Isomorphism Theorem

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Reminder: $\Gamma_e = \quad = \hat{A}_{e-1}, \quad I_e = \mathbb{Z}/e\mathbb{Z} =$ vertices

1.1.1. **Definition** (Hu-Mathas [57, Definition 2.2]). *Fix integers $n \geq 0$ and $\ell \geq 1$. The* ***cyclotomic Hecke algebra of type A****, with* ***Hecke parameter*** *$v \in \mathcal{Z}^\times$ and* ***cyclotomic parameters*** *$Q_1, \dots, Q_\ell \in \mathcal{Z}$, is the unital associative $\mathcal{Z}$-algebra $\mathscr{H}_n = \mathscr{H}_n(\mathcal{Z}, v, Q_1, \dots, Q_\ell)$ with generators $L_1, \dots, L_n$, $T_1, \dots, T_{n-1}$ and relations*

$$\prod_{l=1}^{\ell}(L_1 - Q_l) = 0, \qquad (T_r + v^{-1})(T_r - v) = 0, \qquad L_{r+1} = T_r L_r T_r + T_r,$$
$$L_r L_t = L_t L_r, \qquad T_r T_s = T_s T_r \text{ if } |r - s| > 1,$$
$$T_s T_{s+1} T_s = T_{s+1} T_s T_{s+1}, \qquad T_r L_t = L_t T_r \text{ if } t \neq r, r+1,$$

where $1 \leq r < n$, $1 \leq s < n-1$ and $1 \leq t \leq n$.

$$H_n^\Lambda(\mathbb{F}, v) = H_n(\mathbb{F}, v, v^{k_1}, \dots, v^{k_\ell})$$

$(k_1, \dots, k_\ell) \in I_e^\ell$ a choice of multicharge for Λ

2.2.1. **Definition** (Khovanov and Lauda [74, 75] and Rouquier [121]). *Suppose that $n \geq 0$, $e \geq 1$, and $\beta \in Q^+$. The* ***quiver Hecke algebra****, or* ***Khovanov-Lauda–Rouquier algebra****, $\mathscr{R}_\beta = \mathscr{R}_\beta(\mathcal{Z})$ of type Γ_e is the unital associative $\mathcal{Z}$-algebra with generators*

$$\{\psi_1, \dots, \psi_{n-1}\} \cup \{y_1, \dots, y_n\} \cup \{\, e(\mathbf{i}) \mid \mathbf{i} \in I^\beta \,\}$$

and relations

$$e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{ij}} e(\mathbf{i}), \qquad \textstyle\sum_{\mathbf{i} \in I^\beta} e(\mathbf{i}) = 1,$$
$$y_r e(\mathbf{i}) = e(\mathbf{i}) y_r, \qquad \psi_r e(\mathbf{i}) = e(s_r{\cdot}\mathbf{i})\psi_r, \qquad y_r y_s = y_s y_r,$$

$$\psi_r \psi_s = \psi_s \psi_r, \qquad \text{if } |r - s| > 1,$$
$$\psi_r y_s = y_s \psi_r, \qquad \text{if } s \neq r, r+1,$$

(2.2.2)
$$\psi_r y_{r+1} e(\mathbf{i}) = (y_r \psi_r + \delta_{i_r i_{r+1}}) e(\mathbf{i}),$$
$$y_{r+1} \psi_r e(\mathbf{i}) = (\psi_r y_r + \delta_{i_r i_{r+1}}) e(\mathbf{i}),$$

