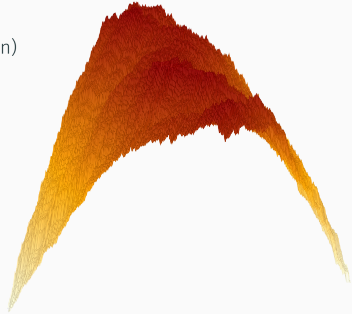
The KPZ scaling limit of the colored asymmetric simple exclusion process

Milind Hegde
(based on joint work with Amol Aggarwal and Ivan Corwin)

Columbia University



CRM-ISM Montreal Probability seminar
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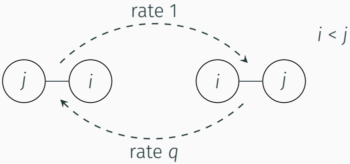
Models and main results

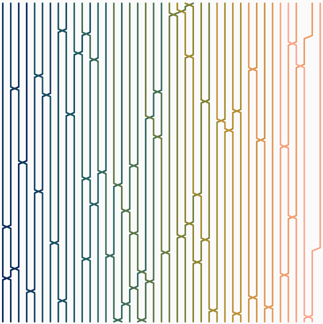
The colored ASEP

Fix $q \in [0, 1)$ and place a particle of "color" $-k$ at location k for every $k \in \mathbb{Z}$.



Particles attempt to swap positions to the left and right with rates q and 1, respectively. Swaps succeed if the initiating particle is of higher color.

Individual particle behavior



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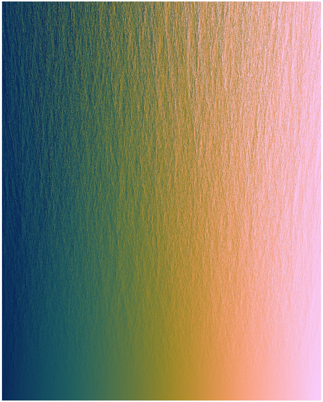

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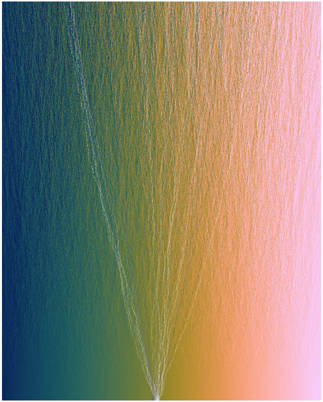


The particles lie in a *rarefaction fan* parametrized by speed $\alpha \in (-1, 1)$.

The colored height function

The colored ASEP height function h^{ASEP} is

$$h^{\mathrm{ASEP}}(x, 0; y, t) := \#\text{ particles of initial position} \leq x \text{ to right of } y \text{ at time } t.$$

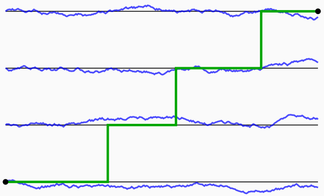
A lot is known about $y \mapsto h^{\mathrm{ASEP}}(0, 0; y, t)$ in the $t \to \infty$ limit: e.g., after rescaling,

- $h^{\mathrm{ASEP}}(0, 0; 0, t)$ converges to the GUE Tracy-Widom distribution of RMT [Tracy-Widom '09]
- $y \mapsto h^{\mathrm{ASEP}}(0, 0; y, t)$ converges to the parabolic Airy_2 process [Quastel-Sarkar '22]

We are interested in the joint limit $(x, y) \mapsto h^{\mathrm{ASEP}}(x, 0; y, t)$.

The limiting object: The Airy sheet

Aity sheet $\mathcal{S}$ arises as the limit of a model of a random directed metric: Brownian last passage percolation (LPP) [Dauvergne-Ortmann-Virág].



