Upper tail scaling limit of continuum path measures in KPZ

Milind Hegde
(based on joint works with Shirshendu Ganguly and Lingfu Zhang)

Columbia University

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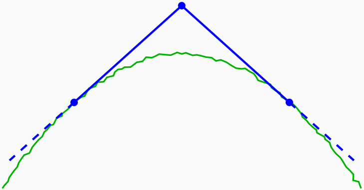


The directed landscape and continuum directed random polymer

- Directed landscape $\mathcal{L} : \mathbb{R}^4 \to \mathbb{R}$ is a random continuous function expected to be a universal KPZ scaling limit.

- It is a last passage percolation problem: continuous paths $\gamma : [s,t] \to \mathbb{R}$ are given a random weight $w(\gamma)$, and

$$\mathcal{L}(x,s;y,t) = \sup_{\substack{\gamma:[s,t]\to\mathbb{R} \\ \gamma(s)=x,\gamma(t)=y}} w(\gamma).$$

  The *geodesic* is the path achieving the maximum.

- $\mathcal{L}(0,0;\cdot,1)$ is the weight profile of the geodesic.

The directed landscape and continuum directed random polymer

- Positive temperature analogue: continuum directed random polymer (CDRP).

- White noise environment ξ on $\mathbb{R} \times \mathbb{R}$, continuous paths $\gamma : [s, t] \to \mathbb{R}$ have a weight $w(\gamma)$,

$$w(\gamma) = \int_s^t \xi(z, \gamma(z))\,\mathrm{d}z.$$

- Polymer measure defined via partition function Z:

$$Z(x, s; y, t) = \mathsf{E}^{x,s;y,t}[\exp(w(\gamma))].$$

  Z is a function of ξ; $\mathsf{E}^{x,s;y,t}$ is over γ only and distributes it as a Brownian bridge from (x, s) to (y, t).

- $\mathfrak{h}(0, 0; \cdot, 1) = \log Z(0, 0; \cdot, 1)$ is the free energy profile.

The directed landscape and continuum directed random polymer

- For both the DL and the CDRP, the location of the geodesic Γ or marginal of the polymer measure μ at a height s are given by convolution formulas:

$$\Gamma(s) = \operatorname*{argmax}_{x \in \mathbb{R}} \mathcal{L}(0,0;x,s) + \mathcal{L}(x,s;0,1)$$

$$\mu(\Gamma(s) = x) = \frac{\exp(\mathfrak{h}(0,0;x,s) + \mathfrak{h}(x,s;0,1))}{\int_{\mathbb{R}} \exp(\mathfrak{h}(0,0;y,s) + \mathfrak{h}(y,s;0,1))\, \mathrm{d}y}.$$

- Much structure of the path measures can be understood via the profile processes $\mathcal{L}$ and $\mathfrak{h}$.

The KPZ equation

- The free energy profile $\mathfrak{h}$ solves the KPZ equation, given by

$$\partial_t h = \frac{1}{4}(\partial_x h)^2 + \frac{1}{4}\partial_x^2 h + \xi,$$

where ξ is space-time white noise on $\mathbb{R} \times (0, \infty)$ and $h : \mathbb{R} \times (0, \infty) \to \mathbb{R}$.

- We will use the Cole-Hopf notion of solution to the KPZ equation, i.e., $\mathfrak{h}$ is defined via $\log Z$ where Z solves the multiplicative SHE:

$$\begin{cases} \partial_t Z(y, t \mid x, s) = \frac{1}{4}\partial_y^2 Z(y, t \mid x, s) + \xi(y, t) Z(y, t \mid x, s) \\ Z(y, s \mid x, s) = \delta_0(x - y) & \text{for all } s > 0 \end{cases}$$

- Introduced by Alberts-Khanin-Quastel, regularity recently studied by Alberts–Janjigian–Rassoul-Agha–Seppäläinen.

Upper tail large deviations of $\mathfrak{h}$ and $\mathcal{L}$

The upper tails and upper large deviations of these two processes have been studied for quite some time, eg.

