KLR Algebras Seminar Outline

Spring 2022

1 Fundamentals

(1) KLR Algebras: Essentials
You might want to replace the notation for a sequence of vertices \( i \in \text{Seq}(\nu) \) by \( \vec{i} \) in your talk. Also, you might want to denote multiplication by \( x_i \) as \( \chi_i \).

(a) Define the algebras \( R(\nu) \), the projective modules \( P_{\vec{i}} \) and \( j_{\vec{i}}P \) and the antiinvolution \( \psi \) of \( R(\nu) \). [3, Section 2.1]

(b) Go over Examples 1) – 3) in [3, Section 2.2]. Example 3) is especially important. Follow/supplement using [5, Section 3]. In particular,

i. Introduce the Schubert basis and state how the Demazure operators act on this basis.
ii. Prove the following chain of isomorphisms
\[ NH_m \cong \text{End}_{\text{Sym}_m}(\mathbb{Z}[x_1, \ldots, x_m]) \cong M_{[m]!}(\text{Sym}_m) \]
and conclude that the center of \( NH_m \) is \( \text{Sym}_m \).
iii. Using the isomorphism above, show that in the ordered basis \( \{\mathcal{S}_{id}, \ldots, \mathcal{S}_{w_0}\} \), \( e_m \) corresponds to \( e_{mm} \), the matrix with a 1 in the \( m \)-th column and row by computing the action on the Schubert basis and thus is an idempotent in \( NH_m \).
iv. Conclude that \( e_m NH_m \) is isomorphic to the polynomial representation \( mP = \mathbb{Z}[x_1, \ldots, x_m] \) as a right \( NH_m \) module (up to a grading shift), and that there is an isomorphism of graded right \( NH_m \) modules
\[ NH_m \cong mP_{\pm [m]!} \]
v. Show that \( mP \) is indecomposable as a \( NH_m \) module and thus up to a grading shift and isomorphism, is the unique, graded, indecomposable projective \( NH_m \) module.
vi. (Show that \( mP \) has a unique graded maximal submodule given by \( \text{Sym}^+_{m} \) and thus up to isomorphism, \( \mathbb{Z}[x_1, \ldots, x_m]/(\text{Sym}^+_{m}) \) is the unique graded irreducible module of \( NH_m \).

(c) Introduce \( P_{\ell}^\nu \) and show that \( R(\nu) \) acts on \( P_{\ell}^\nu \) for your favorite relations of \( R(\nu) \). Introduce \( j_{\vec{i}}B_{\vec{i}} \) and show they span \( j_{\vec{i}}R(\nu)_{\vec{i}} \) as a free abelian group and state that they in fact give a basis and therefore making \( P_{\ell}^\nu \) a faithful module over \( R(\nu) \). [3, Section 2.3]

(d) (Compute more examples. [3, Section 2.2])

(2) KLR Algebras: Properties
You might want to replace the notation \( i \in \text{Seqd}(\nu) \) by \( \vec{(i)} \), where \( (\quad) \) is used to indicate divided powers.

(a) Carefully define \( \text{Sym}(\nu) \) and define \( j_{\vec{1}_i} \) as in [3, Theorem 2.5] and show that the center of \( R(\nu) \) is \( \text{Sym}(\nu) \) and conclude that \( R(\nu) \) is finite free over it’s center. [3, Section 2.4].
(b) Using that any simple $R(\nu)$ module $S$ is finite-dimensional, show that $\text{Sym}^+ (\nu)$ has to act by 0 on $S$, which implies there is a bijection between indecomposable projectives of $R(\nu)$ and simples of $R(\nu)$. [3, Section 2.5]
(c) Show that $P_\ell$ is self dual under the “bar” involution. Introduce the bilinear pairing on $K_0(R(\nu))$ and $G_0(R(\nu))$ and show that $(P_\ell, [S_\nu]) = \delta_\ell^\nu$. Ignore all statements with $\text{ch}(M)$ for now. Define $\bar{\ell}$! and show that $\bar{\ell}P \cong P\bar{\ell}$. [3, Section 2.5]
(d) Prove the “quantum serre relations” [3, Proposition 2.13].
(e) (Go over statements involving $\text{ch}(M)$.)

