## **1** Introduction

My work concerns probability theory, in particular the rigorous study of certain classes of statistical mechanics models. My research has focused mainly on one-dimensional stochastic growth models belonging to the so-called KPZ universality class, after the physicists Kardar, Parisi, and Zhang who first described aspects of it [KPZ86]. A broad range of models are believed to belong to this class and to exhibit certain *universal* behaviors independent of the precise details of the model.

All existing proofs of membership of particular models in the KPZ class rely on certain exact algebraic connections arising from representation theory, which are manifest via certain special measures defined in terms of symmetric functions (see for example the work of Okounkov on Schur measures [Oko01] or Borodin-Corwin on Macdonald measures [BC14]), or from quantum integrable systems. This is the general approach of *integrable probability*. However, such algebraic connections are available in only these special models, while universality suggests the behaviors proven in those cases should be valid much more widely. A theme of my work has been to develop more robust techniques which move in the direction of proving universality in non-integrable settings, and, complementarily, to study integrable objects via non-integrable methods.

This overview has two main strands: the first (Section 2) focuses on models known as *last passage percolation* which are thought to lie in the KPZ class, while the second (Section 3) concerns *Gibbs properties*, which are probabilistic resampling descriptions enjoyed by both a putatively universal limiting object as well as many discrete models. In each I survey my previous work developing non-integrable techniques.

## **2** Geometric aspects of last passage percolation on $\mathbb{Z}^2$

In the model of last passage percolation (LPP) on  $\mathbb{Z}^2$ , we associate to every vertex v of  $\mathbb{Z}^2$  an independent and identically distributed non-negative random variable  $\xi_v$ . We consider *directed* paths in the lattice: at every vertex, the path may only move up or to the right (see Figure 1). The *weight* of a path  $\gamma \subseteq \mathbb{Z}^2$  is  $w(\gamma) = \sum_{v \in \gamma} \xi_v$ . Finally, we define the *last passage value* from (1, 1) to (n, n) by  $X_n = \max_{\gamma:(1,1)\to(n,n)} w(\gamma)$ , where the maximization is over all directed paths from (1, 1) to (n, n). One object of study is  $X_n$ , in particular its random fluctuations, as *n* tends to infinity.



Fig. 1: Paths in LPP.

Here too algebraic connections and integrable techniques are available for specific distributions of the vertex weights  $\{\xi_v : v \in \mathbb{Z}^2\}$ ; namely, under the geometric and exponential distributions. These are called the exactly solvable models. But the geometric approach we pursue instead focuses on the (possibly non-unique) *paths* which achieve the maximization, called *geodesics*; the idea being that path-based arguments will apply regardless of the distribution of the vertex weights. This approach has already witnessed some successes seemingly beyond the reach of integrable methods, such as the solution to the long-standing "slow-bond" problem [BSS14].



Fig. 2: Width of the geodesic.

A basic fact about the model is that the sequence  $\{X_n\}_{n \in \mathbb{N}}$  is *superadditive*, and so, by Kingman's subadditive ergodic theorem, there is a deterministic distributiondependent constant  $\mu$  such that  $\lim_{n\to\infty} X_n/n = \mu$  almost surely. Further, in the exactly solvable models, it is known that the fluctuations of  $X_n$  around  $\mu n$  are of order  $n^{1/3}$  [Joh00a], and that the width of the geodesic is of order  $n^{2/3}$  [Joh00b]. See Figure 2.

**2.1. Geodesic watermelons.** The geodesic has a natural generalization which enjoys interesting integrable connections. For  $1 \le k \le n$ , a *k*-geodesic watermelon is a collection of *k* disjoint paths from the lower left to the upper right corners of  $[1, n]^2 \cap \mathbb{Z}^2$  which maximizes the weight of all such collections; the weight of a

collection of disjoint paths is the sum of their individual weights. Thus the geodesic corresponds to k = 1. The weight of the *k*-geodesic watermelon is denoted by  $X_n^k$ .

Geodesic watermelons are natural objects to study. For instance, when the vertex weights are exponentially distributed,  $X_n^k$  is distributionally equal to the sum of the top *k* eigenvalues of the well-known  $n \times n$  Laguerre Unitary Ensemble (LUE) of random matrix theory [AVMW13]. However, we work in a more general setting which does not have these special connections.

We consider the following question: what are the orders of the fluctuations of  $X_n^k$  and the width of the *k*-geodesic watermelon? In work [BGHH20] with Riddhipratim Basu, Shirshendu Ganguly, and my adviser Alan Hammond, we identify these orders, both as functions of *k* and *n*; they reduce to the known orders in the k = 1 case. This is done under natural assumptions whose important features are that (i) they imply only weak tail decay on the vertex weight distribution and (ii) only concern  $X_n^1$ , not  $X_n^k$  for any  $k \ge 2$ . Further, they are known to hold in the exactly solvable models.

