= R_{2,0} + \lambda_2\int_0^t \mathbb{E}\left((\sigma_s^2 - R_{2,s})\mathbf{1}_{s\leq t\wedge T_M}\right) ds \\ &= R_{2,0} + \lambda_2\int_0^t \mathbb{E}\left(\left\{(\beta_0 + \beta_1 R_{1,s})^2 + 2\left(\beta_0 + \beta_1 R_{1,s}\right)\beta_2\sqrt{R_{2,s}} + (\beta_2^2 - 1)R_{2,s}\right\}\mathbf{1}_{s\leq t\wedge T_M}\right) ds.\end{aligned}$$

Similarly as above, we obtain

$$\mathbb{E}(R_{2,T_M\wedge t}) \leq R_{2,0} + \lambda_2\int_0^t \mathbb{E}\Big(\Big\{2\beta_0^2 + 2\beta_1^2 R^2_{1,s} + \beta_0^2 + \beta_2^2 R_{2,s} + \beta_1^2 R^2_{1,s} + \beta_2^2 R_{2,s} + (\beta_2^2 - 1)R_{2,s}\Big\}\mathbf{1}_{s\leq t\wedge T_M}\Big) ds.$$

We conclude that

$$\begin{aligned}\mathbb{E}(R_{2,T_M\wedge t}) &\leq R_{2,0} + \lambda_2\int_0^t \mathbb{E}\left(\left(3\beta_0^2 + 3\beta_1^2 R^2_{1,s} + (3\beta_2^2 - 1)R_{2,s}\right)\mathbf{1}_{s\leq t\wedge T_M}\right) ds \\ &\leq R_{2,0} + 3\lambda_2\beta_0^2 t + \max\{3\beta_1^2, 3\beta_2^2 - 1\}\int_0^t \mathbb{E}\left(R^2_{1,s\wedge T_M} + R_{2,s\wedge T_M}\right) ds \\ &=: c_{2,1} + c_{2,2}t + c_{2,3}\int_0^t \mathbb{E}\left(R^2_{1,s\wedge T_M} + R_{2,s\wedge T_M}\right) ds,\end{aligned} \tag{5}$$

where again the constants do not depend on M and t.

Writing $c_i = c_{1,i} + c_{2,i}$, combining (4) and (5) yields

$$\mathbb{E}(R^2_{1,T_M\wedge t} + R_{2,T_M\wedge t}) \leq c_1 + c_2 t + c_3\int_0^t \mathbb{E}\left(R^2_{1,s\wedge T_M} + R_{2,s\wedge T_M}\right) ds,$$

and now Grönwall's inequality shows

$$\mathbb{E}(R^2_{1,T_M\wedge t} + R_{2,T_M\wedge t}) \leq (c_1 + c_2 t)\, e^{c_3 t}.$$

This establishes (2) and hence completes the proof. □

Remark 2.5. The results in this section generalize to volatility models having the same dynamics as (2-PDV) for $(R_{1,t}, R_{2,t})$ but a more general functional $\sigma_t = f(R_{1,t}, R_{2,t})$ where

(i) $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is continuous, and Lipschitz on compact subsets of $\mathbb{R} \times (0, \infty)$,

(ii) there are $K_1, K_2 \in \mathbb{R}$ such that

$$f(x, y)^2 \leq K_1(x^2 + y) + K_2. \tag{6}$$

Indeed, continuity and local Lipschitz continuity imply analogues of Lemma 2.1 and Lemma 2.2. Inequality (6) implies that bounding (2) is enough to bound the expectation of σ_t^2. Moreover, it implies that

$$\lambda_1^2 f(x, y)^2 - 2\lambda_1 x^2 \leq K_{1,1}(x^2 + y) + K_{1,2}$$

for some $K_{1,1}, K_{1,2} \in \mathbb{R}$, so that a bound analogous to (4) holds, as well as

$$f(x, y)^2 - y \leq K_1(x^2 + y) + K_2,$$

so that a bound analogous to (5) holds.

2.2 Positivity of σ_t

In this section we show that $\sigma_t > 0$ under certain conditions. More precisely, we exhibit a lower bound $\sigma_t \geq Y_t > 0$ which also shows, e.g., that $1/\sigma_t$ has finite moments of all orders. This strengthens the result in [4, Section 4.1.5] where it is observed that $\sigma_t \geq 0$ since the drift of σ_t would be positive whenever σ_t reaches 0.

Theorem 2.6. *Consider the solution $(R_{1,t}, R_{2,t})$ of (2-PDV) up to a possible time τ of explosion. If the initial values $(R_{1,0}, R_{2,0})$ are such that $\sigma_0 = \beta_0 + \beta_1 R_{1,0} + \beta_2\sqrt{R_{2,0}} > 0$, and if moreover $\lambda_2 < 2\lambda_1$, then $\sigma_t > 0$ for all $t < \tau$. More precisely, we have $\sigma_t \geq Y_t$, where Y_t is the stochastic exponential (10).*

Proof. By Itô's formula, σ_t satisfies the SDE

$$d\sigma_t = \left(-\beta_1\lambda_1 R_{1,t} + \frac{\lambda_2\beta_2}{2}\frac{\sigma_t^2 - R_{2,t}}{\sqrt{R_{2,t}}}\right) dt + \beta_1\lambda_1\sigma_t dW_t\,. \tag{7}$$

