A phase transition in a random loop model on infinite trees

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Spatial Random Permutations

Quick recap of permutations

- Finite permutations decompose as a product of disjoint cycles.
- Infinite permutations do too, but the cycles may be infinite.

Example:

i	1	2	3	4	5
$\sigma(i)$	3	5	1	2	4

Using cycle notation, σ can be decomposed as (1 3)(2 5 4).

What is a spatial random permutation?

A “spatial random permutation” vaguely refers to a random permutation model whose index set possesses some *spatial* or *geometric* structure which affects the permutation structure.

For us, this will be via the index set being the vertices of a graph.

Tóth's model

Introduced in [Tóth '93] to study the quantum Heisenberg ferromagnet:

- Let $\Lambda \subseteq \mathbb{Z}^d$ be a finite box.
- Let μ_T be a measure on permutations on the vertices of Λ.
- Define ν_T by

$$d\nu_T(\sigma) = \frac{1}{Z} \cdot 2^{\#\text{cycles}(\sigma)}\, d\mu_T(\sigma),$$

  i.e. reweight the permutations by the number of cycles present and normalize.

Tóth's model

Let σ_Λ be a sample from ν_T.

Very roughly speaking, Tóth showed that, if $\Lambda \to \mathbb{Z}^d$, then there is a correspondence (with $T =$ time/inverse temperature):

appearance of macrocycles in σ_Λ	$\longleftrightarrow$	physical phase transition in the spin-1/2 q-Heisenberg ferromagnet.

This is a recurring theme with spatial random permutation models connected to physical models. So we want to prove macrocycles exist.

What is μ_{T}? The random stirring process

Consider a graph $G = (V, E)$. The random stirring process (RSP) is a process of permutations on V: $(\sigma_t)_{t \geq 0}$, with $\sigma_0 = \mathrm{Id}$.

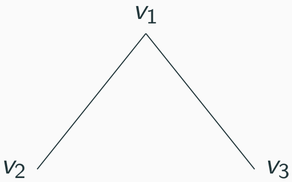
- To each $e \in E$, associate an independent rate 1 Poisson clock.
- Suppose $e = \{u, v\}$ rings at time t. Left compose σ_{t-} with $(u\ v)$:

$$\sigma_t = (u\ v) \circ \sigma_{t-},$$

so we maintain right-continuity.

An example

Suppose $\{v_1, v_2\}$ rings at $t = 1/2$
and $\{v_1, v_3\}$ rings at $t = 1$.

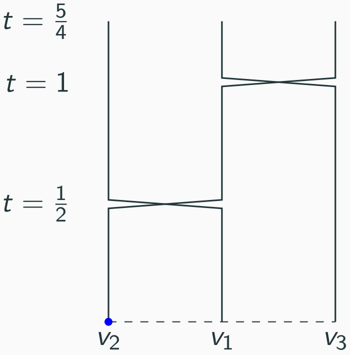


Then σ_t is

$$\sigma_t = \begin{cases} \mathrm{Id} & 0 \leq t < \frac{1}{2} \\ (v_1\ v_2) & \frac{1}{2} \leq t < 1 \\ (v_1\ v_2\ v_3) & t \geq 1. \end{cases}$$

$[(v_1\ v_3) \circ (v_1\ v_2) = (v_1\ v_2\ v_3).]$

Another view

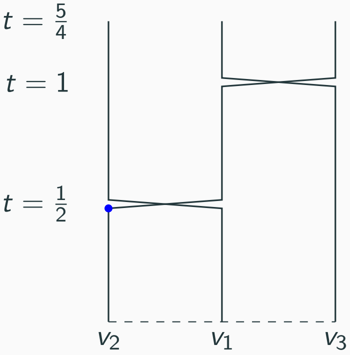


To know $\sigma_{5/4}(v_2)$, place a particle at the vertex v_2, and let it move upwards at unit speed.

When it hits a cross, it jumps over instantly and continues motion up.

The vertex the particle is at at time $t = 5/4$ is exactly $\sigma_{5/4}(v_2)$; in this case, v_3.