**Theorem 1** (Informal). Under the assumptions, there exist positive constants  $c_1$ , c, and  $\delta > 0$  such that, for  $1 \le k \le c_1 n$ , we have:

- (*i*) Weight fluctuation:  $\mathbb{P}\left(X_n^k \mu nk \notin -k^{5/3}n^{1/3} \cdot [\delta, \delta^{-1}]\right) \leq e^{-ck^2}$
- (*ii*) Width fluctuation:  $\mathbb{P}(\text{Width}(n,k) \le \delta \cdot k^{1/3}n^{2/3}) \le e^{-ck^2}$  and  $\mathbb{P}(\text{Width}(n,k) \ge \delta^{-1} \cdot k^{1/3}n^{2/3}) \le e^{-ck}$ .

Importantly, our arguments do not rely on integrable formulas; indeed, these are not available under the general context that our assumptions define. Instead, we develop *deterministic* path properties of geodesic watermelons, such as the fact that the curves of the k- and (k + 1)-geodesic watermelons *interlace*; see Figure 4.

Further, we make use of related deterministic global path properties to obtain concentration estimates for  $X_n^k - X_n^{k-1}$ , which are associated to natural determinantal processes in the exactly solvable cases (eg. the  $k^{\text{th}}$  eigenvalue of LUE). Surprisingly, these estimates are sharper than those directly available via integrable methods. Without the geometric picture, it is unclear why information about merely the top point of these processes, k = 1, should have fairly sharp implications for the lower points,  $k \ge 2$ .



Fig. 4: Interlacing of geodesic watermelons.

**2.2. Bootstrapping in LPP.** A hallmark of belonging to the KPZ universality class is convergence of the relevant observable to the GUE Tracy-Widom distribution  $F_{\text{TW}}$ , first discovered in random matrix theory [TW94]. Again in LPP, this is known for  $X_n$  (suitably centered and scaled) only in the exactly solvable cases, and so the following is a broad goal of the field:

Show that, for a wide class of vertex distributions  $\nu$ , there exist  $\nu$ -dependent constants  $\mu$  and  $\sigma$  such that, as  $n \to \infty$ ,  $(X_n - \mu n)/\sigma n^{1/3} \to F_{\text{TW}}$  in distribution.

Much has been studied about the GUE Tracy-Widom distribution. For example, its tail asymptotics are known to high precision (see for example [RRV11]): as  $t \to \infty$ ,

$$F_{\rm TW}([t,\infty)) = \exp\left(-\frac{4}{3}t^{3/2}(1+o(1))\right) \quad \text{and} \quad F_{\rm TW}((-\infty,-t]) = \exp\left(-\frac{1}{12}t^3(1+o(1))\right). \tag{1}$$

The expected universal convergence of  $X_n$  to  $F_{TW}$  and the tail asymptotics of  $F_{TW}$  suggests the following question: do tail estimates similar to (1) hold for  $X_n$  at finite n?

The answer to this question is known to be affirmative, for example, in the case of exponential LPP. In this case,  $X_n$  has the distribution of the top eigenvalue of the LUE [Joh00a]. Ledoux-Rider proved [LR10] upper



Fig. 3: A simulation of the geodesic watermelon.

bounds on the upper and lower tails of the top eigenvalue with the correct exponents of 3/2 and 3, and a matching-in-exponent lower bound on the upper tail. The missing lower bound on the lower tail was provided by myself in work with Basu, Ganguly, and Manjunath Krishnapur [BGHK19]. But this of course is a highly specific case, and in [GH20c] Ganguly and I prove similar bounds with optimal exponents under similar assumptions as in Theorem 1; a *non-integrable* setting, and a step in the direction of the broad goal.

**Theorem 2** (Informal). Under the assumptions, there exist positive constants  $c_1$ ,  $c_2$ ,  $c_3$ , and  $c_4$  such that the following inequalities hold for nearly sharp ranges of t:

$$\exp\left(-c_{1}t^{3/2}\right) \leq \mathbb{P}\left(X_{n} > \mu n + tn^{1/3}\right) \leq \exp\left(-c_{2}t^{3/2}(\log t)^{-1/2}\right)$$
$$\exp\left(-c_{3}t^{3}\right) \leq \mathbb{P}\left(X_{n} < \mu n - tn^{1/3}\right) \leq \exp\left(-c_{4}t^{3}\right).$$

Because the setting is non-integrable, the special formulas of the exactly solvable cases do not hold. Instead, our arguments are fundamentally geometric in nature. The geometric perspective provides an easy explanation for the discrepancy in the exponents seen in (1): occurrence of the upper tail only requires a single path to have high weight, but the lower tail requires *all* paths to have low weight; hence the lower tail is qualitatively more unlikely (though of course this does not say why the exponents have their precise values).

The arguments to prove Theorem 2 have several elements of interest. Briefly, they leverage superadditivity properties, the idea of considering geodesics at carefully chosen smaller scales, and a natural bootstrapping procedure. They also unearth some intriguing connections: for instance, the upper bound on the lower tail is closely tied to the bound on the k-geodesic watermelon's weight of Theorem 1, and bounds on both tails rely on the concentration of measure phenomenon. These connections point to deeper principles which may play a role in universality, and a broad goal of mine is to better understand what these may be.