Using the assumption that $\lambda_2 < 2\lambda_1$, we can bound the drift of σ_t from below:

$$\begin{aligned}
-\beta_1\lambda_1 R_{1,t} + \frac{\lambda_2\beta_2}{2}\frac{\sigma_t^2 - R_{2,t}}{\sqrt{R_{2,t}}} &= -\lambda_1(\sigma_t - \beta_0 - \beta_2\sqrt{R_{2,t}}) - \frac{\lambda_2\beta_2}{2}\sqrt{R_{2,t}} + \frac{\lambda_2\beta_2}{2}\frac{\sigma_t^2}{\sqrt{R_{2,t}}} \\
&\geq -\lambda_1\sigma_t + \beta_0\lambda_1 + \beta_2\left(\lambda_1 - \frac{\lambda_2}{2}\right)\sqrt{R_{2,t}} + \frac{\lambda_2\beta_2}{2}\frac{\sigma_t^2}{\sqrt{R_{2,t}}} \\
&\geq -\lambda_1\sigma_t
\end{aligned} \tag{8}$$

because all the other terms are nonnegative. Inspired by (8), we define a process Y via

$$dY_t = -\lambda_1 Y_t dt + \beta_1 \lambda_1 Y_t dW_t \ , \quad Y_0 = \sigma_0. \tag{9}$$

Note that Y is simply the stochastic exponential

$$Y_t = \sigma_0 \exp\left(\beta_1 \lambda_1 W_t - \lambda_1 t - \frac{1}{2}\beta_1^2 \lambda_1^2 t\right) \tag{10}$$

and, in particular, $Y_t > 0$ for all $t \geq 0$.

The SDEs (7) and (9) have the same initial condition σ_0, the same volatility function $v(x) = \beta_1 \lambda_1 x$ and their drift functions are ordered according to (8). Moreover, both drift and volatility functions are continuous on the relevant domains, the drift function of Y is Lipschitz, and

$$\int_0^\varepsilon v(x)^{-2} dx = \infty$$

for every $\varepsilon > 0$. In view of these conditions, the comparison result for SDEs[1] [8, Theorem 5.2.18] yields that $\sigma_t \geq Y_t$ for all $t < \tau$. □

3 Analysis of the 4-Factor Model (4-PDV)

In Section 3.1 we prove our main result on the wellposedness of the 4-factor model and the absence of explosions. In Section 3.2 we show that σ_t need not remain positive under the stated conditions.

3.1 Wellposedness and Absence of Explosions

The general wellposedness of (4-PDV) is shown using the same arguments as in the 2-factor model; we therefore omit the proof.

Proposition 3.1. *The SDE* (4-PDV) *satisfies strong existence and uniqueness up to a possible explosion time.*

The next result contains our main contribution.

Lemma 3.2. *A solution* $(R_{1,j,t}, R_{2,j,t})_{j\in\{0,1\}}$ *of* (4-PDV) *cannot explode in finite time. Moreover,* $\sup_{t\leq T} \mathbb{E}(R_{1,0,t}^2 + R_{1,1,t}^2 + R_{2,0,t} + R_{2,1,t}) < \infty$ *for any* $T \in [0,\infty)$.

Proof. We follow the guidance provided by the 2-factor model. Fix $M > 0$ and define

$$T_M^{1,0} := \inf\left\{t \geq 0 : R_{1,0,t}^2 \geq M^2\right\} \qquad T_M^{1,1} := \inf\left\{t \geq 0 : R_{1,1,t}^2 \geq M^2\right\}$$
$$T_M^{2,0} := \inf\left\{t \geq 0 : R_{2,0,t} \geq M^2\right\} \qquad T_M^{2,1} := \inf\left\{t \geq 0 : R_{2,1,t} \geq M^2\right\}$$
$$T_M := T_M^{1,0} \wedge T_M^{1,1} \wedge T_M^{2,0} \wedge T_M^{2,1}.$$

[1] To be precise, the cited theorem is stated for SDEs where the drift and volatility functions depend only on time and the solution process. Here, they are random as they depend on $(R_{1,t}, R_{2,t})$. The proof holds without change.

Fix also $t > 0$, then

$$\begin{aligned} M^2\mathbb{P}(T_M \le t) &= \mathbb{E}(M^2 \mathbf{1}_{T_M \le t}) \\ &\le \mathbb{E}\left(\max\left(R^2_{1,0,t\wedge T_M}, R^2_{1,1,t\wedge T_M}, R_{2,0,t\wedge T_M}, R_{2,1,t\wedge T_M}\right)\right) \\ &\le \mathbb{E}\left(R^2_{1,0,t\wedge T_M} + R^2_{1,1,t\wedge T_M} + R_{2,0,t\wedge T_M} + R_{2,1,t\wedge T_M}\right) = \mathbb{E}(U_{t\wedge T_M}) \end{aligned}$$

where

$$U_t := R^2_{1,0,t} + R^2_{1,1,t} + R_{2,0,t} + R_{2,1,t}\,.$$

We shall prove that

$$\mathbb{E}(U_{t\wedge T_M}) \le C(t) \tag{11}$$

for a continuous $C(t)$ independent of M, and that will imply the claim.

To prove (11), we note as in (3) that

$$\mathbb{E}\left(R^2_{1,j,t\wedge T_M}\right) = R^2_{1,j,0} + \int_0^t \mathbb{E}\left(\{\lambda^2_{1,j}\sigma^2_s - 2\lambda_{1,j}R^2_{1,j,s}\}\mathbf{1}_{s\le t\wedge T_M}\right) ds. \tag{12}$$