- One-point large deviations/upper tails for $\mathcal{L}$ were known from work of Tracy-Widom, see also Rider-Ramirez-Virág.
- Seppäläinen and Johansson studied one-point large deviations of prelimiting zero temp. models (TASEP and geometric LPP resp.)
- Quastel-Tsai studied profile large deviations of TASEP.
- Corwin-Ghosal, Ganguly-H., Tsai-Lin studied upper tails of $\mathfrak{h}$.
- Prelimiting models for $\mathfrak{h}$: ASEP by Das-Zhu and Damron-Petrov-Sivakoff.
- and more...

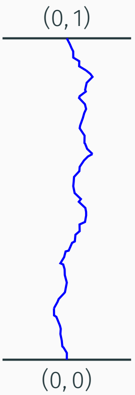
The random path under the upper tail conditioning

Interested in the behaviour of the geodesic or polymer measure when $\mathcal{L}(0,0;0,1)$ or $\mathfrak{h}(0,0;0,1)$ is large, say $> \theta$.

An energy-entropy tradeoff occurs: larger fluctuations give the geodesic more choice of paths, but the cost grows with θ.

So the path measure will become more rigid, i.e., have much smaller transversal fluctuations. (It also becomes a "highway" for geodesics to nearby points.)

Heuristically, a uniformly (on some scale) random path is chosen and made to be the geodesic.



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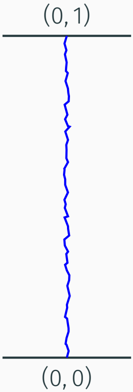

The scaling limit of the geodesic under upper tail conditioning

Let $\Gamma_\theta : [0,1] \to \mathbb{R}$ be the geodesic in the directed landscape from $(0,0)$ to $(0,1)$, conditioned on $\mathcal{L}(0,0;0,1) > \theta$.

Theorem (Ganguly-H.-Zhang)

$\theta^{1/4}\Gamma_\theta \xrightarrow{d} \frac{1}{2}B$ in the uniform topology with B = standard Brownian bridge.

Note that we identify the fluctuation scale to be $\theta^{-1/4}$ as well as the scaling limit.

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This result had been conjectured by Zhipeng Liu, who proved the one-point scale and one-point convergence using exact formulas.

A similar result had earlier been conjectured by Basu-Ganguly for the geodesic in exponential LPP under a large deviation conditioning.

Let $\Gamma_\theta^{\mathrm{ann}} : [0,1] \to \mathbb{R}$ be a sample from the annealed polymer measure from $(0,0)$ to $(0,1)$ in the CDRP, under the conditioning that $\mathfrak{h}(0,0;0,1) > \theta$.

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Liu's methods did not apply to positive temperature, and so one-point convergence was also not previously known.

The scaling limit of the CDRP polymer measure under upper tail conditioning

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What about the quenched situation? The polymer measure concentrates in a $O(\theta^{-1/2})$ window around a random "backbone" $\Gamma_\theta^{\mathrm{back}}$, and $\theta^{-1/4}\Gamma_\theta^{\mathrm{back}} \xrightarrow{d} \frac{1}{2}B$.

A crucial ingredient: an upper tail limit shape

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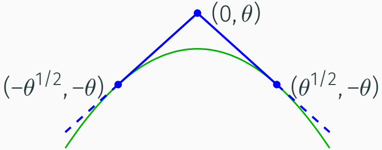

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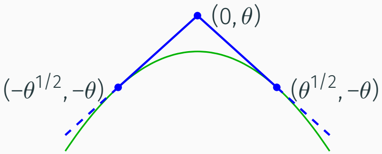

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Theorem (Ganguly-H.)

There exist θ_0 and $c > 0$ such that, for all $\theta > \theta_0$, and $M > 0$,

$$\mathbb{P}\left(\sup_{x\in[-\theta^{1/2},\theta^{1/2}]} |\mathfrak{h}(x) - \mathrm{Tent}_\theta(x)| > M\theta^{1/4} \;\middle|\; \mathfrak{h}(0) = \theta\right) \leq \exp(-cM^2).$$

A second crucial ingredient: an upper tail comparison

From the limit shape, one can obtain sharp asymptotics for the upper tail:

Theorem (Ganguly-H.)

