0. Introduction

How it started:

KLR Algebras $R_n(\Gamma)$

↳ Categorify $U_q^-(\mathfrak{g})$ $(= U_q^-(\widehat{\mathfrak{sl}_e}))$

How it's going:

Integral Cyclotomic Hecke algebra $\mathcal{H}_n^{\Lambda}$ → dominant weight

↳ Cellular basis $c_{\mathfrak{s}\mathfrak{t}}$, $\mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\lambda)$

↳ Specht's S^λ, simples D^μ, Brauer-Humphreys reciprocity.

↳ Semisimplicity criterion

What's ahead:

$R_n^\Lambda(\Gamma)$ = cyclotomic quotient of KLR

↳ $R_n^\Lambda(\Gamma_e) \cong \mathcal{H}_n^\Lambda$ "Graded Isomorphism Theorem"

↳ $R_n^\Lambda(\Gamma_e)$ categorifies the simple $U_q(\widehat{\mathfrak{sl}_e})$-module $L(\Lambda)$ "Graded Categorification Theorem"

↳ Projective indecomposables categorify the canonical basis of $L(\Lambda)$.

Today:

1. Upgrade cellularity to a graded versions
2. Introduce cyclotomic KLR algebras and their first properties.
3. Describe R_n^Λ fully in the semisimple case, via a graded cellular basis.

   ↳ Corollary: Graded Categorification Theorem in this case

Previous Notions:

A : cellular algebra

$(\mathcal{P}, \trianglerighteq)$ weight poset.

$\lambda \in \mathcal{P} \Rightarrow T(\lambda)$ finite set

$C = \{ c_{st} : s,t \in T(\lambda),\ \lambda \in \mathcal{P} \}$ cellular basis

$*$ antiinvolution

$C^\lambda \in \mathrm{Mod}_A$ cell module, basis $\xleftrightarrow{1:1} T(\lambda)$

$\langle , \rangle$ on C^λ, $D^\lambda = C^\lambda / \mathrm{rad}\, C^\lambda \overset{\mathbb{Z}=F}{\Leftarrow}$ All the simples (if $\neq 0$)

$P^\mu \longrightarrow D^\mu$ projective cover

Brauer-Humphreys recip.: $[P^\mu : C^\lambda] = [C^\lambda : D^\mu]$

Example: integral cyclotomic Hecke

$A = \mathcal{H}_n^\Lambda = \mathbb{Z} \cdot \{ \underset{\text{Hecke}}{v^{\pm 1}}, \underset{S_n}{T_i}, \underset{\text{JM}}{L_i}, \underset{\text{cyclotomic}}{Q_i = v^{\kappa_i}} \} / \text{rels}$

$\Lambda = \Lambda_{\kappa_1 (\mathrm{mod}\, e)} + \dots + \Lambda_{\kappa_\ell (\mathrm{mod}\, e)}$

$\mathcal{P}_n^\Lambda$ = ℓ-multipartitions of n $\boldsymbol{\lambda}$ $\left(\begin{array}{|c|c|c|} \hline \ & \ & \ \\ \hline \ \\ \cline{1-1} \end{array}\ ,\ \begin{array}{|c|c|} \hline \ & \ \\ \hline \ & \ \\ \hline \end{array} \right)$

$T(\boldsymbol{\lambda}) = \mathrm{Std}(\boldsymbol{\lambda})$ standard $\boldsymbol{\lambda}$-tableaux $\left(\begin{array}{|c|c|c|} \hline 2 & 3 & 6 \\ \hline 5 \\ \cline{1-1} \end{array}\ ,\ \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 7 & 8 \\ \hline \end{array} \right)$

$C = \{ m_{st} : s,t \in \mathrm{Std}(\boldsymbol{\lambda}),\ \boldsymbol{\lambda} \vdash^{\ell} n \}$

$m_{st} = u_\lambda x_{\boldsymbol{\lambda}}$, $x_{\boldsymbol{\lambda}}$ ↳ Young sym

$T_i^* = T_i$, $L_i^* = L_i$, $v^* = v^{-1}$, $c_{st}^* = c_{st}$

$S^{\boldsymbol{\lambda}}$ Specht module, basis $\{ m_t : t \in \mathrm{Std}(\boldsymbol{\lambda}) \}$

$D^{\boldsymbol{\lambda}} = S^{\boldsymbol{\lambda}} / \mathrm{rad}(S^{\boldsymbol{\lambda}})$

1. Graded Everything

Again take $\mathcal{Z}$ be a commutative domain. Every module and algebra will be free over $\mathcal{Z}$ with $\dim_{\mathcal{Z}} < \infty$.