We first focus on $j = 0$. Inserting the definition of σ^2_s and then the one of $R^2_{1,s}$, we have

$$\begin{aligned} &\lambda^2_{1,0}\sigma^2_s - 2\lambda_{1,0}R^2_{1,0,s} \\ &\quad= \lambda^2_{1,0}(\beta_0 + \beta_2\sqrt{R_{2,s}})^2 + 2\lambda^2_{1,0}\beta_1(\beta_0 + \beta_2\sqrt{R_{2,s}})R_{1,s} + \lambda^2_{1,0}\beta^2_1 R^2_{1,s} - 2\lambda_{1,0}R^2_{1,0,s} \\ &\quad\le 3\lambda^2_{1,0}\beta^2_0 + 3\lambda^2_{1,0}\beta^2_2 R_{2,s} + 3\lambda^2_{1,0}\beta^2_1 R^2_{1,s} - 2\lambda_{1,0}R^2_{1,0,s} \\ &\quad\le 3\lambda^2_{1,0}\beta^2_0 + 3\lambda^2_{1,0}\beta^2_2(R_{2,0,s} + R_{2,1,s}) + 3\lambda^2_{1,0}\beta^2_1(R^2_{1,0,s} + R^2_{1,1,s}) - 2\lambda_{1,0}R^2_{1,0,s} \\ &\quad= 3\lambda^2_{1,0}\beta^2_0 + 3\lambda^2_{1,0}\beta^2_2 R_{2,0,s} + 3\lambda^2_{1,0}\beta^2_2 R_{2,1,s} + 3\lambda^2_{1,0}\beta^2_1 R^2_{1,1,s} + (3\lambda^2_{1,0}\beta^2_1 - 2\lambda_{1,0})R^2_{1,0,s} \end{aligned}$$

where we used the elementary convexity inequalities

$$R^2_{1,s} = ((1-\theta_1)R_{1,0,s} + \theta_1 R_{1,1,s})^2 \le (1-\theta_1)R^2_{1,0,s} + \theta_1 R^2_{1,1,s} \le R^2_{1,0,s} + R^2_{1,1,s}\,, \tag{13}$$

$$R_{2,s} = (1-\theta_2)R_{2,0,s} + \theta_2 R_{2,1,s} \le R_{2,0,s} + R_{2,1,s}\,. \tag{14}$$

Using this inequality in (12) yields

$$\begin{aligned} \mathbb{E}\left(R^2_{1,0,t\wedge T_M}\right) \le R^2_{1,0,0} &+ \int_0^t \mathbb{E}\Big(3\lambda^2_{1,0}\beta^2_0 + 3\lambda^2_{1,0}\beta^2_2 R_{2,0,s\wedge T_M} \\ &+ 3\lambda^2_{1,0}\beta^2_2 R_{2,1,s\wedge T_M} + 3\lambda^2_{1,0}\beta^2_1 R^2_{1,1,s\wedge T_M} + (3\lambda^2_{1,0}\beta^2_1 - 2\lambda_{1,0})R^2_{1,0,s\wedge T_M}\Big) ds \\ =:&\; c_{1,0,0}(t) + c_{1,0,1}\int_0^t \mathbb{E}(R_{2,0,s\wedge T_M})ds + c_{1,0,1}\int_0^t \mathbb{E}(R_{2,1,s\wedge T_M})ds \\ &+ c_{1,0,1}\int_0^t \mathbb{E}(R^2_{1,1,s\wedge T_M})ds + c_{1,0,2}\int_0^t \mathbb{E}(R^2_{1,0,s\wedge T_M})ds \end{aligned}$$

where $c_{1,0,0}(t)$ is affine in t.

Symmetrically, we have for $j = 1$ that

$$\begin{aligned} \lambda^2_{1,1}\sigma^2_s - 2\lambda_{1,1}R^2_{1,1,s} \le\;& 3\lambda^2_{1,1}\beta^2_0 + 3\lambda^2_{1,1}\beta^2_2 R_{2,0,s} \\ &+ 3\lambda^2_{1,1}\beta^2_2 R_{2,1,s} + 3\lambda^2_{1,1}\beta^2_1 R^2_{1,0,s} + (3\lambda^2_{1,1}\beta^2_1 - 2\lambda_{1,1})R^2_{1,1,s} \end{aligned}$$

and then

$$\begin{aligned}\mathbb{E}\left(R^2_{1,1,t\wedge T_M}\right) \le R^2_{1,1,0} + \int_0^t \mathbb{E}\Big(3\lambda^2_{1,1}\beta_0^2 + 3\lambda^2_{1,1}\beta_2^2 R_{2,0,s\wedge T_M} \\ + 3\lambda^2_{1,1}\beta_2^2 R_{2,1,s\wedge T_M} + 3\lambda^2_{1,1}\beta_1^2 R^2_{1,0,s\wedge T_M} + (3\lambda^2_{1,1}\beta_1^2 - 2\lambda_{1,1}) R^2_{1,1,s\wedge T_M}\Big) ds \\ =: c_{1,1,0}(t) + c_{1,1,1}\int_0^t \mathbb{E}(R_{2,0,s\wedge T_M})ds + c_{1,1,1}\int_0^t \mathbb{E}(R_{2,1,s\wedge T_M})ds \\ + c_{1,1,1}\int_0^t \mathbb{E}(R^2_{1,0,s\wedge T_M})ds + c_{1,1,2}\int_0^t \mathbb{E}(R^2_{1,1,s\wedge T_M})ds.\end{aligned}$$

Combining the results for $j = 0$ and $j = 1$ yields

$$\begin{aligned}\mathbb{E}\left(R^2_{1,0,t\wedge T_M} + R^2_{1,1,t\wedge T_M}\right) \\ \le c_{1,0}(t) + c_{1,1}\int_0^t \mathbb{E}(R_{2,0,s\wedge T_M} + R_{2,1,s\wedge T_M})ds \\ + c_{1,2}\int_0^t \mathbb{E}\left(R^2_{1,0,s\wedge T_M} + R^2_{1,1,s\wedge T_M}\right) ds\end{aligned} \tag{15}$$

where the constants have the obvious definitions.