(2.2.3)
$$\psi_r^2 e(\mathbf{i}) = \begin{cases} (y_{r+1} - y_r)(y_r - y_{r+1})e(\mathbf{i}), & \text{if } i_r \rightleftarrows i_{r+1}, \\ (y_r - y_{r+1})e(\mathbf{i}), & \text{if } i_r \to i_{r+1}, \\ (y_{r+1} - y_r)e(\mathbf{i}), & \text{if } i_r \leftarrow i_{r+1}, \\ 0, & \text{if } i_r = i_{r+1}, \\ e(\mathbf{i}), & \text{otherwise,} \end{cases}$$

and $(\psi_r \psi_{r+1} \psi_r - \psi_{r+1} \psi_r \psi_{r+1}) e(\mathbf{i})$ is equal to

$$\Big\{ (y_r + y_{r+2} - 2y_{r+1})e(\mathbf{i}), \qquad \text{if } i_{r+2} = i_r \rightleftarrows i_{r+1},$$

$$R_n^\Lambda(\Gamma_e, \mathbb{F}) = {}_{ht(\beta)=n}\frac{R_\beta(\mathbb{F})}{\langle y_1^{\langle \Lambda, \alpha_{i_1} \rangle} e(\vec{i}) \mid \vec{i} \in Seq(\beta) \rangle}$$

Thrm 1 (Graded Isomorphism): $\exists$ alg iso

$$\Phi : R_n^\Lambda(\Gamma_e, \mathbb{F}) \xrightarrow{\sim} H_n^\Lambda(\mathbb{F}, v)\big|_{qchar(v) = e}$$

"Pf": - Recall in Pavel's talk in s.s case

$$H_n^\Lambda = \bigoplus_{\vec{\lambda} \in Std(P_n^\Lambda)} H_{\vec{\lambda}}, \quad H_{\vec{\lambda}} = \{ h \mid L_r h = v^{c_r^{\mathbb{Z}}(\vec{\lambda})} h \}$$

- In general, only have gen eigenspace decomp

$$H_n^\Lambda = \bigoplus_{\vec{i} \in I_e^n} \tilde{H}_{\vec{i}}, \quad \tilde{H}_{\vec{i}} = \{ h \mid (L_r - v^{i_r})^m h = 0 \}$$

$\rightsquigarrow$ gives idempotents $F_{\vec{i}}$ in H_n^Λ

- Explicitly, $F_{\vec{i}} = \sum_{Res(\vec{\lambda}) = \vec{i}} F_{\vec{\lambda}}$ $\leftarrow$ defined in Pavel's talk

$$Res(\vec{\lambda}) = (c_1^{\mathbb{Z}}(\vec{\lambda}) \bmod e, \dots, c_n^{\mathbb{Z}}(\vec{\lambda}) \bmod e)$$

Rem: In Pavel's talk s.s $\longleftrightarrow$ content separated so each $\vec{\lambda}$ has unique $\vec{i}$, i.e. $F_{Res(\vec{\lambda})} = F_{\vec{\lambda}}$

$$\left.\begin{matrix} e(\mathbf{i}), & \textit{otherwise}, \end{matrix}\right.$$

and $(\psi_r\psi_{r+1}\psi_r - \psi_{r+1}\psi_r\psi_{r+1})e(\mathbf{i})$ *is equal to*

$$(2.2.4) \qquad \begin{cases} (y_r + y_{r+2} - 2y_{r+1})e(\mathbf{i}), & \textit{if } i_{r+2} = i_r \rightleftarrows i_{r+1}, \\ -e(\mathbf{i}), & \textit{if } i_{r+2} = i_r \to i_{r+1}, \\ e(\mathbf{i}), & \textit{if } i_{r+2} = i_r \leftarrow i_{r+1}, \\ 0, & \textit{otherwise}, \end{cases}$$

Isomorphism Theorem 2

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Defining Φ: • $\Phi(e(\vec{i})) = F_{\vec{i}}$

• $\Phi(y_r e(\vec{i})) = v^{-i_r}(L_r - v^{i_r}) F_{\vec{i}}$

• $\Phi(\psi_s e(\vec{i})) = (T_s + P_s(\vec{i})) \frac{1}{Q_s(\vec{i})} F_{\vec{i}}$

$P_s(\vec{i}), Q_s(\vec{i})$ power series in $\Phi(y_r e(\vec{i}))$ and $\Phi(y_{r+1} e(\vec{i}))$, becomes poly b/c $\Phi(y_r e(\vec{i}))$ nilpotent

- BK then checked all relations hold by hand, similarly w/ inverse map
- Mathas reduces to s.s. case via a modular system

