Chabauty-Coleman Notes

Overview of idea

In $\mathbb{C}$, looking at $S^1 = X$.

have $\omega = \frac{dz}{z}$ & want to compute

$$\int_1^t \frac{dz}{z} \quad \text{for } t \in S^1.$$

Locally, expand $\frac{1}{z}$ & integrate term-by-term, so this works for t in some disc around 1. Can do the same for other disks, & if two disks intersect, can match up constants of integration so that the integrals agree on overlap.
↳ results in some $2\pi i$ monodromy

This fails for p-adic manifolds, as discs of radius 1 are disjoint or equal.

Idea "analytic continuation along Frobenius"

If $\varphi(x) = x^p$ for $x \in K$ ← say $\mathbb{Q}_p$
↖ lift of Frobenius

$\varphi : X \to X$

$\varphi^* \omega = p\omega.$

So, $\underset{\substack{\| \\ \text{“}\int_1^t \frac{dz}{z}\text{”}}}{\varphi^* F_\omega(t)} = F_\omega(t^p) = pF_\omega(t) + C$

If we force φ-equivariance, so that $C=0$, then taking $\alpha \in X$ s.t. $\alpha^{p^k} = \alpha$, i.e. $\varphi^k(\alpha) = \alpha$,

$$F_w(\alpha^{p^k}) = F_w(\alpha) = p^k F_w(\alpha)$$
$$\Rightarrow F_w(\alpha) = 0.$$

& since all $z \in X$ satisfy $|z-\alpha|<1$ for some such α, $F_w(z)$ is determined as local log expansion.

Quick Background on p-adic geometry

For rigid geom., replace poly. rings w/ power series rings where functions converge on some appropriate space.

e.g. $T_n = K\langle t_1, \dots, t_n \rangle = \{ \sum a_I t^I$ s.t. $a_I \in K$ & $\lim_{I \to \infty} |a_I| = 0 \}$

e.g. $\mathbb{Q}_p$ multi-index p-adic valuation

corresponds to funcs converging on unit polydisc

$$B_n = \{ (z_1, \dots, z_n) \in \overline{K}^n \mid |z_i| \le 1 \}.$$

$T_n \leftarrow$ Tate algebra

$T_n \to A \to 0$

"affinoid algebra".

Look at $\mathrm{mSpec}(A) = X$

gives this a Grothendieck top. by restricting open sets & open covers to some admissible things.

Problems w/ deRham cohomology

$d : T_1 \longrightarrow T_1\, dt$ has infinite cokernel

$$\int \sum a_i t^i\, dt = \sum \frac{a_i}{i+1} t^i$$

can take a_i to converge to 0 sufficiently slowly so that doesn't converge on unit disc.

Fix this by taking overconvergent series

$$T_n^* = \left\{ \sum a_I t^I,\ a_I \in R \ \&\ \exists r > 1 \text{ s.t. } \lim_{I \to \infty} |a_I| r^I = 0 \right\}$$

$A^* \longleftarrow$ weakly complete fin. gen. (wcfg)

if $A^* = T_n^* / (f_1, \dots, f_m)$

Then, $\Omega^1_{A^*} = \left(\bigoplus_{i=1}^{n} A^* dt_i \right) / (df_1, \dots, df_m)$

Then $\Omega^n_{A^*} := \wedge^n \Omega^1_{A^*}$. & deRham cmplx

is $\Omega^\bullet_{A^*}$.

<u>Lemma</u> (pf ommited)

If $k = R/\pi$, then any smooth & fin. gen k-alg. is

$$\bar{A} := A^*/\pi$$

$\uparrow$ some w.c.f.g. alg. (lift of $\bar{A}$)

A

1) any two such lifts are isom.
2) $\bar{f} : \bar{A} \to \bar{B}$ lifts
3) $A^* \to B^*$ w/ $A^*/\pi \cong B^*/\pi$ induce homotopic maps $\Omega^\bullet_{A^*} \otimes_R K \to \Omega^\bullet_{B^*} \otimes_R K$.

<u>Defn</u> (Monsky–Washnitzer cohomology).
The MW coh. of $\bar{A}$ is the cohomology of $\Omega^\bullet_{A^*} \otimes_R K$.

i.e. $H^i_{MW}(\bar{A}/K) = H^i(\Omega^\bullet_{A^*} \otimes K)$

$\uparrow$ fin. dim. (due to Berthelot).

$\varphi(x) = x^p$ on $\bar{A}$ lifts by lemma (2) to $\varphi : A^* \to A^*$ which is σ-linear ~~as a~~ map of R-algs.

$\uparrow$ lift of Frob. on K.

~~φ~~

Note if $|K| = p^r$, then φ^r is R-linear as the lift is

So φ^r induces endomorph.

$$\varphi^r : H^i_{MW}(\bar{A}/K) \longrightarrow H^i_{MW}(\bar{A}/K)$$

important thm. { & by using some crystalline, & eventually Weil conj, get eigenvalues of φ^r on $H^i_{MW}(\bar{A}/K)$ are alg. ints w/ abs. val. q or $\sqrt{q}$ for any emb. into $\mathbb{C}$.