Next, we derive a similar bound for $R_{2,j}$ instead of $R^2_{1,j}$. From the SDE for $R_{2,j}$,

$$\mathbb{E}\left(R_{2,j,t\wedge T_M}\right) = R_{2,j,0} + \lambda_{2,j}\int_0^t \mathbb{E}\left((\sigma_s^2 - R_{2,j,s})\mathbf{1}_{s\le t\wedge T_M}\right) ds. \tag{16}$$

Focusing again on $j = 0$ first, we insert the definitions of σ_s^2 and $R_{2,s}$ and estimate

$$\begin{aligned}&\sigma_s^2 - R_{2,0,s} \\ &\quad = (\beta_0 + \beta_1 R_{1,s})^2 + 2\beta_2(\beta_0 + \beta_1 R_{1,s})\sqrt{R_{2,s}} + \beta_2^2\theta_2 R_{2,1,s} + (\beta_2^2(1-\theta_2) - 1)R_{2,0,s} \\ &\quad \le 2\beta_0^2 + 2\beta_1^2 R^2_{1,s} + \beta_0^2 + \beta_2^2 R_{2,s} + \beta_1^2 R^2_{1,s} + \beta_2^2 R_{2,s} + \beta_2^2\theta_2 R_{2,1,s} + (\beta_2^2(1-\theta_2) - 1)R_{2,0,s} \\ &\quad \le 3\beta_0^2 + 3\beta_1^2 R^2_{1,s} + 3\beta_2^2 R_{2,1,s} + (3\beta_2^2 - 1)R_{2,0,s} \\ &\quad \le 3\beta_0^2 + 3\beta_1^2 R^2_{1,0,s} + 3\beta_1^2 R^2_{1,1,s} + 3\beta_2^2 R_{2,1,s} + (3\beta_2^2 - 1)R_{2,0,s}\,.\end{aligned}$$

Applying this in (16), we conclude that

$$\begin{aligned}\mathbb{E}\left(R_{2,0,t\wedge T_M}\right) \le R_{2,0,0} + \lambda_{2,0}\int_0^t \mathbb{E}\Big(3\beta_0^2 + 3\beta_1^2 R^2_{1,0,s\wedge T_M} \\ + 3\beta_1^2 R^2_{1,1,s\wedge T_M} + 3\beta_2^2 R_{2,1,s\wedge T_M} + (3\beta_2^2 - 1)R_{2,0,s\wedge T_M}\Big)ds \\ =: c_{2,0,0}(t) + c_{2,0,1}\int_0^t \mathbb{E}\left(R^2_{1,0,s\wedge T_M}\right) ds + c_{2,0,1}\int_0^t \mathbb{E}\left(R^2_{1,1,s\wedge T_M}\right) ds \\ + c_{2,0,1}\int_0^t \mathbb{E}\left(R_{2,1,s\wedge T_M}\right) ds + c_{2,0,2}\int_0^t \mathbb{E}\left(R_{2,0,s\wedge T_M}\right) ds.\end{aligned}$$

Symmetrically, we obtain for $j = 1$ that

$$\sigma_s^2 - R_{2,1,s} \le 3\beta_0^2 + 3\beta_1^2 R^2_{1,0,s} + 3\beta_1^2 R^2_{1,1,s} + 3\beta_2^2 R_{2,0,s} + (3\beta_2^2 - 1)R_{2,1,s}$$

and then

$$
\begin{aligned}
\mathbb{E}\left(R_{2,1,t\wedge T_M}\right) \leq R_{2,1,0} + \lambda_{2,0}\int_0^t \mathbb{E}\Big(3\beta_0^2 + 3\beta_1^2 R_{1,0,s}^2 \\
+ 3\beta_1^2 R_{1,1,s}^2 + 3\beta_2^2 R_{2,0,s} + (3\beta_2^2-1)R_{2,1,s}\Big) ds \\
=: c_{2,1,0}(t) + c_{2,1,1}\int_0^t \mathbb{E}\left(R_{1,0,s\wedge T_M}^2\right) ds + c_{2,1,1}\int_0^t \mathbb{E}\left(R_{1,1,s\wedge T_M}^2\right) ds \\
+ c_{2,1,1}\int_0^t \mathbb{E}\left(R_{2,0,s\wedge T_M}\right) ds + c_{2,1,2}\int_0^t \mathbb{E}\left(R_{2,1,s\wedge T_M}\right) ds.
\end{aligned}
$$

Adding the two inequalities, we deduce

$$
\begin{aligned}
&\mathbb{E}\left(R_{2,0,t\wedge T_M} + R_{2,1,t\wedge T_M}\right) \\
&\quad \leq c_{2,0}(t) + c_{2,1}\int_0^t \mathbb{E}(R_{1,0,s\wedge T_M}^2 + R_{1,1,s\wedge T_M}^2) ds \\
&\quad\quad + c_{2,2}\int_0^t \mathbb{E}\left(R_{2,0,s\wedge T_M} + R_{2,1,s\wedge T_M}\right) ds.
\end{aligned} \tag{17}
$$

We can now add (15) and (17) to obtain

$$
\mathbb{E}(U_{t\wedge T_M}) \leq c_0(t) + c_1\int_0^t \mathbb{E}(U_{s\wedge T_M}) ds
$$

where $c_0(t)$ is affine and nondecreasing in t and $c_0(t), c_1$ do not depend on M. Grönwall's inequality then yields

$$
\mathbb{E}(U_{t\wedge T_M}) \leq c_0(t) e^{c_1 t}
$$

which is the desired bound (11). □

Remark 3.3. The results in this section generalize to volatility models having the same dynamics as (4-PDV) for $(R_{1,0,t}, R_{1,1,t}, R_{2,0,t}, R_{2,1,t})$ but a more general functional

$$
\sigma_t = \tilde{f}(R_{1,t}, R_{2,t}) = f(R_{1,0,t}, R_{1,1,t}, R_{2,0,t}, R_{2,1,t}),
$$

where

(i) $f : \mathbb{R}^2 \times \mathbb{R}_+^2 \to \mathbb{R}$ is continuous, and Lipschitz on compact subsets of $\mathbb{R}^2 \times (0,\infty)^2$,