There exist $C < \infty$ and θ_0 such that, for all $\theta > \theta_0$,

$$\exp\left(-\frac{4}{3}\theta^{3/2} - C\theta^{3/4}\right) \leq \frac{1}{\mathrm{d}\theta}\mathbb{P}\left(\mathfrak{h}(0) \in \mathrm{d}\theta\right) \leq \exp\left(-\frac{4}{3}\theta^{3/2} + C\theta^{3/4}\right).$$

As an immediate consequence, the same bounds also hold for $\mathbb{P}(\mathfrak{h}(0) > \theta)$.

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By more refined coupling arguments, we also get a comparison statement:

Theorem (Ganguly-H.-Zhang)

There exist $C < \infty$ and θ_0 such that, for all $\delta > 0$ and $\theta > \theta_0$,

$$\frac{\mathbb{P}\left(\mathfrak{h}(0) \geq \theta + \delta\right)}{\mathbb{P}\left(\mathfrak{h}(0) \geq \theta\right)} = \exp\left(-2\delta\theta^{1/2} + O(\delta L^{-1/4})\right).$$

Heuristics and proof ideas

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- $x \mapsto \mathfrak{h}(0, 0; x, 1) + x^2$ and $x \mapsto \mathcal{L}(0, 0; x, 1) + x^2$ are stationary.
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- So the geodesic/polymer fluctuating by ε means it suffers a loss of $O(\varepsilon^2)$.
- Under the conditioning of being $> \theta$, this loss has to be made up; akin to $\mathfrak{h}(0, 0; 0, 1) > \theta + O(\varepsilon^2)$ (by stationarity).
- But $\dfrac{\mathbb{P}(\mathfrak{h}(0, 0; 0, 1) > \theta + O(\varepsilon^2))}{\mathbb{P}(\mathfrak{h}(0, 0; 0, 1) > \theta)} \approx \exp(-C\varepsilon^2\theta^{1/2})$.
- This is $O(1)$ exactly when $\varepsilon = O(\theta^{-1/4})$.

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- Scale $x \mapsto x\theta^{-1/4}$. By the comparison theorem, this ratio is

$$\exp\left(-2\theta^{1/2}(x\theta^{-1/4})^2[s^{-1} + (1-s)^{-1}]\right) = \exp\left(-\frac{x^2}{2 \times \frac{1}{4}s(1-s)}\right),$$

i.e., the exponent of the density of $\frac{1}{2}B(s) = N(0, \frac{1}{4}s(1-s))$.

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The comparison theorem is not sharp enough to obtain eg. constants.

The actual argument compares probabilities: we look at

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The main insight is that these events essentially imply that there is a tent peaked at x or y, so the above is approximately

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The tent picture allows us to say that

$$\mathbb{P}\left(\Gamma^{\theta}(s) = x, \mathcal{L}(0,0;0,1) > \theta \mid \mathcal{L}(0,0;x,s) = h_1, \mathcal{L}(x,s;0,1) = h_2\right)$$

is essentially the same as the y-analogue. The ratio of probabilities of the conditioning events gives the *ratio* of densities, as before.

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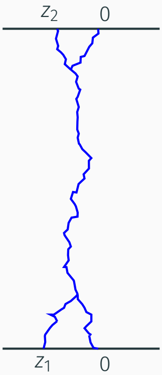

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- $(\Gamma(s), \Gamma(t)) = \underset{z_1, z_2}{\operatorname{argmax}} \mathcal{L}(0, 0; z_1, s) + \mathcal{L}(z_1, s; z_2, t) + \mathcal{L}(z_2, t; 0, 1)$
- Coalescence gives quadrangle equality:
  $\mathcal{L}(z_1, s; z_2, t) + \mathcal{L}(0, s; 0, t) = \mathcal{L}(z_1, s; 0, t) + \mathcal{L}(0, s; z_2, t)$
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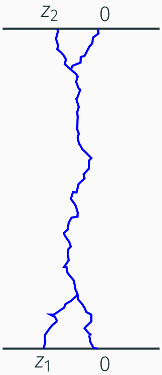

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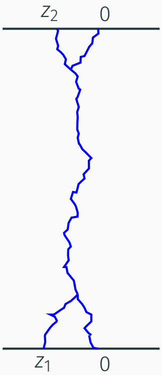
- So the double argmax decouples.
- Heuristically, coalescence also implies the two process on the RHS are (approximately) independent.
- The proof of independence relies crucially on shift invariance of $\mathcal{L}$ or free energy fields.