With $B = (B_1, \ldots, B_n)$ i.i.d. Brownian motions,

$$B[(x,n) \to (y,1)] = \sup_{\gamma} B[\gamma],$$

where the weight $B[\gamma]$ of a directed path γ is the integral of B over γ, i.e., sum of increments along the B_i.

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Theorem (Dauvergne-Ortmann-Virág)

As $\varepsilon \to 0$,

$$\varepsilon^{1/3}\left(B[(2\varepsilon^{-2/3}x, \varepsilon^{-1}) \to (\varepsilon^{-1} + 2\varepsilon^{-2/3}y, 1)] - 2\varepsilon^{-1} + 2(x-y)\varepsilon^{-2/3}\right)$$

converges in distribution to the Airy sheet $\mathcal{S}(x, y)$ as a continuous function on $\mathbb{R}^2$ uniformly on compact sets.

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converges in distribution to the Airy sheet $\mathcal{S}(x, y)$ *as a continuous function on* $\mathbb{R}^2$ *uniformly on compact sets.*

The scaling exponents $\frac{1}{3}$ and $\frac{2}{3}$ are characteristic of the Kardar-Parisi-Zhang universality class.

Main result: Colored ASEP to Airy sheet

Recall

$$h^{\mathrm{ASEP}}(x, 0; y, t) = \#\text{ particles of initial position } \leq x \text{ to right of } y \text{ at time } t.$$

Theorem (Aggarwal-Corwin-H.)

Fix $q \in [0, 1)$ and $\alpha = 0$. The rescaled colored ASEP height function

$$\varepsilon^{1/3}\left(\varepsilon^{-1} + 2(x-y)\varepsilon^{-2/3} - 2h^{\mathrm{ASEP}}(2\varepsilon^{-2/3}x, 0; 2\varepsilon^{-2/3}y, 2\varepsilon^{-1}(1-q)^{-1})\right)$$

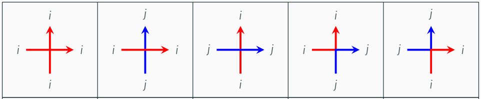
converges in distribution, as $\varepsilon \to 0$, to the Airy sheet $\mathcal{S}(x, y)$ as continuous functions on $\mathbb{R}^2$ uniformly on compact sets.

The case of general $\alpha \in (-1, 1)$ holds too, with explicit α-dependent scaling coefficients.

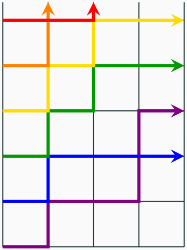
Colored S6V model

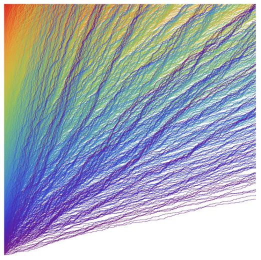
Introduced by [Jimbo '86], [Bazhanov '85], [Gwa-Spohn '93], [Kuniba--Mangazeev-Maruyama-Okado '16], [Borodin-Wheeler '22].

Quantum parameter $q \in [0, 1)$, spectral parameter $z \in (0, 1)$. At most one arrow per edge.



1	$\frac{q(1-z)}{1-qz}$	$\frac{1-z}{1-qz}$	$\frac{1-q}{1-qz}$	$\frac{z(1-q)}{1-qz}$





Simulation by Leo Petrov

Main result: Colored S6V to Airy sheet

The colored S6V height function $h^{\mathrm{S6V}}(x, 0; y, t)$ is the number of arrows of color $\geq x$ exiting horizontally from vertical line t at height y or higher.

Theorem (Aggarwal-Corwin-H.)