Another view



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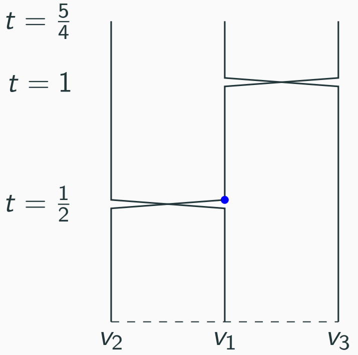

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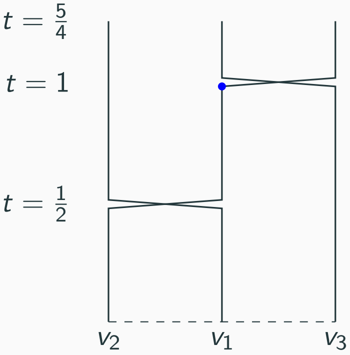

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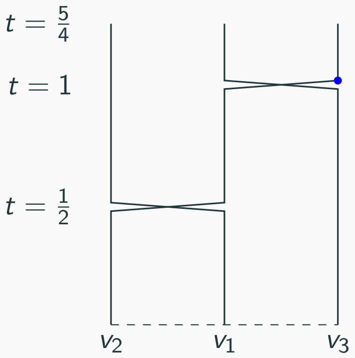

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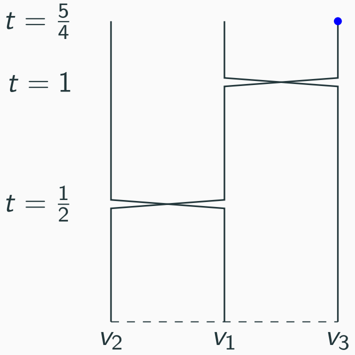

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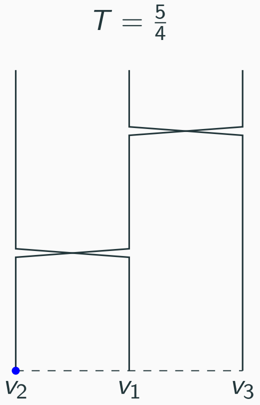
The Cyclic Time Random Meander (CyTRM)

We can take this viewpoint in general. Fix T. To study σ_T, we study a related process, called the cyclic time random meander of parameter T: CyTRM(T). Defined on $(0, \infty)$.

For a graph $G = (V, E)$, associate to each $e \in E$ an independent rate 1 Poisson point process on $[0, T)$. These are the crosses from earlier.

Visualize a vertical pole of height T at each $v \in V$. Poles are connected by crosses at the points of the point processes.

The Cyclic Time Random Meander (CyTRM)



Let $X = \text{CyTRM}(T)$ started at a vertex v. It is defined on $[0, \infty)$, with $X(0) = v$.

The motion is as before, except when we reach the top of a pole.

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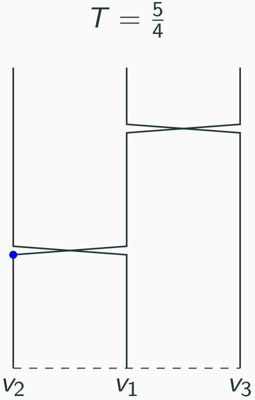

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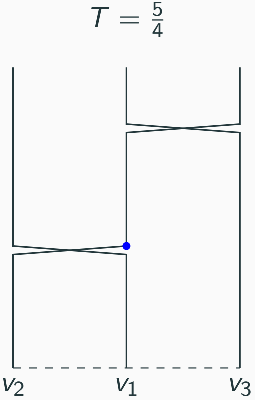

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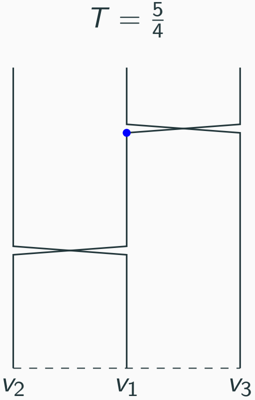

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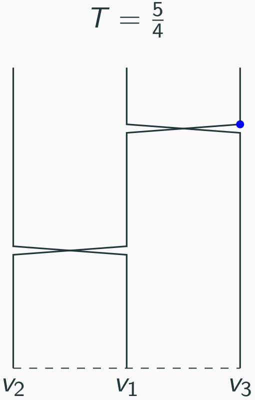

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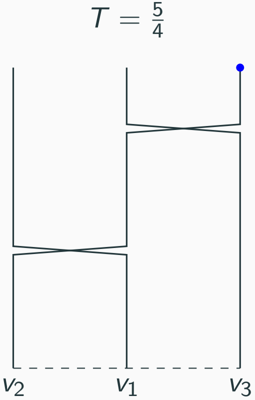

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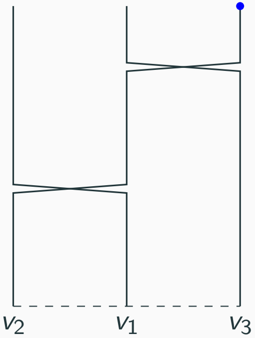

When the particle reaches the top of a pole, it instantly appears at the bottom and continues motion: it is *cyclic*.