Cor 2: $\exists$ non-trivial grading on H_n^Λ

$|F_{\vec{i}}| = 0$, $|\Phi(y_r e(i))| = 2$, $|\Phi(\psi_s e(i))| = -C_{i_s, i_{s+1}}$

Cor 3: Let $v, v' \in \mathbb{F}$ s.t. $\mathrm{qchar}(v) = \mathrm{qchar}(v') = e$

$$H_n^\Lambda(\mathbb{F}, v) \cong H_n^\Lambda(\mathbb{F}, v')$$

Rem: If $\mathbb{F} = \overline{\mathbb{F}_p}$, $v = 1$, $\mathrm{qchar}(v) = p$

If $\mathbb{F} = \mathbb{C}$, $v' = e^{2\pi i/p}$, $\mathrm{qchar}(v') = p$. So

$$H_n^\Lambda(\overline{\mathbb{F}_p}, 1) \overset{?}{\cong} H_n^\Lambda(\mathbb{C}, e^{2\pi i/p})$$

No! Φ depends on $\mathbb{F}$! But very close

2. $U_q(\hat{\mathfrak{sl}}_e)$ and it's Fock space

The **quantum group** $U_q(\widehat{\mathfrak{sl}}_e)$ associated with the quiver Γ_e is the $\mathbb{Q}(q)$-algebra generated by $\{ E_i, F_i, K_i^{\pm} \mid i \in I \}$, subject to the relations:

$$K_i K_j = K_j K_i, \qquad K_i K_i^{-1} = 1, \qquad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$K_i E_j K_i^{-1} = q^{c_{ij}} E_j, \qquad K_i F_j K_i^{-1} = q^{-c_{ij}} F_j,$$

$$\sum_{0 \le c \le 1 - c_{ij}} (-1)^c \begin{bmatrix} 1 - c_{ij} \\ c \end{bmatrix}_q E_i^{1 - c_{ij} - c} E_j E_i^c = 0,$$

$$\sum_{0 \le c \le 1 - c_{ij}} (-1)^c \begin{bmatrix} 1 - c_{ij} \\ c \end{bmatrix}_q F_i^{1 - c_{ij} - c} F_j F_i^c = 0,$$

Def: Λ-Fock space $\mathcal{F}_{\mathbb{Z}}^\Lambda$ is the free $\mathbb{Z}$-mod w/ basis $\{ |\vec{\lambda}\rangle \mid |\vec{\lambda}\rangle \in P^\Lambda = \bigcup_{n \ge 0} P_n^\Lambda \}$

Def: For $\vec{\mu} \in P_n^\Lambda$, a node A is an addable node of λ if $\vec{\mu} \cup \{A\} \in P_{n+1}^\Lambda$. Similarly w/ removable.

Def: node A is a i-node if $\text{cont}(A) \bmod e = i$

If A is an add/removable i-node of $\vec{\mu} \in P_n^\Lambda$, let

- $d_R(\vec{\mu}, A) = |\{B \in Add_i(\vec{\mu}) \mid B > A\}| - |\{B \in Rem_i(\vec{\mu}) \mid B > A\}|$

  is below A in partition corr to A or in another partition to the right of partition corr to A

- $d^L(\vec{\mu}, A) = |\{C \in Add_i(\vec{\mu}) \mid C < A\}| - |\{C \in Rem_i(\vec{\mu}) \mid C < A\}|$

- $d_i(\vec{\mu}) = |Add_i(\vec{\mu})| - |Rem_i(\vec{\mu})|$

Ex: $\vec{\mu} = \left(\begin{array}{cccc} 0 & 1 & 2 & \boxed{0} \\ 2 & 0 & & \\ 1 & & & \\ \hline \boxed{0} & & & \end{array} \right)$ $\leftarrow A$ (the node 0 in the second row), $\quad e = 3 \quad d_0(\vec{\mu}) = -1$

$\Lambda = \Lambda_0$, $k_1 = 0$

$\Rightarrow d^R(A) = 1 - 0$

$d^L(A) = 0 - 1$

Thrm 4 (Hayashi): Suppose $\Lambda \in P^+$. Then $\mathcal{F}^\Lambda_{\mathbb{Q}(q)}$ is an integrable $U_q(\hat{sl}_e)$ module where