(ii) there are $K_1, K_2 \in \mathbb{R}$ such that

$$
f(x_0, x_1, y_0, y_1)^2 \leq K_1 + K_2(x_0^2 + x_1^2 + y_0 + y_1). \tag{18}
$$

These conditions for f are satisfied in particular if $\tilde{f}$ satisfies the conditions of Remark 3.3; this can be seen using inequalities (13) and (14).

Indeed, continuity and local Lipschitz continuity imply the analogue of Proposition 3.1. Inequality (18) implies that bounding (11) is enough to bound the expectation of σ_t^2. Moreover, it implies that, for $j \in \{0,1\}$, f satisfies

$$
\lambda_{1,j}^2 f(x_0, x_1, y_0, y_1)^2 - 2\lambda_{1,j} x_j^2 \leq K_{1,j,0} + K_{1,j,1}(x_0^2 + x_1^2 + y_0 + y_1) \tag{19}
$$

for some $K_{1,j,0}, K_{1,j,1} \in \mathbb{R}$, so that a bound analogous to (15) holds, as well as, for $j \in \{0,1\}$,

$$f(x_0, x_1, y_0, y_1)^2 - y_j \le K_1 + K_2(x_0^2 + x_1^2 + y_0 + y_1),$$

so that a bound analogous to (17) holds.

3.2 Failure of Positivity of σ_t in (4-PDV)

In [4] it is reported that for realistic parameter values, the volatility σ_t remained positive in simulations of the 4-factor model. Nevertheless, existence of a reasonable sufficient condition for $\sigma_t > 0$ (or even just $\sigma_t \ge 0$) in (4-PDV) remains open. Below, we explain that a direct generalization of Theorem 2.6 fails. Indeed, in the 4-factor model, σ_t follows the SDE

$$d\sigma_t = \left(-\beta_1 \bar{\lambda}_1 \bar{R}_{1,t} + \frac{\bar{\lambda}_2 \beta_2}{2} \frac{\sigma_t^2 - \bar{R}_{2,t}}{\sqrt{R_{2,t}}} \right) dt + \beta_1 \bar{\lambda}_1 \sigma_t dW_t \tag{20}$$

where

$$\bar{\lambda}_i := (1-\theta_i)\lambda_{i,0} + \theta_i \lambda_{i,1}\,, \qquad \bar{R}_{i,t} := \frac{(1-\theta_i)\lambda_{i,0} R_{i,0,t} + \theta_i \lambda_{i,1} R_{i,1,t}}{\bar{\lambda}_i}$$

as seen in [4]. The analogue of the condition in Theorem 2.6 is $\bar{\lambda}_2 < 2\bar{\lambda}_1$.

Proposition 3.4. *Under* (4-PDV)*, it may happen that* $\sigma_0 > 0$ *but* $\mathbb{P}(\sigma_t < 0) > 0$ *for some* $t > 0$*, even if* $\bar{\lambda}_2 < 2\bar{\lambda}_1$*.*

Proof. We choose initial conditions $R_{1,0,0} < 0$ and $R_{1,1,0} > 0$, and coefficients $\theta_1, \lambda_{1,0}, \lambda_{1,1}$, such that $R_{1,0} > 0$ and $\bar{R}_{1,0} < 0$. Next, choose $\beta_1 = -1$ (say), and then $\beta_2 > 0$ such that $\beta_1 R_{1,0} + \beta_2 \sqrt{R_{2,0}} = 0$. Consider for the moment $\beta_0 := 0$, then the preceding identity means that $\sigma_0 = 0$. Inspecting (20), we see that at $t = 0$, the volatility vanishes while the drift rate is

$$-\beta_1 \bar{\lambda}_1 \bar{R}_{1,t} - \frac{\bar{\lambda}_2 \beta_2}{2} \frac{\bar{R}_{2,t}}{\sqrt{R_{2,t}}} < 0.$$

By continuity of the paths, it follows that $\mathbb{P}(\sigma_t < 0) > 0$ for all $t > 0$ sufficiently small.

Next, we modify the above by choosing β_0 strictly positive, so that $\sigma_0 = \beta_0 > 0$. We may see β_0 as a parameter of the SDE determining σ_t. If the solution is continuous with respect to β_0, it follows that $\mathbb{P}(\sigma_t < 0) > 0$ for $\beta_0 > 0$ and t sufficiently small. Continuity is a standard result for SDEs with Lipschitz coefficients (e.g., [9, Section 4.5]). To see that the Lipschitz result is sufficient, note that for the present purpose of showing that $\mathbb{P}(\sigma_t < 0) > 0$ for some small $t > 0$, we may truncate the non-Lipschitz coefficients in (4-PDV); that is, we replace $\sqrt{R_{2,t}}$ by $\sqrt{R_{2,t} \vee \delta}$ and σ_t^2 by $\sigma_t^2 \wedge \delta^{-1}$ for a small constant $\delta > 0$. □