Fix $q \in [0, 1)$, $z \in (0, 1)$, $\alpha \in (z, z^{-1})$. For explicit scaling coefficients μ, σ and β (depending on α), the rescaled colored S6V height function

$$\sigma^{-1}\varepsilon^{1/3}\Big(h^{\mathrm{S6V}}(\beta\varepsilon^{-2/3}x, 0; \alpha\varepsilon^{-1} + \beta\varepsilon^{-2/3}y, \varepsilon^{-1}) - \mu\varepsilon^{-1} - \mu'\beta(y - x)\varepsilon^{-2/3} + \beta x\varepsilon^{-2/3}\Big)$$

converges in distribution, as $\varepsilon \to 0$, to the Airy sheet $\mathcal{S}(x, y)$ as continuous functions on $\mathbb{R}^2$ uniformly on compact sets.

Proof ingredients

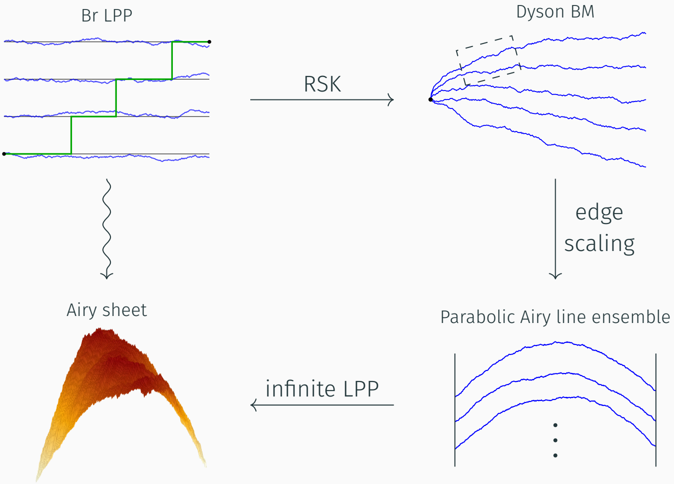
Definition of the Airy sheet

The Airy line ensemble $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots)$ is an $\mathbb{N}$-indexed collection of random non-intersecting curves on $\mathbb{R}$ [Prähofer-Spohn ’02, Corwin-Hammond ’14]:

It arises as the edge scaling limit of Dyson Brownian motion.

$\mathcal{S}$ was defined by Dauvergne-Ortmann-Virág via an infinite LPP problem in $\mathcal{P}$.

The path from Brownian LPP to the Airy sheet



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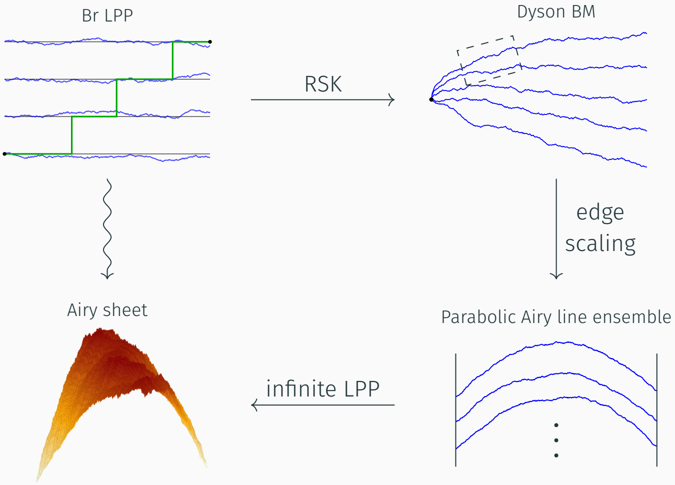
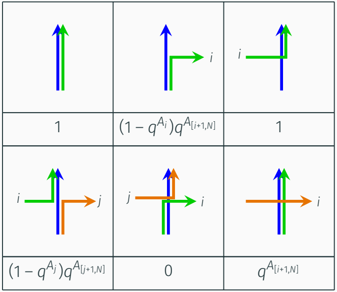
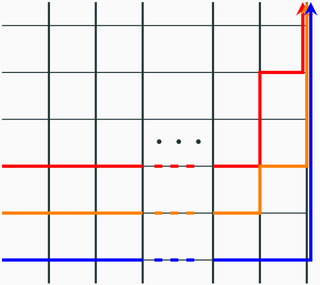


RSK isn't applicable to S6V or ASEP.