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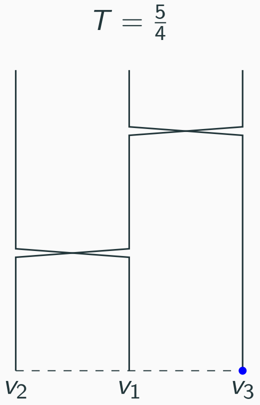

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Thus $X(T) = \sigma_T(v)$.
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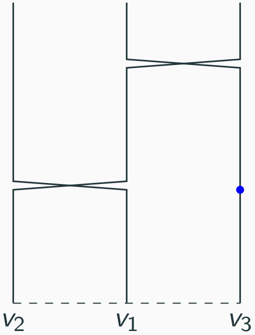

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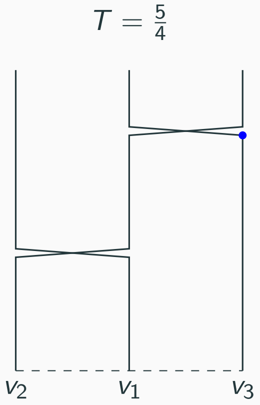

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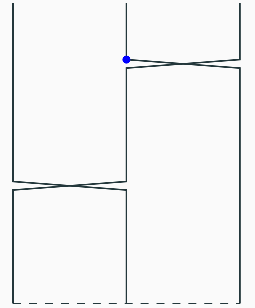

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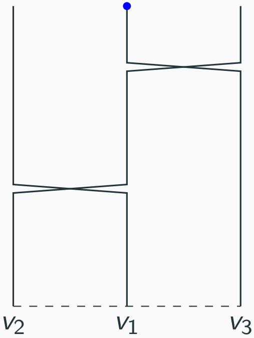


When the particle reaches the top of a pole, it instantly appears at the bottom and continues motion: it is *cyclic*.

Thus $X(T) = \sigma_T(v)$.
Here, with $T = \frac{5}{4}$, $X\left(\frac{5}{4}\right) = v_3$.

By the cyclic nature,
$X(2T) = \sigma_T^2(v)$.
Here, $X\left(\frac{5}{2}\right) = v_1$.

The Cyclic Time Random Meander

Similarly, if X is started at v,

$$X(kT) = \sigma_T^k(v).$$

So if v lies in an infinite cycle in σ_T, $X(kT) \neq v$ for any k.

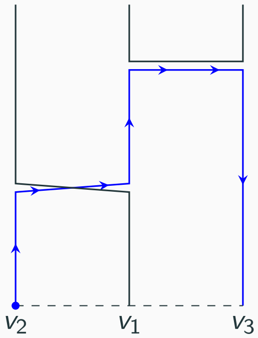
The logic can be extended to say that

transience of CyTRM(T) started at v	$\iff$	$v \in$ infinite cycle in σ_T

We want to analyse transience of CyTRM(T) as a function of T.

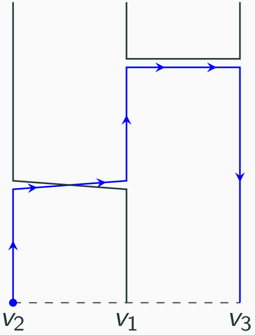
The Actual Model

Introduced by Ueltschi in [Ueltschi '13]. Instead of just crosses, we also have *double bars*: when the particle encounters a double bar, it jumps over instantly, but its direction of motion is *reversed*.



The Actual Model

If the particle hits the bottom while moving down, it cycles to the top instantly. In either case, the direction of motion is maintained.



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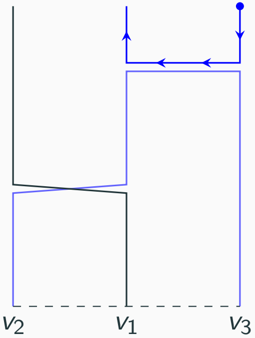


The Actual Model: CyTRM(u, T)

- Collectively, crosses and double bars will be called *bridges*.
- New parameter $u \in [0, 1]$: probability that a bridge is a cross. Otherwise, a double bar.
- The $u = 1$ case was the original model first described.
- Denote the modified model CyTRM(u, T).
- As before, our interest is in transience of this process.