Let $A = \bigoplus_{d \in \mathbb{Z}} A_d$ be a graded algebra. Let $\underline{A}$ = ungraded A.

A graded A-module M is an $\underline{A}$-module with a decomposition $M = \bigoplus_{d \in \mathbb{Z}} M_d$ s.t. $A_{d'} M_d \subseteq M_{d'+d}$.

Sadly we set $M\langle s \rangle_d = M_{d-s}$.

The graded Hom is $\mathrm{Hom}^{\bullet}_A(M,N) := \bigoplus_{s \in \mathbb{Z}} \mathrm{Hom}_A(M, N\langle s \rangle)$.

graded dim is $\dim_q(M) = \sum_{s \in \mathbb{Z}} q^s \dim_{\mathcal{Z}} \mathrm{Hom}_A(M, N\langle s \rangle)$

Given a simple graded A-module D, the graded decomposition number is $[M:D]_q = \sum_{s \in \mathbb{Z}} q^s [M : D\langle s \rangle]$

Let $\underline{M_0}$ be an ungraded $\underline{A}$-module. A graded lift is an A-module M s.t. $\underline{M} \cong \underline{M_0}$.

Proposition: any two graded lifts of a f.g. indecomposable module over A are unique up to shift.

Proof: Recall the Fitting Lemma: endomorphisms of finite length indecomposables are either isomorphisms or nilpotent.

Now let M, N be A-modules with $\underline{M}, \underline{N} \cong U$. Now $\mathrm{Hom}^{\bullet}(M,N) = \mathrm{Hom}(\underline{M}, \underline{N}) = \mathrm{End}(U)$. In particular, the identity map on U can be written as a finite sum $\mathrm{id}_U = \sum_{d \in \mathbb{Z}} \varphi_d$ with φ_d of degree d.

Assume every φ_d is nilpotent. Then $(\mathrm{id}_U)^{\gg N} = 0$, absurd. So at least one φ_d is an isomorphism. (In fact, only one) ◻

Definition (Graded cell datum): A graded cell datum $(\mathcal{P}, T, C)$ is one that comes equipped with a map (of sets) $T(\lambda) \to \mathbb{Z}$ $\forall \lambda \in \mathcal{P}$, and such that c_{st} is homogeneous of degree $\deg(s) + \deg(t)$.

Remark: most things carry over: C^λ is the graded cell module, its bilinear form has degree 0, $\mathrm{rad}\, C^\lambda$ is a graded submodule, $D^\lambda = C^\lambda / \mathrm{rad}\, C^\lambda$ is graded, etc. As a corollary of the Proposition, we get that $\{ D^\lambda \langle s \rangle : D^\lambda \neq 0 \}$ is a complete set of graded simples.

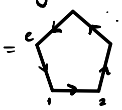
Relevant for us is the following: a graded cell filtration of M is $M = M_0 > \ldots > M_m = 0$ with $M_i / M_{i+1} = C^\lambda \langle s_i \rangle$.

The cell module multiplicities and shifts are independent of the filtration and so we define $[M : C^\lambda]_q = \sum_{i : M_i/M_{i+1} \cong C^\lambda \langle s_i \rangle} q^{s_i}$

We also have a notion of graded Brauer-Humphreys reciprocity: $[P^\mu : C^\lambda]_q = [C^\lambda : D^\mu]_q$.

2. Cyclotomic KLR algebras aka cyclotomic quiver Hecke algebras.

In the first lecture we defined KLR algebras for a loop-less unoriented quiver with no multiple edges.

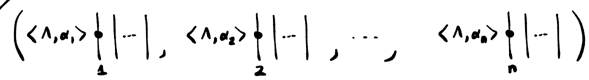
Recall that we had $R_\nu(\Gamma) = \bigoplus_{\vec{i},\vec{j} \in \mathrm{Seq}(\nu)} {}_{\vec{j}}R_\nu(\Gamma)_{\vec{i}}$ (e.g. cyclic quiver $\Gamma_e =$ )

We also had bases for ${}_{\vec{i}}R_\nu(\Gamma)_{\vec{j}} = \{\text{diagrams from } \vec{i} \text{ to } \vec{j}\}$ given by diagrams with dots at the bottom.