4 Martingale Property of the Price Process

In this section we study the martingale property of the price process $(X_t)_{t \ge 0}$; cf. Theorem 1.3. Following [10, Proof of Lemma 4.2] and [7, Section 5.3], the idea is to rephrase the martingale property into non-explosiveness of another process, and establish the latter. Given a continuous adapted process $(\nu_t)_{t\geq 0}$, consider the exponential local martingale $(X_t)_{t\geq 0}$ given by

$$dX_t = \nu_t X_t dW_t, \quad X_0 = x_0 > 0. \tag{21}$$

For $M > 0$, we define the stopping time

$$T_M := \inf \{t \geq 0 : |\nu_t| \geq M\}. \tag{22}$$

Given also $t \geq 0$, Novikov's condition allows us to define a probability measure $\tilde{\mathbb{P}}_{M,t} \ll \mathbb{P}$ by

$$\frac{d\tilde{\mathbb{P}}_{M,t}}{d\mathbb{P}} = \exp\left\{ \int_0^{T_M\wedge t} \nu_s dW_s - \frac{1}{2}\int_0^{T_M\wedge t} \nu_s^2 ds \right\}.$$

We have $T_M \to \infty$ a.s. given that $(\nu_t)_{t\geq 0}$ does not explode in finite time. The martingale property of $(X_t)_{t\geq 0}$ is characterized as follows.

Lemma 4.1. *For the local martingale $(X_t)_{t\geq 0}$ given by* (21)*, the following are equivalent:*

(i) $(X_t)_{t\geq 0}$ *is a martingale,*

(ii) $\liminf_{M\to\infty} \mathbb{E}(X_{T_M} \mathbf{1}_{T_M<t}) = 0$ *for all* $t > 0$*,*

(iii) $\liminf_{M\to\infty} \tilde{\mathbb{P}}_{M,t}(T_M < t) = 0$ *for all* $t > 0$*.*

Proof. As $(X_t)_{t\geq 0}$ is a nonnegative local martingale, it is a supermartingale by Fatou's lemma, hence a martingale if and only if $x_0 = \mathbb{E}(X_t)$ for all $t > 0$. The bounded process $(X_{t\wedge T_M})_{t\geq 0}$ is a martingale, hence

$$x_0 = \mathbb{E}(X_{t\wedge T_M}) = \mathbb{E}(X_t \mathbf{1}_{T_M\geq t}) + \mathbb{E}(X_{T_M}\mathbf{1}_{T_M<t}).$$

Using monotone convergence for $\mathbb{E}(X_t\mathbf{1}_{T_M\geq t})$, this yields

$$x_0 = \mathbb{E}(X_t) + \liminf_{M\to\infty} \mathbb{E}(X_{T_M}\mathbf{1}_{T_M<t})$$

and now the equivalence of (i) and (ii) follows. For fixed $M > 0$ and $t \geq 0$,

$$\begin{aligned}
\mathbb{E}(X_{T_M}\mathbf{1}_{T_M<t}) &= \mathbb{E}\left(x_0 \exp\left\{ \int_0^{T_M} \nu_s dW_s - \frac{1}{2}\int_0^{T_M} \nu_s^2 ds \right\} \mathbf{1}_{T_M<t} \right) \\
&= x_0 \mathbb{E}\left(\exp\left\{ \int_0^{T_M\wedge t} \nu_s dW_s - \frac{1}{2}\int_0^{T_M\wedge t} \nu_s^2 ds \right\} \mathbf{1}_{T_M<t} \right) \\
&= x_0 \tilde{\mathbb{P}}_{M,t}(T_M < t),
\end{aligned} \tag{23}$$

showing the equivalence of (ii) and (iii). □

Proposition 4.2. *Let $(\sigma_t)_{t\geq 0}$ be given by* (2-PDV)*. For a fixed constant $C \geq 0$, define the stopping time $\tau := \inf\{t \geq 0 : \sigma_t < -C\}$ and set $\nu_t := \sigma_{t\wedge\tau}$. Then the exponential local martingale $(X_t)_{t\geq 0}$ of* (21) *is a true martingale.*

Proof. Define the stopping time

$$S_M := \inf \{t \geq 0 : |\sigma_t| \geq M\}.$$

Fix $t, M > 0$. By Girsanov's theorem,

$$\tilde{W}_s = W_s - \int_0^s \sigma_r dr \ , \quad 0 \leq s < t \wedge S_M$$

is a Brownian motion under $\tilde{\mathbb{P}}_{M,t}$ up to time $t \wedge S_M$. The system (2-PDV) for $(R_{1,s}, R_{2,s})$ can be stated up to time $t \wedge S_M$ as

$$\begin{aligned} \sigma_s &= \beta_0 + \beta_1 R_{1,s} + \beta_2 \sqrt{R_{2,s}} \\ dR_{1,s} &= \lambda_1 \sigma_s d\tilde{W}_s + \lambda_1(\sigma_s^2 - R_{1,s})ds \\ dR_{2,s} &= \left(\lambda_2 \sigma_s^2 - \lambda_2 R_{2,s}\right) ds. \end{aligned}$$

By the same arguments as in Lemma 2.1 and Lemma 2.2, this system has a unique strong solution up to time $t \wedge S_M$ under $\tilde{\mathbb{P}}_{M,t}$. The solution (σ_s) of (2-PDV) up to time $t \wedge S_M$ under $\tilde{\mathbb{P}}_{M,t}$ has the same distribution as the solution of (2-PDV˜) in Lemma 4.3 below under $\mathbb{P}$. For $M > C$, we also note that $t \wedge S_M \geq t \wedge T_M \wedge \tau$. By the assertion of Lemma 4.3, it then follows that $\lim_{M\to\infty} \tilde{\mathbb{P}}_{M,t}(T_M < t) = 0$ for any $t \geq 0$, which by Lemma 4.1 shows that $(X_t)_{t\geq 0}$ is a martingale. □

Lemma 4.3. *The following SDE under $\mathbb{P}$ has a unique strong solution $(\sigma_s)_{s\geq 0}$ up to a possible time of explosion:*

$$\begin{aligned} \sigma_s &= \beta_0 + \beta_1 R_{1,s} + \beta_2 \sqrt{R_{2,s}} \\ dR_{1,s} &= \lambda_1 \sigma_s dW_s + \lambda_1(\sigma_s^2 - R_{1,s})ds \\ dR_{2,s} &= \left(\lambda_2 \sigma_s^2 - \lambda_2 R_{2,s}\right) ds. \end{aligned} \tag{2-PDV˜}$$

Given $C \geq 0$, define $\tau = \inf\{t \geq 0 : \sigma_s < -C\}$. Then the process $(\sigma_{s\wedge\tau})_{s\geq 0}$ a.s. does not explode in finite time.