- $E_i|\vec{\lambda}\rangle = \sum_{A \in Rem_i(\vec{\lambda})} q^{d_R(\vec{\lambda}, A)} |\vec{\lambda} - A\rangle$

- $F_i|\vec{\lambda}\rangle = \sum_{A \in Add_i(\vec{\lambda})} q^{-d^L(\vec{\lambda}, A)} |\vec{\lambda} + A\rangle$

- $K_i|\vec{\lambda}\rangle = q^{d_i(\vec{\lambda})} |\vec{\lambda}\rangle$

Ex: $\vec{\lambda} = \left(\begin{array}{cc} 0 & 1 \\ 2 & \boxed{0} \\ 1 & \\ \hline \boxed{0} & \end{array} \right)$ $\quad F_0(\vec{\lambda}) = q|(2,2,1,1)\rangle$

$d_0(\vec{\lambda}) = 0 \qquad E_0(\vec{\lambda}) = q|(2,1,1)\rangle$

- $E_0 F_0 |\vec{\lambda}\rangle = q^{-1}(q|(2,1,1,1)\rangle) + q|(2,2,1)\rangle$
- $F_0 E_0 |\vec{\lambda}\rangle = q|(2,2,1)\rangle + q^{-1}(q|(2,1,1,1)\rangle)$

$\Rightarrow [E_0, F_0]|\vec{\lambda}\rangle = 0$

- $K_0|\vec{\lambda}\rangle = q^0|\vec{\lambda}\rangle$

$\Rightarrow \dfrac{K_0 - K_0^{-1}}{q - q^{-1}}|\vec{\lambda}\rangle = 0$

Rem: Write $\Lambda = \Lambda_{k_1} + \dots + \Lambda_{k_e}$. Then as $U_q(\hat{sl}_e)$-mod

$$\mathcal{F}^\Lambda_{\mathbb{Q}(q)} \cong \mathcal{F}^{\Lambda_{k_1}}_{\mathbb{Q}(q)} \otimes \dots \otimes \mathcal{F}^{\Lambda_{k_e}}_{\mathbb{Q}(q)}$$

L(\Lambda)

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Def $v \in \mathcal{F}^{\Lambda}_{\mathbb{Q}(q)}$ has weight $wt(v)=\alpha$ if

$$K_i v = q^{(\alpha,\alpha_i)} v \quad \forall i \in I_e$$

-Note for $|\phi_\ell\rangle = (\phi|\dots|\phi) \in \mathcal{P}^{\Lambda}_n$ of level ℓ

$$K_i|\phi_\ell\rangle = q^{d_i(|\phi_\ell\rangle)}|\phi_\ell\rangle = q^{\#\Lambda_i \text{ appears in } \Lambda}|\phi_\ell\rangle$$

$$(\boxed{K_i}|\dots|) = q^{(\Lambda,\alpha_i)}|\phi_\ell\rangle \Rightarrow |\phi_\ell\rangle \text{ has wt } \Lambda$$

- Clear that $E_i|\phi_\ell\rangle = 0 \quad \forall i$ as nothing to remove

Def $L(\Lambda) = U_q(\widehat{sl_e})|\phi_\ell\rangle$

Lemma 5 (1) $L(\Lambda)$ is h.w. of weight Λ

(2) $L(\Lambda)$ is integrable $\Rightarrow$ simple

(3) $L(\Lambda)$ is the unique int $U_q(\widehat{sl_e})$-mod of wt Λ

Pf: (1)✓ (2) Int = $E_i^n v = F_i^n v = 0 \quad \forall v, m, n > 0$

E_i = remove nodes ✓ F_i = add i-nodes, but $|\vec{\lambda}+A\rangle$ for i-node A has fewer addable i-nodes ✓