Main Result

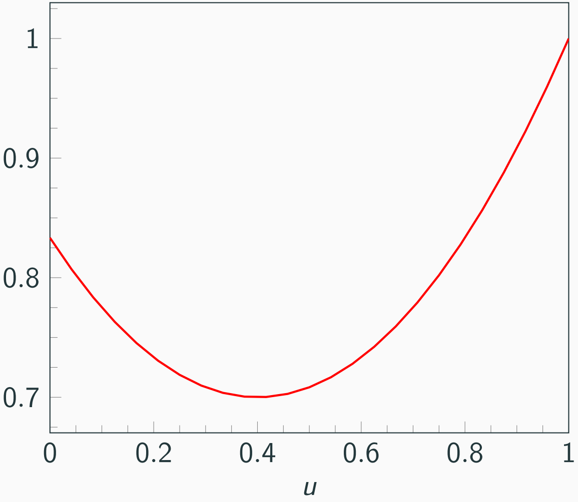
Theorem

1. *Let G be a rooted tree of bounded degree with at least d_0 offspring at every vertex. Then there exists a T_0 such that* $\mathrm{CyTRM}(u, T)$ *is transient when $T > T_0$.*
   We may take $T_0 = 0.495$ and $d_0 = 16$.
2. *If G has exactly d offspring at every vertex, then there exists $T_c(u, d)$ such that* $\mathrm{CyTRM}(u, T)$ *is transient for $T > T_c$ and recurrent for $0 < T < T_c$.*

Asymptotic formula for T_c from [Björnberg-Ueltschi '18]:

$$T_c(u, d) = \frac{1}{d} + \frac{1 - u(1-u) - \frac{1}{6}(1-u)^2}{d^2} + o(d^{-2}).$$

Plot of $1 - u(1-u) - \frac{1}{6}(1-u)^2$



Previous Work

Breaking up the parameter space

Let G be the regular tree of offspring number d.

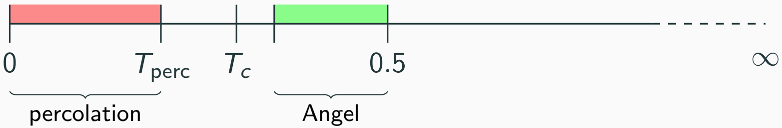
The percolation probability for G is d^{-1}. If T is such that the probability of at least one bridge on an edge is less than d^{-1}, we have recurrence.

So we have recurrence for $T < T_{\text{perc}} := \log \frac{d}{d-1} = \frac{1}{d} + \frac{1}{2d^2} + o(d^{-2})$.

(by equating $1 - e^{-T}$ with d^{-1})

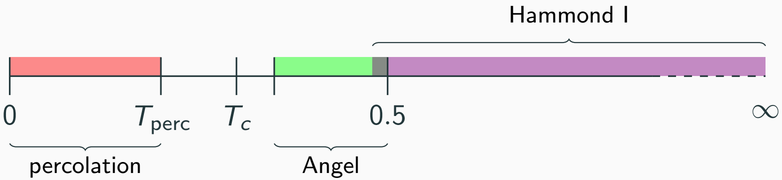
Previous work: Angel '03

- $u = 1$ case studied on infinite regular trees; $u \in [0, 1]$ easily adapted.
- Established transience in a finite interval slightly above $T_c(1, d)$: $[d^{-1} + 2d^{-2}, \frac{1}{2}]$
- Outline: Identified a local configuration which forces transience, and showed that vertices with the local configuration form a Galton-Watson tree. In the mentioned interval, the GW mean offspring number is greater than 1.
- The local configuration requires a small number of bridges, which is unlikely for T high; the argument works only for low T.



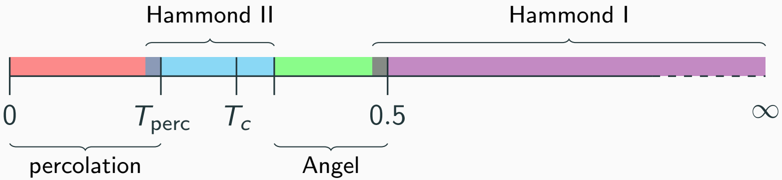
Previous work: Hammond I

- Only $u = 1$ case.
- Establishes a similar result as ours: there exists $T_0 (= 429 d^{-1})$ such that for sufficiently large d and $T > T_0$, CyTRM$(1, T)$ is transient.
- A large part of our work is extending and simplifying this argument; will speak more later.



Previous work: Hammond II

- Only $u = 1$ case, but also applies to $u \in [0, 1]$, as observed in [Björnberg-Ueltschi '18].
- Establishes monotonicity in a small interval around critical point: if $d^{-1} < T < T' < d^{-1} + 2d^{-2}$ and CyTRM(u, T) is transient, then so is CyTRM(u, T').