Proof. Existence and uniqueness again follows as in Lemma 2.1 and Lemma 2.2. For $M > C$, let

$$\begin{aligned} T_M &:= \inf\{s \geq 0 : |\sigma_{s\wedge\tau}| \geq M\} = \inf\{s \geq 0 : \sigma_s \geq M\}, \\ S_M &:= \inf\{s \geq 0 : |\sigma_s| \geq M\}. \end{aligned}$$

Note that $\tau \leq T_M$ implies $T_M = \infty$. For fixed $t \geq 0$, we can then write

$$\begin{aligned} M\mathbb{P}(T_M \leq t) &= \mathbb{E}(\sigma_{t\wedge T_M} \mathbf{1}_{T_M \leq t}) = \mathbb{E}(\sigma_{t\wedge T_M} \mathbf{1}_{T_M \leq t} \mathbf{1}_{\tau > T_M}) \\ &= \mathbb{E}(\sigma_{t\wedge S_M} \mathbf{1}_{T_M \leq t} \mathbf{1}_{\tau > T_M}) \leq \mathbb{E}((C + \sigma_{t\wedge S_M}) \mathbf{1}_{T_M \leq t} \mathbf{1}_{\tau > T_M}) \\ &\leq \mathbb{E}(C + \sigma_{t\wedge S_M}) = C + \mathbb{E}(\sigma_{t\wedge S_M}). \end{aligned} \tag{24}$$

Below, we show a bound $\mathbb{E}(\sigma_{t\wedge S_M}) \leq K_0 + K_1 t$ that is uniform in M. Then, (24) implies that $\lim_{M\to\infty} \mathbb{P}(T_M \leq t) = 0$, which is the claim.

Choose $\hat{\beta}_2 > 0$ small enough such that $\beta_1\lambda_1 + \hat{\beta}_2\lambda_2 < 0$, and then choose $\bar{\beta}_2 > 0$ such that $\beta_2\sqrt{x} \le \bar{\beta}_2 + \hat{\beta}_2 x$ for all $x \ge 0$. The first equation in (2-PDV˜) then yields

$$\mathbb{E}(\sigma_{t\wedge S_M}) \le \beta_0 + \bar{\beta}_2 + \mathbb{E}(\beta_1 R_{1,t\wedge S_M} + \hat{\beta}_2 R_{2,t\wedge S_M}) \tag{25}$$

and we can focus on bounding the last expectation. Using the last two equations in (2-PDV˜), we have

$$\begin{aligned}\mathbb{E}(\beta_1 R_{1,t\wedge S_M} + \hat{\beta}_2 R_{2,t\wedge S_M}) = \beta_1 R_{1,0} + \hat{\beta}_2 R_{2,0} + \mathbb{E}\Big(\int_0^\infty \Big\{(\beta_1\lambda_1 + \hat{\beta}_2\lambda_2)\sigma_s^2 \\ - \lambda_1\beta_1 R_{1,s} - \lambda_2\hat{\beta}_2 R_{2,s}\Big\}\mathbf{1}_{s\le t\wedge S_M}\,ds\Big).\end{aligned} \tag{26}$$

Defining $\alpha := \beta_1\lambda_1 + \hat{\beta}_2\lambda_2 < 0$, the term under the integral is

$$\begin{aligned}\alpha\sigma_s^2 - \lambda_1\beta_1 R_{1,s} - \lambda_2\hat{\beta}_2 R_{2,s} &= \alpha\beta_0^2 + \alpha\beta_2^2 R_{2,s} + \alpha\beta_1^2 R_{1,s}^2 + 2\alpha\beta_0\beta_1 R_{1,s} \\ &\quad + 2\alpha\beta_0\beta_2\sqrt{R_{2,s}} + 2\alpha\beta_1\beta_2 R_{1,s}\sqrt{R_{2,s}} - \lambda_1\beta_1 R_{1,s} - \lambda_2\hat{\beta}_2 R_{2,s} \\ &= C + B\sqrt{R_{2,s}} + AR_{2,s}\end{aligned}$$

where

$$\begin{aligned}C &= \alpha\beta_0^2 + \alpha\beta_1^2 R_{1,s}^2 + 2\alpha\beta_0\beta_1 R_{1,s} - \lambda_1\beta_1 R_{1,s}\ , \\ B &= 2\alpha\beta_2(\beta_0 + \beta_1 R_{1,s})\ , \quad A = \alpha\beta_2^2 - \lambda_2\hat{\beta}_2\end{aligned}$$

and $A < 0$ due to $\alpha < 0$ and $\lambda_2\hat{\beta}_2 > 0$. Thus, the term is bounded from above by