(3) Find any book on crystal basis

3. Cat of $L(\Lambda)$

-Recall we had Ind, Res functors from

$$R(\alpha)\otimes R(\beta) \hookrightarrow R(\alpha+\beta)$$

-Now let i-Ind, i-Res be Ind, Res from

$$R^{\Lambda}(\alpha)\otimes R^{\Lambda}((i)) \hookrightarrow R^{\Lambda}((\alpha,i))$$

Lemma 6: (1) i-Ind, i-Res are exact

(2) i-Ind is biadjoint to i-Res

-Let $[\mathrm{Proj}^{\Lambda}_n(e)] = K_0\left(R^{\Lambda}_n(\mathcal{P}_e, \mathbb{F})\text{-gpmod}\right)\otimes_{\mathbb{Z}}\mathbb{Q}(q)$

$[\mathrm{Rep}^{\Lambda}_n(e)] = K_0\left(R^{\Lambda}_n(\mathcal{P}_e,\mathbb{F})\text{-gmod}\right)\otimes_{\mathbb{Z}}\mathbb{Q}(q)$

Thm 7 (Cat Theorem): Let $\Lambda \in P^+$. Then letting $E_i = i$-Res, $F_i = q \circ i\text{-Ind} \circ K_i^{-1}$, we have

$$\bigoplus_{n\geq 0}[\mathrm{Proj}^{\Lambda}_n(e)] \cong L(\Lambda) \cong \bigoplus_{n\geq 0}[\mathrm{Rep}^{\Lambda}_n(e)]$$

as $U_q(\widehat{sl_e})$-mod

Categorification of L(\Lambda)

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Pf: Let $K_n^\Lambda = \{\vec\lambda \in P_n^\Lambda \mid D^{\vec\lambda} \neq 0\}$ ← simples. Then $[\mathrm{Proj}_n^\Lambda(e)]$ has basis $\{[P^{\vec\mu}] \mid \vec\mu \in K_n^\Lambda\}$. Consider

$$\begin{array}{ccc} [\mathrm{Proj}_n^\Lambda(e)] & \xrightarrow{e_q} & \mathcal{F}_n^\Lambda \\ & & \downarrow d_q \\ & & [\mathrm{Rep}_n^\Lambda(e)] \end{array}$$

- $e_q([P^{\vec\mu}]) = \sum_{\vec\lambda \in P_n^\Lambda} [P^{\vec\mu} : S^{\vec\lambda}] |\vec\lambda\rangle$
- $d_q(|\vec\lambda\rangle) = [S^{\vec\lambda}]$

Step 1: d_q is a $U_q(\hat{sl}_e)$-morph

Prop (BK): $[i\text{-Res}\, S^{\vec\lambda}] = \sum_{A \in \mathrm{Rem}_i(\vec\lambda)} q^{d_A(\vec\lambda)} [S^{\vec\lambda - B}]$

$[i\text{-Ind}\, S^{\lambda}(1-d_i |\vec\lambda\rangle)] = \sum_{A \in \mathrm{Add}_i(\vec\lambda)} q^{-d_A(\vec\lambda)} [S^{\vec\lambda + A}]$

Step 2: e_q is a $U_q(\hat{sl}_e)$-morph

- Notice $d_q(|\vec\lambda\rangle) = \sum_{\vec\mu \in K_n^\Lambda} [S^{\vec\lambda} : D^{\vec\mu}] [D^{\vec\mu}]$

$= \sum_{\vec\mu \in K_n^\Lambda} d_{\vec\lambda\vec\mu}(q) [D^{\vec\mu}]$. By BH-reciproc

$[P^{\vec\mu} : S^{\vec\lambda}] = [S^{\vec\lambda} : D^{\vec\mu}] = d_{\vec\lambda\vec\mu}$

$\Rightarrow e_q = d_q^T = d_q^* : [\mathrm{Rep}_n^\Lambda(e)] \longrightarrow \mathcal{F}_n^{\Lambda *}$ $\quad \mathcal{F}_n^\Lambda = \mathcal{U}(q) P_n^\Lambda$

after $[\mathrm{Rep}_n^\Lambda(e)] \xrightarrow{\sim} [\mathrm{Proj}_n^\Lambda(e)]$ via Cartan