The colored q-Boson model

The colored q-Boson model is a colored vertex model on a semi-infinite strip of fixed height. It allows *arbitrarily* many arrows on vertical edges.

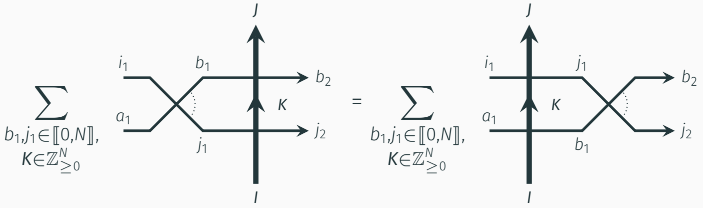




Arrows enter at $-\infty$ and travel straight except for finitely many columns.

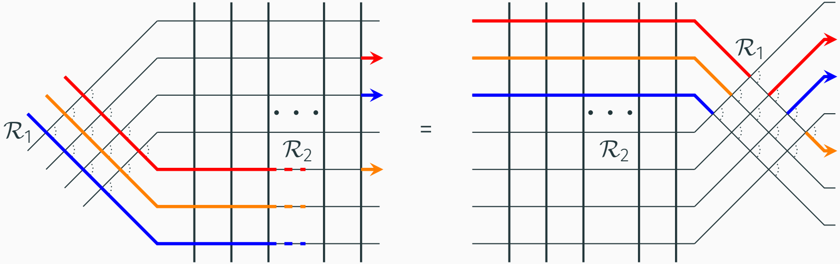
The Yang-Baxter equation

The colored q-Boson model and the colored S6V model are related via the Yang-Baxter equation.



Gives a way to manipulate vertex models graphically while preserving partition functions/distributions, and is the source of "integrability."

A matching via Yang-Baxter



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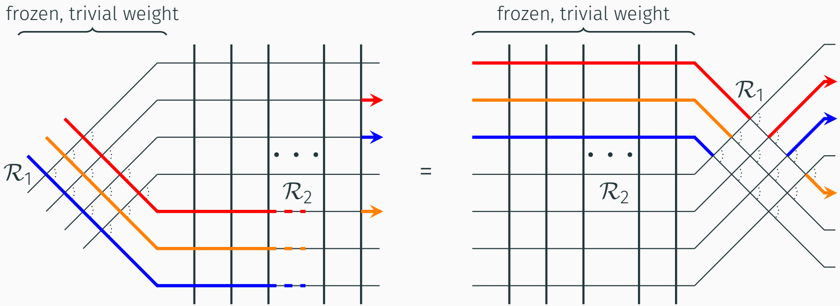

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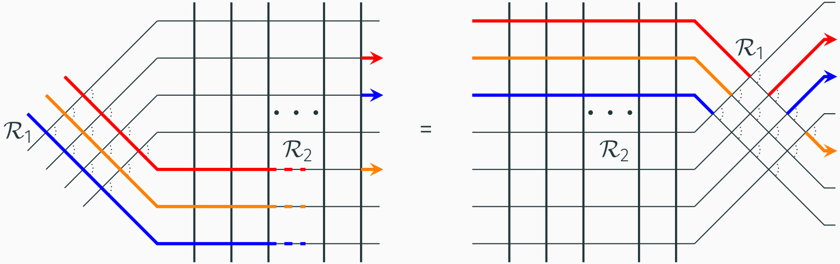
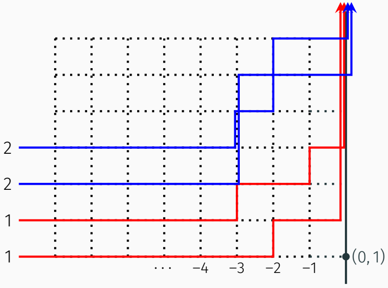
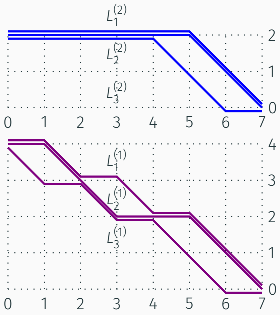


So the colored S6V height function is distributed as colored arrow counts in the last column of q-Boson.