$$\begin{aligned}C - \frac{B^2}{4A} &= \alpha\beta_0^2 + \alpha\beta_1^2 R_{1,s}^2 + 2\alpha\beta_0\beta_1 R_{1,s} - \lambda_1\beta_1 R_{1,s} - \frac{4\alpha^2\beta_2^2(\beta_0^2 + \beta_2^2 R_{1,s}^2 + 2\beta_0\beta_1 R_{1,s})}{4(\alpha\beta_2^2 - \lambda_2\hat{\beta}_2)} \\ &= C' + B' R_{1,s} + A' R_{1,s}^2\end{aligned}$$

with

$$C' = \alpha\beta_0^2 + \frac{\alpha^2\beta_0^2\beta_2^2}{\lambda_2\hat{\beta}_2 - \alpha\beta_2^2}, \quad B' = 2\alpha\beta_0\beta_1 - \lambda_1\beta_1 + \frac{2\alpha^2\beta_2^2\beta_0\beta_1}{\lambda_2\hat{\beta}_2 - \alpha\beta_2^2},$$
$$A' = \alpha\beta_1^2 + \frac{\alpha^2\beta_1^2\beta_2^2}{\lambda_2\hat{\beta}_2 - \alpha\beta_2^2}.$$

Here again $A' < 0$ due to $\alpha < 0$ and $\lambda_2\hat{\beta}_2 > 0$, so that the term is bounded from above by the constant $L := C' - \frac{B'^2}{4A'}$. Using this bound in (26) yields

$$\begin{aligned}\mathbb{E}(\beta_1 R_{1,t\wedge S_M} + \hat{\beta}_2 R_{2,t\wedge S_M}) &\le \beta_1 R_{1,0} + \hat{\beta}_2 R_{2,0} + \mathbb{E}\left(\int_0^\infty L\mathbf{1}_{s\le t\wedge S_M}\,ds\right) \\ &\le \beta_1 R_{1,0} + \hat{\beta}_2 R_{2,0} + |L|t\end{aligned}$$

and thus

$$\mathbb{E}(\sigma_{t\wedge S_M}) \le \beta_0 + \bar{\beta}_2 + \beta_1 R_{1,0} + \hat{\beta}_2 R_{2,0} + |L|t =: K_0 + K_1 t$$

as claimed. □

Remark 4.4. Proposition 4.2 and its proof generalize to volatility models having the same dynamics as (2-PDV) for $(R_{1,t}, R_{2,t})$ but a more general functional $\sigma_t = f(R_{1,t}, R_{2,t})$, where f satisfies the conditions of Remark 2.5 and in addition there exist constants $L_1, L_2, L_3, L \in \mathbb{R}$ such that

$$
\begin{aligned}
f(x,y) &\le L_0 + L_1 x + L_2 y, \\
(L_1\lambda_1 + L_2\lambda_2) f(x,y)^2 - \lambda_1 L_1 x - \lambda_2 L_2 y &\le L.
\end{aligned}
$$

The constant L_1 typically needs to be negative, as in (2-PDV) where $L_1 = \beta_1 < 0$.

Our final remark details why the above proof does not extend to the 4-factor model.

Remark 4.5. The general line of argument given above may extend to (4-PDV). Indeed, the following SDE under $\mathbb{P}$ has a unique strong solution $(\sigma_s)_{s\ge 0}$ up to a possible time of explosion:

$$
\begin{aligned}
\sigma_t &= \beta_0 + \beta_1 R_{1,t} + \beta_2 \sqrt{R_{2,t}} \\
R_{1,t} &= (1-\theta_1) R_{1,0,t} + \theta_1 R_{1,1,t} \\
R_{2,t} &= (1-\theta_2) R_{2,0,t} + \theta_2 R_{2,1,t} \\
dR_{1,j,t} &= \lambda_{1,j}\sigma_t dW_t + \lambda_{1,j}(\sigma_t^2 - R_{1,j,t})dt, \quad j \in \{0,1\} \\
dR_{2,j,t} &= \lambda_{2,j}\left(\sigma_t^2 - R_{2,j,t}\right)dt, \quad j \in \{0,1\}.
\end{aligned}
\tag{4-PDV\~{}}
$$

Given $C \ge 0$, define $\tau = \inf\{t \ge 0 : \sigma_s < -C\}$. If the process $(\sigma_{s\wedge\tau})_{s\ge 0}$ a.s. does not explode in finite time, then the assertion of Theorem 1.3 extends to (4-PDV). However, the proof for non-explosiveness given in Lemma 4.3 does not extend directly to the present setting.

Indeed, the system (4-PDV˜) satisfies bounds analogous to (24) and (25). Applying the same procedure as in the proof of Lemma 4.3, and recalling the notation used in (20),

$$
\begin{aligned}
\mathbb{E}(\beta_1 R_{1,t\wedge S_M} + \hat\beta_2 R_{2,t\wedge S_M}) =& \beta_1 R_{1,0} + \hat\beta_2 R_{2,0} + \mathbb{E}\Big(\int_0^\infty \Big\{ (\beta_1\bar\lambda_1 + \hat\beta_2\bar\lambda_2)\sigma_s^2 \\
&- \lambda_{1,1}\theta_1\beta_1 R_{1,1,s} - \lambda_{1,0}(1-\theta_1)\beta_1 R_{1,0,s} \\
&- \lambda_{2,1}\theta_2\hat\beta_2 R_{2,1,s} - \lambda_{2,0}(1-\theta_2)\hat\beta_2 R_{2,0,s} \Big\} \mathbf{1}_{s\le t\wedge S_M} ds \Big).
\end{aligned}
$$

The integrand is bounded from above by a quadratic form in $(R_{1,0,s}, R_{1,1,s}, \sqrt{R_{2,0,s}}, \sqrt{R_{2,1,s}})$; however, the matrix defining the form is not negative definite. Hence, we cannot bound it uniformly as we did in Lemma 4.3.

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