KLR Algebras Seminar Outline

Spring 2022

1 Fundamentals

(1) **KLR Algebras: Essentials**
You might want to replace the notation for a sequence of vertices $\mathbf{i} \in \mathrm{Seq}(\nu)$ by $\vec{i}$ in your talk. Also, you might want to denote multiplication by x_i as χ_i.

(a) Define the algebras $R(\nu)$, the projective modules $P_{\vec{i}}$ and ${}_{\vec{j}}P$ and the antiinvolution ψ of $R(\nu)$. [3, Section 2.1]

(b) Go over Examples 1) − 3) in [3, Section 2.2]. Example 3) is especially important. Follow/supplement using [5, Section 3]. In particular,

i. Introduce the Schubert basis and state how the Demazure operators act on this basis.

ii. Prove the following chain of isomorphisms

$$NH_m \cong \mathrm{End}_{\mathrm{Sym}_m}(\mathbb{Z}[x_1, \ldots, x_m]) \cong M_{[m]_{q^2}!}(\mathrm{Sym}_m)$$

and conclude that the center of NH_m is Sym_m.

iii. Using the isomorphism above, show that in the ordered basis $\{\mathfrak{S}_{\mathrm{id}}, \ldots, \mathfrak{S}_{w_0}\}$, e_m corresponds to e_{mm}, the matrix with a 1 in the $m-$th column and row by computing the action on the Schubert basis and thus is an idempotent in NH_m.

iv. Conclude that $e_m NH_m$ is isomorphic to the polynomial representation ${}_mP = \mathbb{Z}[x_1, \ldots, x_m]$ as a right NH_m module (up to a grading shift), and that there is an isomorphism of graded right NH_m modules

$$NH_m \cong {}_mP^{\oplus [m]_{q^2}!}$$

v. Show that ${}_mP$ is indecomposable as a NH_m module and thus up to a grading shift and isomorphism, is the unique, graded, indecomposable projective NH_m module.

vi. (Show that ${}_mP$ has a unique graded maximal submodule given by (Sym_m^+) and thus up to isomorphism, $\mathbb{Z}[x_1, \ldots, x_m]/(\mathrm{Sym}_m^+)$ is the unique graded irreducible module of NH_m.)

(c) Introduce $\mathcal{P}o\ell_\nu$ and show that $R(\nu)$ acts on $\mathcal{P}o\ell_\nu$ for your favorite relations of $R(\nu)$. Introduce ${}_{\vec{j}}B_{\vec{i}}$ and show they span ${}_{\vec{j}}R(\nu)_{\vec{i}}$ as a free abelian group and state that they in fact give a basis and therefore making $\mathcal{P}o\ell_\nu$ a faithful module over $R(\nu)$. [3, Section 2.3]

(d) (Compute more examples. [3, Section 2.2])

(2) **KLR Algebras: Properties**
You might want to replace the notation $\mathbf{i} \in \mathrm{Seqd}(\nu)$ by $(\vec{i})$, where ( ) is used to indicate divided powers.

(a) Carefully define $\mathrm{Sym}(\nu)$ and define ${}_{\vec{j}}1_{\vec{i}}$ as in [3, Theorem 2.5] and show that the center of $R(\nu)$ is $\mathrm{Sym}(\nu)$ and conclude that $R(\nu)$ is finite free over it's center. [3, Section 2.4].

(b) Using that any simple $R(\nu)$ module S is finite-dimensional, show that $\mathrm{Sym}^+(\nu)$ has to act by 0 on S, which implies there is a bijection between indecomposable projectives of $R(\nu)$ and simples of $R(\nu)$. [3, Section 2.5]

(c) Show that $P_{\vec{i}}$ is self dual under the "bar" involution. Introduce the bilinear pairing on $K_0(R(\nu))$ and $G_0(R(\nu))$ and show that $\left(\left[P_{\vec{b}}\right], [S_{\vec{a}}]\right) = \delta_{\vec{b}\vec{a}}$. Ignore all statements with $\mathrm{ch}(M)$ for now. Define $(\vec{i})!$ and show that ${}_{\widehat{(\vec{i})}}P \cong_{(\vec{i})} P^{\oplus (\vec{i})!}$. [3, Section 2.5]

(d) Prove the "quantum serre relations" [3, Proposition 2.13].