The techniques used to prove Theorem 2, being highly geometric in nature, should prove to be adaptable to other non-integrable contexts. In particular, there is promise that similar ideas may yield similar results under comparable assumptions in the notoriously difficult non-integrable model of first passage percolation (FPP); FPP is a model similar to LPP wherein weights over paths with given endpoints are *minimized* instead of maximized, and there is no orientation or directedness constraint on the paths.

Unconditional progress in FPP over the last three decades has been quite limited, with most significant results being proven under natural but as yet unproven assumptions on curvature and fluctuations, similar to what we do in Theorem 2. Examples include Chatterjee's proof of the KPZ relation [Cha13] and Newman-Piza's work on divergence of geodesic fluctuations [NP95]. An immediate obstacle to adapting our arguments to FPP is our usage of LPP's directedness constraint on paths, while paths can and do *backtrack* in FPP. But I expect this issue to be surmountable, likely via more careful discretizations.

## **3** Probabilistic resampling via Gibbs properties

Observables in many KPZ models, under particular initial conditions, are known to converge to the Airy process minus a parabola, which we denote  $\mathcal{L}_1$ . The Airy process is a stationary process whose finite dimensional distributions can be expressed in terms of the classical special function, the Airy function, and whose one point marginal is the GUE Tracy-Widom distribution [PS02]. The methods which obtain these formulas fall within the ambit of integrable probability. In LPP, the observable in question is the profile of last passage values from (1, 1) to (n - x, n + x) as a function of x, appropriately centered and scaled.

Within probability theory, Brownian motion is a ubiquitous and deeply understood object. Partly for this reason, it was expected that the Airy process should be comparable to Brownian motion; and having a quantitative form of comparison would be a valuable tool to estimate Airy probabilities. Thus an important question is the degree of comparison to Brownian motion that  $\mathcal{L}_1$  can withstand, especially on unit-order scales, i.e., for example, on the interval [-1, 1].

 $\mathcal{L}_1$  can be embedded as the top curve of a random infinite collection of curves known as the *parabolic Airy line ensemble*  $\mathcal{L}$  (see Figure 5). Though ostensibly more complicated, this ensemble enjoys a probabilistic resampling property known as the *Brownian Gibbs* property, proven in the fundamental paper [CH14].



It was with this tool that the first version of unit-order Brownianity of  $\mathcal{L}_1$  was proved in [CH14]: the distribution of the process  $\mathcal{L}_k(\cdot) - \mathcal{L}_k(-d)$  on an interval [-d, d] is absolutely continuous with respect to the distribution  $\mathcal{B}$  of Brownian motion of rate two on the same interval. This is a *qualitative* comparison; it says the Radon-Nikodym derivative of the two measures exists.



In the following result with Jacob Calvert and Alan Hammond, we strengthened this comparison considerably in a *quantitative* form. For fixed d > 0, let C be the space of continuous functions on the interval [-d, d] that vanish at -d.

**Theorem 3.** Let A be any Borel measurable subset of C and  $\varepsilon = \mathcal{B}(A)$ . Then  $\mathbb{P}(\mathcal{L}_k(\cdot) - \mathcal{L}_k(-d) \in A) \leq \varepsilon^{1-o(1)}$ . Thus, the Radon-Nikodym derivative of  $\mathcal{L}_k(\cdot) - \mathcal{L}_k(-d)$  with respect to B lies in all  $L^p$  spaces with  $1 \leq p < \infty$ .

The theorem's utility is that it translates the problem of understanding probabilities for the Airy process, which is often difficult to do directly, to the much more tractable one of understanding the analogous Brownian probabilities. Indeed, it has already been a crucial tool in other KPZ studies [BGZ19,DSV20,GH20a,GH20b]. Its proof builds on a more sophisticated application of the Brownian Gibbs property developed in [Ham].

**3.1. Other initial data.** Though the Brownian Gibbs property naturally arises under very particular initial conditions (akin to a fundamental solution in PDE), recently it has had a central role in understanding other initial conditions: the construction of richer objects like the Airy sheet and directed landscape [DOV18] and in the convergence of the KPZ equation to the KPZ fixed point [Vir20]. The KPZ fixed point  $t \mapsto \mathfrak{h}_t$  is a Markov process, with  $h_t$  a function on  $\mathbb{R}$  for each  $t \ge 0$ ;  $\mathfrak{h}_0$  is a given initial condition.

The theme of using the Brownian Gibbs property to study the KPZ universality class under more general initial data is still being developed. In this line, I have worked with Ivan Corwin, Alan Hammond, and Konstantin Matetski to use both Brownian Gibbs and integrable methods to identify the Hausdorff dimension of the exceptional times where the KPZ fixed point has multiple (spatial) maximizers [CHHM21]. More precisely, let  $T_T = \{t \in [0, T] : \mathfrak{h}_t$  has at least two distinct maximizers}. These times are exceptional, i.e., random, because  $\mathfrak{h}_t$  almost surely has a unique maximizer for any fixed *t*. Thus it is interesting to understand just how rare they are.

**Theorem 3.1.** For a broad class of initial conditions,  $T_T \neq \emptyset$  with positive probability. Conditional on this event, the Hausdorff dimension of  $T_T$  is almost surely  $\frac{2}{3}$ .

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