The source of $\theta^{-1/2}$ window around backbone

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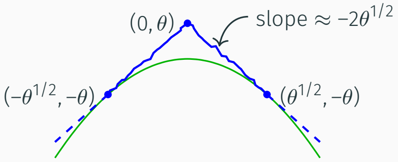
- Recall that when $\mathfrak{h}(0,0;0,1) > \theta$, the profile has slope approximately $-2\theta^{1/2}$.
- So at distance $O(\theta^{-1/2})$, the loss in free energy is $O(1)$; all such locations are therefore competitive for the polymer measure.
- Different scale in zero temp: the argmax location will be on scale θ^{-1}, as then the slope loss and Brownian fluctuations are of the same order, $\theta^{-1/2}$.



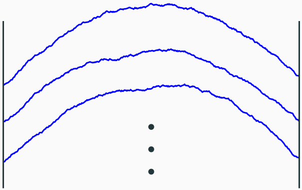
Summary

- Using geometric methods + Brownian Gibbs properties, we can obtain the shape of the weight and free energy profiles under upper tail events.
- These also give sharp upper tail asymptotics and probability comparison statements.
- With these + “tent” picture, can prove that geodesic/polymer measure rescaled by $\theta^{-1/4}$ converges to a Brownian bridge, under upper tail.
- Further, the polymer measure fluctuates on scale $\theta^{-1/2}$ around a random “backbone” curve.

The Brownian Gibbs property

The resampling property

Both $\mathfrak{h}(0,0;\cdot,1)$ and $\mathcal{L}(0,0;\cdot,1)$ can be embedded as the top/lowest-indexed curve in a $\mathbb{N}$-indexed ensemble of random continuous curves.



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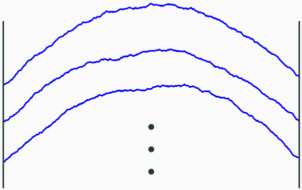

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For the $\mathcal{L}$ ensemble, it says the conditional distribution of $\mathcal{L}$ on an interval is a non-intersecting Brownian bridge (of rate 2).

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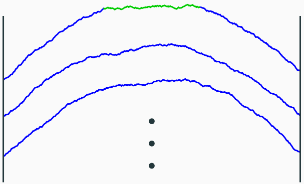

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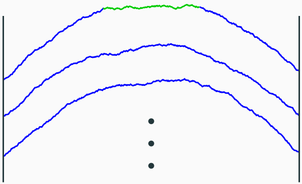


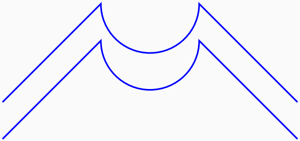
A useful heuristic to keep in mind:

$\mathfrak{h}$ and $\mathcal{L}$ are like Brownian bridges conditioned to stay above a parabola $-x^2$ with which they share endpoints.

A convex consequence

A key idea is that the resampling in terms of Brownian bridges implies that the limit shapes should be convex.

Indeed, suppose the limit shape of the top curve is not convex in some neighbourhood. This pushes the second curve down on the interval.



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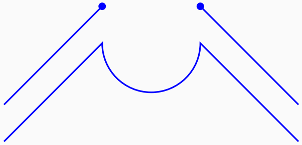
Then resample the top curve on that interval. *Since the non-convexity means the second curve is far away, Brownian bridge naturally avoids it.*



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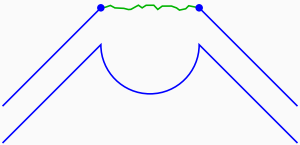

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