$\mathcal{F}_n^{\Lambda *} \xrightarrow{\sim} \mathcal{F}_n^\Lambda$, via dual $|\lambda\rangle^* \mapsto |\lambda\rangle$

- Check $\langle F_i |\lambda\rangle^*, |\mu\rangle\rangle_{\mathrm{dual}} \overset{(?)}{=} \langle |\lambda\rangle^*, E_i |\mu\rangle\rangle_{\mathrm{dual}}$

LHS: $q=1$ $\langle \sum_{A \in \mathrm{Add}_i(\lambda)} |\lambda + A\rangle^*, |\mu\rangle\rangle_d = \begin{cases} 1 & \text{if } \lambda + A = \mu \\ 0 & \text{otherwise} \end{cases}$

RHS: $q=1$ $\langle |\lambda\rangle^*, \sum_{B \in \mathrm{Rem}_i(\mu)} |\vec\mu - B\rangle\rangle = \begin{cases} 1 & \text{if } \lambda = \mu - B \\ 0 & \text{otherwise} \end{cases}$

$\langle e_q(E_i \cdot Y), X\rangle_{\mathrm{dual}} \overset{\mathrm{dual}}{=} \langle E_i \cdot Y, d_q(X)\rangle_C$

$\overset{\mathrm{def}}{=} \langle i\text{-Res}(Y), X\rangle_C \overset{\text{bi-adj}}{=} \langle Y, i\text{-Ind}\, d_q(X)\rangle_C$

$\overset{\text{Step 1}}{=} \langle Y, d_q(i\text{-Ind} X)\rangle_C \overset{\mathrm{dual}}{=} \langle d_q^* Y, F_i X\rangle_{\mathrm{dual}}$

$= \langle E_i \cdot e_q(Y), X\rangle_{\mathrm{dual}}$

Step 3: e_q is an iso

- Note that $d_q(|\vec{\lambda}\rangle) = \sum_{\vec{\mu} \in K_n^\Lambda} [S^{\vec{\lambda}} : D^{\vec{\mu}}] [D^{\vec{\mu}}]$

written in basis $\{|\lambda\rangle\}_{\lambda \in P_n^\Lambda}$ of $\mathcal{F}_n^\Lambda$ and basis $\{[D^\mu]\}_{\mu \in K_n^\Lambda}$ is literally the matrix

D^T where $D = (d_{\lambda\mu})_{\substack{\lambda \in P_n^\Lambda \\ \mu \in K_n^\Lambda}} \Rightarrow e_q = d_q^T = D$

- From Dinushi's talk D has 1's on diagonal $\Rightarrow$ full rank $\Rightarrow e_q$ is inj

- Because $e_q(p_\varnothing) = |\phi_e\rangle$ and e_q is a $U_q(\hat{sl_e})$-morph + $[\mathrm{Proj}^\Lambda(e)]$ is cyclic $U_q(\hat{sl_e})$-mod??

$\Rightarrow \operatorname{im} e_q \subseteq L(\Lambda)$.

$L(\Lambda)$ simple $\Rightarrow \operatorname{im} e_q = L(\Lambda)$. Dualize to get corr statement for $[\mathrm{Rep}_n^\Lambda(e)]$

Thrm 8 (Ariki): When char $\mathbb{F}=0$, $q=1$, the iso in Cat Theorem sends the basis $\{[P^{\vec{\mu}}]\}$ of indecomp projective $\bigoplus_{n \geq 0} H_n^\Lambda(\mathbb{F})$-modules to canonical basis of $L_1(\Lambda)$ of $U(\hat{sl_e})$.

Thrm 9 (BK): When char $\mathbb{F}=0$, the iso in Cat Theorem sends the basis $\{q^{-\mathrm{def}\,\vec{\mu}}[P^{\vec{\mu}}] \mid \vec{\mu} \in K^\Lambda\}$ of indecomp self-dual projective $\bigoplus_{n \geq 0} H_n^\Lambda(\mathbb{F})$-gmod to canonical basis of $L(\Lambda)$ of $U_q(\hat{sl_e})$.