Previous Work

[Björnberg-Ueltschi '18]

- Found an asymptotic expansion of T_c:

$$T_c(u,d) = \frac{1}{d} + \frac{1 - u(1-u) - \frac{1}{6}(1-u)^2}{d^2} + o(d^{-2}).$$

- They show transience for $T \in \left(T_c, \frac{1}{d} + \frac{A}{d^{-2}}\right]$ for $d > d_0 = d_0(A)$
- But in principle transience may not hold for arbitrarily large T...
- Shows that $T_c(u,d) > T_{\text{perc}}$ asymptotically.

Previous Work

Björnberg-Ueltschi '18 preprint

- Extend their asymptotic formula when cycles are reweighted by $\theta > 0$ (as in Tóth's model, where $\theta = 2$).

$$T_c(u, \theta, d) = \frac{\theta}{d} + \frac{\theta\left[1 - \theta u(1-u) - \frac{1}{6}\theta^2(1-u)^2\right]}{d^2} + o(d^{-2}).$$

Previous Work

Betz-Ehlert-Lees-Roth '18 preprint

- Further the expansion of $T_c(u, d)$ to order 4 in d^{-1}.
- Obtain sharper bounds for T_c for finite d.
  - The gap between T_{perc} and T_c is established for all $d \geq 3$.

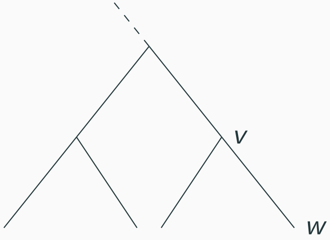
Proof Overview

Proof Overview

- We need to show that for high T, CyTRM(u, T) escapes to infinity with positive probability.
- This is simple when $u = 1$ and "$T = \infty$": it's just simple random walk on the tree.
- So why is SRW on a tree transient?

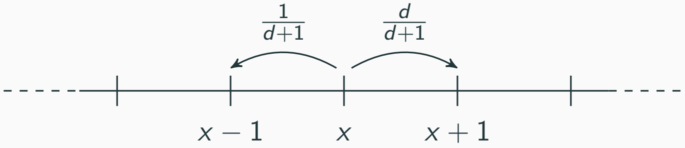
Simple Case: SRW on tree

- Uniformly positive probability p of departing to new territory at each step—a "frontier departure".
- Then it either never returns, or, if it returns, two possibilities:
    - moves to new territory again—an "acceptable return".
    - moves back into old territory
- If not an acceptable return, positive probability of moving to new territory next time.



Simple Case: SRW on tree

So the distance from the root stochastically dominates the following random walk on $\mathbb{Z}$:



This has positive drift and so escapes to $+\infty$ with positive probability, which implies the original SRW is transient.

T finite and $u \in [0, 1]$

Now we don't have complete independence. But after a frontier departure, we have some independence for duration T.

We introduce a proxy for the distance from the root: the number of "useful bridges" at time t.

Think of them as barriers the particle must undo to return to the root.

Main property of useful bridges: if an edge supports a useful bridge at time t, it has been crossed only once until that time.

$$\implies \#\text{useful bridges} \leq \text{distance from root.}$$

T finite and $u \in [0, 1]$

We have to redefine an "acceptable return":

A (first) return to a previously visited edge e is acceptable if the particle then leaves to an unvisited vertex *and moves forward consecutively N times by duration T*.

Note that the type of bridge crossed doesn't matter, as long as the direction is away from the root.

T finite and $u \in [0, 1]$

We can lower bound probabilities of

(i) frontier departure

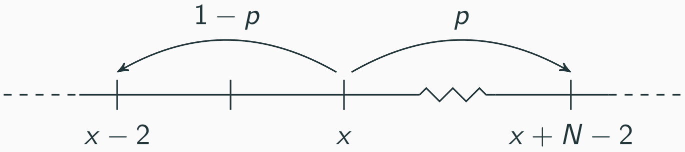
(ii) moving forward N times in time T given a frontier departure.

This gives a lower bound $p(N, T, d)$ for the probability of an acceptable return.

When a return is acceptable, gain $N - 2$ useful bridges at least.
When not acceptable, lose 2 useful bridges at most.

Completing the argument

Looking at the number of useful bridges at suitable stopping times, it dominates the following random walk on $\mathbb{Z}$ with $p = p(N, T, d)$:



$$\text{Drift} = N \times \frac{d-1}{d+1}(1 - e^{-(d+1)T/2}) \times \left(1 - \frac{1}{d+1}\right)^N \left[1 - e^{N-(d+1)T}\left(\frac{(d+1)T}{N}\right)^N\right] - 2.$$

Play with the parameters to make it positive $\implies$ transience.

Selected References

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Thank you