(e) (Go over statements involving $\mathrm{ch}(M)$.)

(3) **KLR Algebras: Categorification and a generalization**

(a) Follow [3, Section 2.6] up until Proposition 2.19.

(b) Prove injectivity of the Categorification Theorem [3, Section 3.1]. For surjectivity, follow [1, Theorem 3.3, Corollary 3.4, Page 16]. In particular, define $\mathrm{Ch}(M)$ if not already defined.

(c) State [1, Theorem 3.11].

(d) Define quiver Hecke algebras following [6, Remark 2.2.6, Definition 2.2.1] and show that we recover the KLR algebra defined in [3] by setting $Q_{ij}(u,v) = u^{-c_{ij}} + v^{-c_{ij}}$ for $i \neq j$.

(e) (Go over statements involving $\mathrm{ch}(M)$, in particular [3, Lemma 2.20], the "Shuffle Lemma.")

2 Cyclotomic Quiver and Hecke Algebras in type A

(4) **Cellular Algebras**

(a) Define cellular algebras, give examples, and prove Lemma 2.3 in [7, Chapter 2]. Actually just follow [7, Chapter 2] completely until you prove Corollary 2.17.

(b) (Prove the rest of the results in [7, Chapter 2] up to Corollary 2.21.)

(c) (Use [7, Theorem 2.20] to prove Brauer-Humphreys reciprocity)

$$[P^\lambda : C^\mu] = [C^\mu : D^\lambda]$$

(5) **Integral Cyclotomic Hecke Algebras and the Murphy Basis**
You might want to replace the notation for a multipartition λ by $\vec{\lambda}$ and the notation for a $\vec{\lambda}-$tableau by $\vec{t}$.

(a) Introduce Hu-Mathas definition of the cyclotomic Hecke algebra of type A and show that for $\ell = 1$ you recover the finite Hecke algebra of type A_{n-1}. State the Basis theorem [6, Equation 1.1.2] and introduce the Ariki-Koike algebras and say how they are related. [6, Section 1.1]

(b) We will make some modifications to [6, Section 1.2] to make it more readable at the cost of possibly being off by a constant. In particular,

i. First, define the quantum characteristic of v to be the smallest element such that $(e)_v = 0$ where $(n)_v = 1 + v + \ldots + v^{n-1}$. Then follow [6, Section 1.2] up to the part where you define the fundamental weights Λ_i. Always use $e = 0$ instead of $e = \infty$.

ii. For $\Lambda = \sum_{i \in I_e} c_i \Lambda_i$ the level is defined to be $\sum_{i \in I_e} c_i$. Now fix a weight Λ of level ℓ and fix a multicharge for Λ, aka an ordered $\ell-$tuple, $(\kappa, \ldots, \kappa_\ell) \in I_e^\ell$ so that $i \in I_e$ appears c_i times. Follow [6, Definition 1.2.1] except replace $[\kappa_r]_v$ with v^{κ_r}. Note that $\mathscr{H}_n^\Lambda$ does not depend on the choice of multicharge for Λ.

iii. Finally state [6, Theorem 1.2.2].

(c) Follow [6, Section 1.4]until you reach the definition of $d(\vec{t})$ but skip the paragraph with strong dominance order. Then follow [6, Section 1.5] but change the definition of $u_{\vec{\lambda}}$ so that it is instead

$$u_{\vec{\lambda}} = \prod_{1\le x<\ell} \prod_{r=1}^{|\lambda^{(1)}|+\ldots+|\lambda^{(x)}|} (L_r - v^{\kappa_{x+1}})$$

Also $\mathfrak{S}_{\vec{\lambda}}$ is the row stabilizer of the standard tableaux $\vec{t}^{\vec{\lambda}}$. State [6, Theorem 1.5.1, Corollary 1.5.2].

(d) Define the Gelfand-Zetlin subalgebra of $\mathscr{H}_n$. Change the definition of $c_r^{\mathcal{Z}}(\vec{t})$ to be $c_r^{\mathcal{Z}}(\vec{t}) = v^{c_r^{\mathbb{Z}}(\vec{t})}$. State [6, Lemma 1.6.2, Corollary 1.6.3].