The uncolored case was shown in [Borodin-Bufetov-Wheeler '16], and the colored case in [Aggarwal-Borodin '24].

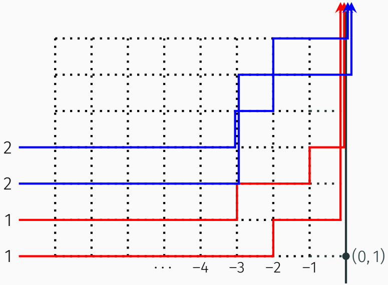


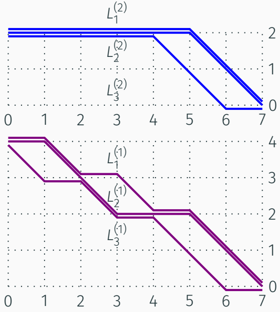


Colored line ensemble $\mathbf{L}^{\text{col}} = (\mathbf{L}^{(1)}, \dots, \mathbf{L}^{(N)})$, with $\mathbf{L}^{(k)} = (L_1^{(k)}, L_2^{(k)}, \dots)$ a line ensemble defined by

$$L_i^{(k)}(y) = \#\left\{y' > y : \text{color exiting horizontally from } (-i, y') \text{ is } \geq k\right\}.$$

The colored Hall-Littlewood line ensemble from the colored q-Boson model





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Yang-Baxter: $L_1^{(k)}$ is distributed as the color k height function $h^{\mathrm{S6V}}(k, 0; \bullet, t)$.

Two main ingredients

Recall that $\mathcal{S}$ is represented as a last passage percolation (LPP) problem in the parabolic Airy line ensemble $\mathcal{P}$.

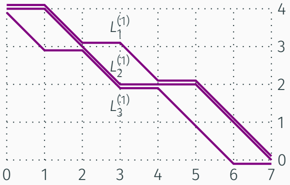
Proving convergence to the Airy sheet comes down to two main components:

1. Show colored height function (i.e., $L_1^{(k)}$) is *approximately* LPP in $\boldsymbol{L}^{(1)}$.
2. Show convergence of $\boldsymbol{L}^{(1)}$ to $\mathcal{P}$.

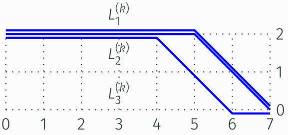
The colored and uncolored line ensembles each have Gibbs properties. Colored Gibbs is the tool for (1) and uncolored Gibbs the tool for (2).

An approximate LPP representation

Given $L^{(1)}$



, $L^{(k)}$



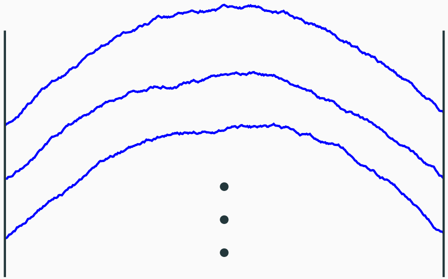
satisfies a (colored Hall-Littlewood) Gibbs property. Can be represented in terms of a variational problem: when $q = 0$, it holds that

$$L_i^{(k)} = \mathrm{PT}\left(L_i^{(1)}, L_{i+1}^{(k)}\right), \qquad \mathrm{PT}\left(f, g\right)(x) = f(x) + \max_{0 \le y \le x}\left(g(y) - f(y)\right),$$

and for $q > 0$,

$$\mathbb{P}\left(\max_y \left|L_i^{(k)}(y) - \mathrm{PT}\left(L_i^{(1)}, L_{i+1}^{(k)}\right)(y)\right| \geq m\right) \leq q^{cm^2}.$$