From wcfg alg A^*, get affinoid A by completing $A^* \otimes K$ under Gauss norm
↳ max of abs. value of coeffs

There is a specialization map
$$X = \mathrm{mspec}(A) \longrightarrow X_k = \mathrm{Spec}(\bar{A})$$
↳ also for X^{geo} (geom. points)

Can define class of funcs A_{loc} ≤ $X^{geo} \longrightarrow \bar{K}$
↳ $\mathrm{Gal}(\bar{K}/K)$ equiv. & given by convergent power series on residue discs.

<u>Coleman Theory (finally)</u>

<u>Thm</u> $K/\mathbb{Q}_p$ fin. ext., there exists unique K-lin. integration map.

$$\int : (\Omega^1_{A^*} \otimes K)^{d=0} \longrightarrow A_{loc}/K. \quad \text{s.t.}$$

1) $d \circ \int = \text{canonical}\left((\Omega^1_{A^\dagger} \otimes K)^{d=0} \to \Omega^1_{loc} \right)$

2) $\int \circ d = \text{canonical}\left(A^\dagger_K \to A_{loc}/K \right)$

3) $\varphi \circ \int = \int \circ \varphi$ ← important cond!

$\int$ is indep. of φ chosen.

__Pf Sketch__ $H^1_{MW}(\overline{A}/K)$ is fin. dim, w/ basis say

$$\omega_1, \omega_2, \ldots, \omega_n \in \Omega^1_{A^\dagger} \otimes K.$$

Suffices by (2) to define $F_{\omega_i} = \int \omega_i$

Let $\omega = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix}$, & $F_\omega = \begin{bmatrix} F_{\omega_1} \\ \vdots \\ F_{\omega_n} \end{bmatrix}$.

Then,

$$\varphi\omega = M\omega + dg$$

$M \in M_n(K)$, $g \in (A^\dagger_K)^n$.

Integrating & using (2) & (3), get

$$\varphi F_\omega = MF_\omega + g + c$$

c: vect. of const.

Can get c to vanish by adjusting F_ω.

Lemma $\sigma - M : K^n \to K^n$ is bijective
↳ use that $I - \tilde{M}$ is invertible!
$\uparrow$ power of matrix giving φ^r on $H^1_{MW}(\bar{X})$

Using this, can kill c.
Now claim f_ω is determined.
$df_\omega = \omega$, so we can determine f_ω on residue disks up to vect. of const. by locally expanding ω & integrating term-by-term.

Therefore suffices to determine f_ω on a single point of each residue disk

On funcs, φ acts by
$$(\varphi \cdot f)(x) = \sigma f(\varphi(x)).$$
$\uparrow$ $\varphi(x)$ "=" $\sigma^{-1} \circ \_ \circ \varphi$.
K-linear func. giving x.

So, $\sigma(f_\omega(\varphi(x))) = M f_\omega(x) + g(x)$
$\varphi(x)$ & x in same residue disk, so
$$e = f_\omega(\varphi(x)) - f_\omega(x) = \int_x^{\varphi(x)} \omega$$ ← expand & compute.

So,
$$\sigma(f_\omega(x)) + \sigma(e) = M f_\omega(x) + g(x)$$
Again use $\sigma - M$ bijective, so
$f_\omega(x)$ is determined.
Check that this has all required properties

<u>Note</u> can also do iterated integrals similarly!

Application

<u>Recall</u> (classic Chabauty).

$\mathrm{Alb}: C \hookrightarrow J(C)$ via some rat'l point.

There is some p-adic log map

$$\log_J : J(\mathbb{Q}_p) \to T_0 J(\mathbb{Q}_p) = \Omega^1_J(\mathbb{Q}_p)^*.$$

kernel being torsion points

(inversion of $\exp: \mathfrak{g} \to G$ locally & then extending to all of $J(\mathbb{Q}_p)$).

$$\mathrm{Alb}^* : \Omega^1_C(\mathbb{Q}_p) \xrightarrow{\sim} \Omega^1_J(\mathbb{Q}_p)$$

So get

$$J(\mathbb{Q}_p) \times \Omega^1_C(\mathbb{Q}_p) \to \mathbb{Q}_p \to \langle -, - \rangle.$$

which is non-deg. if we restrict to a suff. small open in $J(\mathbb{Q}_p)$, say $\mathcal{U}$.

Now, $\operatorname{rank} J(\mathbb{Q}) = r < g = \dim_{\mathbb{Q}_p} \Omega^1_J(\mathbb{Q}_p)$

so we can find $\omega \in \Omega^1_C(\mathbb{Q}_p)$

so that $\langle -, \omega \rangle$ vanishes on $\overline{J(\mathbb{Q})}$

$$\langle R, \omega \rangle = \int_0^R \omega. \leftarrow \text{p-adic analytic func. in } R.$$
$$\text{“}\lambda(R)$$

So, get $\lambda(P) = 0$ for all $P \in C(\mathbb{Q})$
↳ count these in residue discs mod p

Coleman's innovation was to actually bound these p-adic integrals better

↳ estimate #zeroes of λ in residue discs via #zeroes of λ' of diff. forms on $C(\mathbb{Q}_p)$

want prime of good reduction so that residue discs are legitimately isom. to p-adic discs.

Reference

"Heidelberg Lectures on Coleman Integration"
↳ Amnon Besser

"Effective Chabauty" → R. Coleman

"Rat'l Points on Curves" → M. Stoll