(3) KLR Algebras: Categorification and a generalization
(a) Follow [3, Section 2.6] up until Proposition 2.19.
(b) Prove injectivity of the Categorification Theorem [3, Section 3.1]. For surjectivity, follow [1, Theorem 3.3, Corollary 3.4, Page 16]. In particular, define $\text{Ch}(M)$ if not already defined.
(c) State [1, Theorem 3.11].
(d) Define quiver Hecke algebras following [6, Remark 2.2.6, Definition 2.2.1] and show that we recover the KLR algebra defined in [3] by setting $Q_{ij}(u, v) = u^{c_{ij}} + v^{c_{ij}}$ for $i \neq j$.
(e) (Go over statements involving $\text{ch}(M)$, in particular [3, Lemma 2.20], the “Shuffle Lemma.”)

2 Cyclotomic Quiver and Hecke Algebras in type A

(4) Cellular Algebras
(a) Define cellular algebras, give examples, and prove Lemma 2.3 in [7, Chapter 2]. Actually just follow [7, Chapter 2] completely until you prove Corollary 2.17.
(b) (Prove the rest of the results in [7, Chapter 2] up to Corollary 2.21.)
(c) (Use [7, Theorem 2.20] to prove Brauer-Humphreys reciprocity)
$$[P^\lambda : C^\mu] = [C^\mu : D^\lambda]$$

(5) Integral Cyclotomic Hecke Algebras and the Murphy Basis
You might want to replace the notation for a multipartition $\lambda$ by $\vec{\lambda}$ and the notation for a $\vec{\lambda}$–tableau by $\vec{t}$.
(a) Introduce Hu-Mathas definition of the cyclotomic Hecke algebra of type $A$ and show that for $\ell = 1$ you recover the finite Hecke algebra of type $A_{n-1}$. State the Basis theorem [6, Equation 1.1.2] and introduce the Ariki-Koike algebras and say how they are related. [6, Section 1.1]
(b) We will make some modifications to [6, Section 1.2] to make it more readable at the cost of possibly being off by a constant. In particular,
   i. First, define the quantum characteristic of $v$ to be the smallest element such that $(e)_v = 0$ where $(n)_v = 1 + v + \ldots + v^{n-1}$. Then follow [6, Section 1.2] up to the part where you define the fundamental weights $\Lambda_i$. Always use $e = 0$ instead of $e = \infty$.
   ii. For $\Lambda = \sum_{i \in I_\ell} c_i \Lambda_i$ the level is defined to be $\sum_{i \in I_\ell} c_i$. Now fix a weight $\Lambda$ of level $\ell$ and fix a multicharge for $\Lambda$, aka an ordered $\ell$–tuple, $(\kappa, \ldots, \kappa_\ell) \in I_\ell^\ell$ so that $i \in I_\ell$ appears $c_i$ times. Follow [6, Definition 1.2.1] except replace $[\kappa_r]$ with $v^{\kappa_r}$. Note that $\mathcal{H}_n^\Lambda$ does not depend on the choice of multicharge for $\Lambda$. 
iii. Finally state [6, Theorem 1.2.2].

(c) Follow [6, Section 1.4] until you reach the definition of \(d(\vec{t})\) but skip the paragraph with strong dominance order. Then follow [6, Section 1.5] but change the definition of \(u_{\vec{\lambda}}\) so that it is instead

\[
 u_{\vec{\lambda}} = \prod_{1 \leq x < \ell} \prod_{r=1}^{|\lambda^{(1)}| + \ldots + |\lambda^{(x)}|} (L_r - v^{\kappa_{x+1}})
\]

Also \(S_{\vec{\lambda}}\) is the row stabilizer of the standard tableaux \(\vec{t}A\). State [6, Theorem 1.5.1, Corollary 1.5.2].

(d) Define the Gelfand-Zetlin subalgebra of \(H_n\). Change the definition of \(c^{\vec{\rho}}(\vec{t})\) to be \(c^{\vec{\rho}}(\vec{t}) = v^{\vec{c}(\vec{t})}\). State [6, Lemma 1.6.2, Corollary 1.6.3].