Rem: There are efficient algor to compute canonical basis of $L(\Lambda)$ such as LLT

Cor 10: $\exists$ explicit combinatorial description for $K_n^\Lambda = \{\vec{\mu} \mid D^{\vec{\mu}} \neq 0\}$

Pf: $[P^{\vec{\mu}}] \longleftrightarrow$ canonical basis = (lower) crystal basis and crystal graph of $L(\Lambda)$ is well-known/studied.

Decomposition Multiplicities

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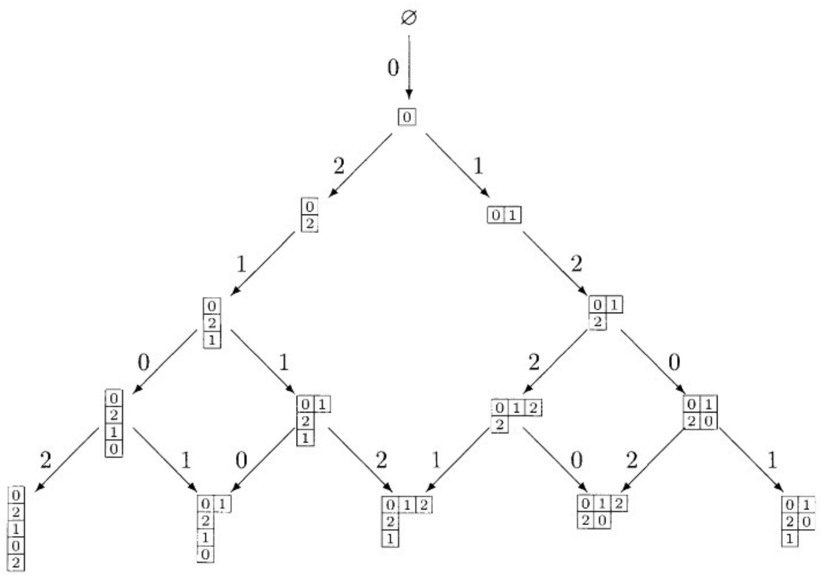
6.20 Example Suppose that $e = 3$. Then the first six layers of the crystal graph of $L(\Lambda_0)$ are as follows.



Cor 11: Let $\{b_{\vec{\lambda}}\}$ be the canonical basis for $L(\Lambda)$. Write $b_{\vec{\lambda}} = |\vec{\lambda}\rangle + \sum_{\substack{\vec{\mu} \in \ell \kappa_n \\ \vec{\mu} < \vec{\lambda}}} b_{\vec{\mu}\vec{\lambda}}(q) |\vec{\mu}\rangle$

Then $[S^{\vec{\mu}} : D^{\vec{\lambda}}]_q = b_{\vec{\mu}\vec{\lambda}}(q)$.

Pf: Recall if $C = ([P^{\vec{\mu}} : S^{\vec{\lambda}}])$, then $C = D^t D$. $D = ([S^{\vec{\lambda}} : D^{\vec{\mu}}])$. By Cat Thm

$$\begin{tikzcd}
{[\mathrm{Proj}^\Lambda(e)]} \arrow[r, "e_q = D", "\sim"'] \arrow[dr, "\sim", "{c_q = C \ (= \text{natural inclusion})}"'] & L(\Lambda) \arrow[d, "d_q = D^t", "\wr"'] \\
 & {[\mathrm{Rep}^\Lambda(e)]}
\end{tikzcd}$$

By graded BH reciprocity

$$[S^{\vec{\mu}} : D^{\vec{\lambda}}]_q = [P^{\vec{\lambda}} : S^{\vec{\mu}}]_q$$

$$= [e_q^{-1} c_q^{-1}(P^{\vec{\lambda}}) : d_q^{-1}(S^{\vec{\mu}})]_q$$

$$= [e_q^{-1}(P^{\vec{\lambda}}) : |\mu\rangle]_q = [b_{\vec{\lambda}} : |\mu\rangle]_q$$

$$= b_{\vec{\mu}\vec{\lambda}}(q)$$