(e) Now ignore the rest of [6, Section 1.6] until part (f) below and instead just show that the set

$$F_{\vec{s}} := \prod_{r=1}^{n} \prod_{\substack{c\in\mathscr{C}_n \\ c\neq c_r^{\mathcal{Z}}(\vec{s})}} \frac{L_r - c}{c_r^{\mathcal{Z}}(\vec{s}) - c}, \qquad \vec{s}\in \mathrm{Std}(\mathcal{P}_n^{\Lambda})$$

form a complete set of pairwise orthogonal idempotents in $\mathscr{H}_n^{\Lambda}$. Hint: By [6, Corollary 1.6.3] $\mathscr{H}_n^{\Lambda}$ has an eigenbasis $\{f_{\vec{u}\vec{v}}\}$ for the operators $\{L_r\}$. Using that the regular representation is faithful, it suffices to show

$$F_{\vec{s}} F_{\vec{t}} f_{\vec{u}\vec{v}} = \delta_{\vec{s}\vec{t}} F_{\vec{s}} f_{\vec{u}\vec{v}}$$

(f) Introduce the $i-$string of length n and state [6, Corollary 1.6.11]. (Mention that $(\Lambda, \alpha_{i,n}) \le 1 \iff \kappa_r + d \neq \kappa_s \forall 1\le r<s\le\ell, \forall d$ s.t. $-n<d<n$, aka the fundamental weights that appear are spaced far enough apart.)

(6) **Representations of Cyclotomic Quiver Hecke Algebras**

(a) Quickly review some notions from graded algebras, state Fitting's Lemma, and show that if $\underline{m}$ is indecomposable, then it's graded lift, if it exists is unique up to grading shift [6, Section 2.1]. Follow the rest of [6, Section 2.1] and use [6, Corollary 2.1.6] to show graded Brauer-Humphreys reciprocity

$$[P^{\lambda} : C^{\mu}]_q = [C^{\mu} : D^{\lambda}]_q$$

(b) Define $\mathscr{R}_\beta, \mathscr{R}_n, \mathscr{R}_n^{\Lambda}$ but use the notation $\mathscr{R}_\beta(\Gamma_e), \mathscr{R}_n(\Gamma_e), \mathscr{R}_n^{\Lambda}(\Gamma_e)$ instead. [6, Section 2.2] (Also see [3, Section 3.4])

(c) Follow [6, Section 2.3]. Then state [6, Proposition 2.4.3], but use [4, Theorem 1.17] for the definition of the action and the proof. (Separating is equivalent to $(\Lambda, \alpha_{i,n}) \le 1$ by [6, Theorem 1.6.10].) The definition of $\mathbf{i}^{\mathbf{t}^{\lambda}}$ can be found at the start of [6, Section 2.4] (or alternatively it's just the content modulo e.)

(d) State [6, Theorem 2.4.8, Corollary 2.4.9, Corrollary 2.4.11]. .

(e) (Define blocks and state [6, Corollary 1.8.2].)

(7) **Brundan-Kleshchev Graded Isomorphism and Categorification Theorem**

(a) State [6, Theorem 3.1.1, Corollary 3.1.3]. [TODO]

(b) Follow [6, Section 3.5] until you finish proving [6, Proposition 3.5.6] but remove X_μ from all statements.

(c) Introduce the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_e})$ and the Fock space $\mathscr{F}_{\mathcal{A}}^{\Lambda}$ and state [6, Theorem 3.5.9]. Show that $|\vec{0}_\ell\rangle$ is highest weight of weight Λ.

(d) Introduce $i-$Ind and $i-$Res and state [6, Theorem 3.4.2]. Then prove [6, Proposition 3.5.12], in particular note that $d_q(|\vec{\lambda}\rangle) = [S^{\vec{\lambda}}]$.

(e) Prove [2, Corollary 5.11] (Replace $\{D_{\vec{\mu}}\}$ the notation for dual canonical basis by $|D_{\vec{\mu}}\rangle$, note $S_\lambda = |\lambda\rangle$ in our notation above) and use this to show that $d_q(D_\lambda) = []$ s of $L(\Lambda)$, $\epsilon^{-1} = d_q$

3 KLRW Algebras and Knot Invariants

You are on your own here, good luck!

(8) [10]

(9) [10]

(10) [10]

4 DG KLR Algebras and Categorification of Verma Modules

[9], [8].

References

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