(e) Now ignore the rest of [6, Section 1.6] until part (f) below and instead just show that the set

\[
 F_{\vec{s}} := \prod_{r=1}^{n} \prod_{c \in \mathcal{C}_n, c \neq c^{\vec{\rho}}(\vec{s})} \frac{L_r - c}{c^{\vec{\rho}}(\vec{s}) - c}, \quad \vec{s} \in \text{Std} (\mathcal{P}_n^\Lambda)
\]

form a complete set of pairwise orthogonal idempotents in \(H_n^\Lambda\). Hint: By [6, Corollary 1.6.3] \(H_n^\Lambda\) has an eigenbasis \(\{f_{\vec{u}}\}\) for the operators \(\{L_r\}\). Using that the regular representation is faithful, it suffices to show

\[
 F_{\vec{s}} F_{\vec{t}} f_{\vec{u}} = \delta_{\vec{s}\vec{t}} F_{\vec{s}} f_{\vec{u}}
\]

(f) Introduce the \(i-\)string of length \(n\) and state [6, Corollary 1.6.11]. (Mention that \((\Lambda, \alpha_{i,n}) \leq 1 \iff \kappa_r + d \neq \kappa_s \forall 1 \leq r < s \leq \ell, \forall d \text{ s.t. } -n < d < n, \text{ aka the fundamental weights that appear are spaced far enough apart}.)

(6) **Representations of Cyclotomic Quiver Hecke Algebras**

(a) Quickly review some notions from graded algebras, state Fitting’s Lemma, and show that if \(m\) is indecomposable, then it’s graded lift, if it exists is unique up to grading shift [6, Section 2.1]. Follow the rest of [6, Section 2.1] and use [6, Corollary 2.1.6] to show graded Brauer-Humphreys reciprocity

\[
 [P^\lambda : C^m]_q = [C^\mu : D^\lambda]_q
\]

(b) Define \(R_\beta, R_n, R_n^\Lambda\) but use the notation \(R_\beta(\Gamma_e), R_n(\Gamma_e), R_n^\Lambda(\Gamma_e)\) instead. [6, Section 2.2]

(Also see [3, Section 3.4])

(c) Follow [6, Section 2.3]. Then state [6, Proposition 2.4.3], but use [4, Theorem 1.17] for the definition of the action and the proof. (Separating is equivalent to \((\Lambda, \alpha_{i,n}) \leq 1\) by [6, Theorem 1.6.10].) The definition of \(i^\lambda\) can be found at the start of [6, Section 2.4] (or alternatively it’s just the content modulo \(e\).)

(d) State [6, Theorem 2.4.8, Corollary 2.4.9, Corollary 2.4.11].

(e) (Define blocks and state [6, Corollary 1.8.2].)

(7) **Brundan-Kleshchev Graded Isomorphism and Categorification Theorem**

(a) State [6, Theorem 3.1.1, Corollary 3.1.3]. [TODO]
(b) Follow [6, Section 3.5] until you finish proving [6, Proposition 3.5.6] but remove $X_\mu$ from all statements.

(c) Introduce the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_e})$ and the Fock space $F_\Lambda$ and state [6, Theorem 3.5.9]. Show that $|\vec{0}_e\rangle$ is highest weight of weight $\Lambda$.

(d) Introduce $i-\text{Ind}$ and $i-\text{Res}$ and state [6, Theorem 3.4.2]. Then prove [6, Proposition 3.5.12], in particular note that $d_q(|\vec{\lambda}\rangle) = [S^\Lambda]$.

(e) Prove [2, Corollary 5.11] (Replace $\{D_\mu\}$ the notation for dual canonical basis by $|D_\mu\rangle$, note $S_\lambda = |\lambda\rangle$ in our notation above) and use this to show that $d_q(D_\lambda) = []$ of $L(\Lambda)$, $\epsilon^{-1} = d_q$

3 KLRW Algebras and Knot Invariants

You are on your own here, good luck!

(8) [10]
(9) [10]
(10) [10]

4 DG KLR Algebras and Categorification of Verma Modules

[9], [8].

References