The (uncolored) Hall-Littlewood Gibbs property of $L^{(1)}$



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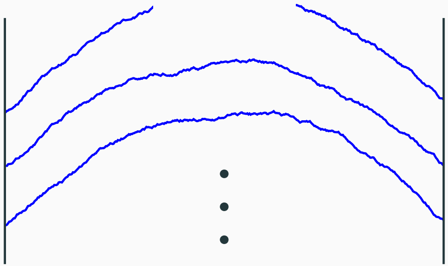

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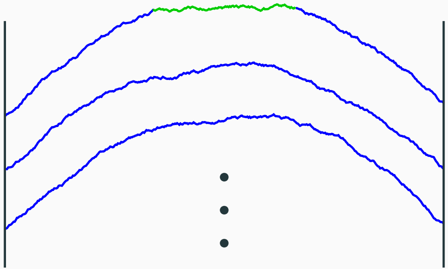
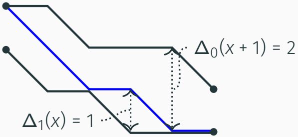


$\mathcal{P}$ has Brownian Gibbs property: given by non-intersecting Brownian bridges.

$L^{(1)}$'s Gibbs property is more complicated.



Law of top k curves of $L^{(1)}$ on $[a, b]$ is a collection of non-crossing Bernoulli random walk bridges, reweighted by a RN derivative

$$\prod_{i=0}^{k} \prod_{x=a+1}^{b} \left(1 - q^{\Delta_i(x-1)} \mathbb{1}_{\Delta_i(x)=\Delta_i(x-1)-1}\right),$$

where $\Delta_i(x)$ is separation of $(i-1)^{\text{st}}$ and i^{th} curve at x [Corwin-Dimitrov '18].

Convergence of $L^{(1)}$ to $\mathcal{P}$ and a lack of monotonicity

Showing $\boldsymbol{L}^{(1)} \to \mathcal{P}$ comes down to establishing

1. tightness of $\boldsymbol{L}^{(1)}$ at the edge, and
2. showing all subsequential limits have Brownian Gibbs.

Then can use [Aggarwal-Huang '23] which characterizes $\mathcal{P}$ as the unique law among Brownian Gibbsian line ensembles with parabolic decay of $-x^2$.

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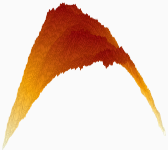
Many works have proved tightness of line ensembles, but all rely heavily on monotone coupling properties of the line ensembles.

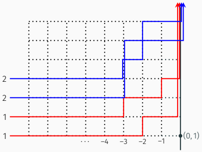
These do not exist for the Hall-Littlewood line ensemble!

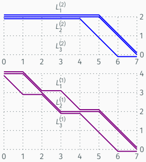
We give a new proof framework for tightness using only "weak monotonicity" of partition functions [Corwin-Dimitrov '18].

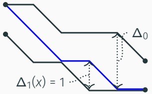
Next directions

- Including time in the scaling limit (✓ for ASEP and S6V)
- Scaling limit under general initial conditions (✓ for ASEP)
- Extend to other models
- Use to investigate other phenomena, e.g. mixing times, stationary measures, scaling limits of particle trajectories...









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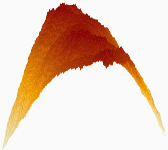
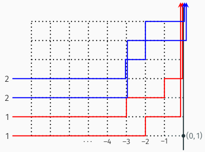
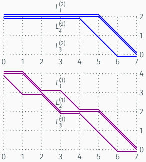
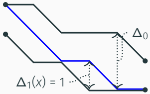
- Line ensemble tightness via only uncolored Gibbs & weak monotonicity.

Summary









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Thank you!