Define $R_n(\Gamma) = \bigoplus_{|\nu|=n} R_\nu(\Gamma)$

Definition. Let $\Lambda = \Lambda_{\bar{k}_1} + \dots + \Lambda_{\bar{k}_n}$. The cyclotomic quiver Hecke algebra (or cyclotomic KLR algebra) is

$$R_n^\Lambda(\Gamma) := \bigoplus_{|\nu|=n} R_\nu^\Lambda(\Gamma) \Big/ \left(\dots \right)$$



Examples: • Let $n=1$, $\Lambda = a_1\Lambda_1 + \dots + a_e\Lambda_e$

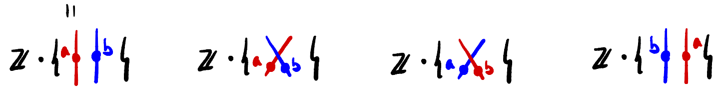
Then $R_n^\Lambda(\Gamma) = R^\Lambda_{\alpha_1}(\Gamma) \big/ \langle \, \rangle \oplus \dots \oplus R^\Lambda_{\alpha_e}(\Gamma) \big/ (\,) = \mathbb{Z}[\,]/(\,) \oplus \dots \oplus \mathbb{Z}[\,]/(\,)$

Coincidentally, $H_1^\Lambda = \mathbb{Z}[L_1] \big/ (L_1 - v)^{a_1} \cdots (L_1 - v^e)^{a_e} \cong \mathbb{Z}[L_1]/L_1^{a_1} \oplus \dots \oplus \mathbb{Z}[L_1]/L_e^{a_e}$

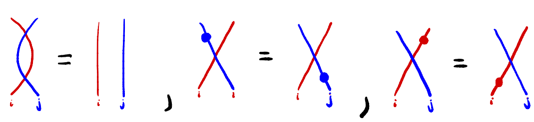
• Let $n=2$, $\Lambda = \Lambda_1 + \Lambda_2$, $\nu = \alpha_1 + \alpha_2$, $\Gamma = \overset{\bullet}{\alpha_1} \;\; \overset{\bullet}{\alpha_2}$

Recall that $R := R_{\alpha_1+\alpha_2}(\Gamma)$, was

${}_{\alpha_1\alpha_2}R_{\alpha_1\alpha_2} \oplus {}_{j\,i}R_{\alpha_1\alpha_2} \oplus {}_{\alpha_1\alpha_2}R_{\alpha_2\alpha_1} \oplus {}_{\alpha_2\alpha_1}R_{\alpha_2\alpha_1}$



subject to:



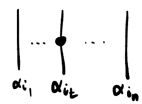
$\cong M_2(\mathbb{Z}[x,y])$

Now $R_\nu^{\Lambda_1+\Lambda_2}(\Gamma) = M_2(\mathbb{Z}[x,y]) \big/ \langle [\,,\,] \rangle = M_2(\mathbb{Z})$.

Coincidentally, this has $\dim_{\mathbb{Z}} = 4 < \infty$.

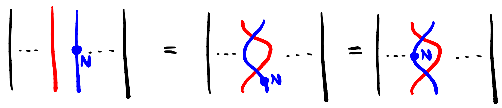
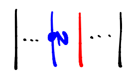
3. R_n^Λ is finite dimensional

Proposition: the title of the section holds.

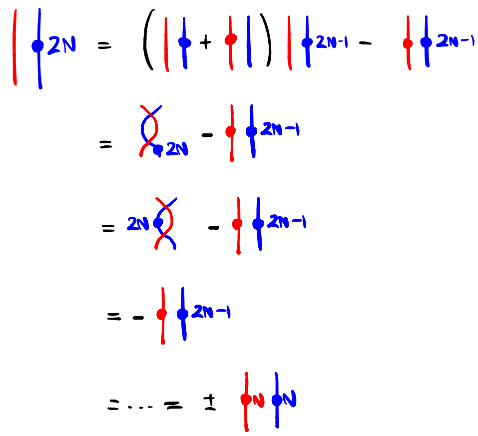
Proof: Since we have bases with dots on the bottom, it suffices to show that every "dot diagram"  is nilpotent.

We induct on t, $t=1$ being clear from the definition of cyclotomic quotient.

For the inductive step, we have 3 cases:

- $\alpha_{i_t} \not - \alpha_{i_{t+1}}$:  . But  $=0$ for $N \gg 0$, we win.

  Observe that it is enough to prove these for the two relevant strands.

- $\alpha_{i_t} — \alpha_{i_{t+1}}$: 

- $\alpha_{i_t} = \alpha_{i_{t+1}}$: exercise.

$\square$

Remark: The above proof uses the conventions in [KL], but for this section we are following [M], which gives the following (Rouquier) presentation:

$$e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{ij}}e(\mathbf{i}), \qquad \textstyle\sum_{\mathbf{i}\in I^\beta} e(\mathbf{i}) = 1,$$
$$y_r e(\mathbf{i}) = e(\mathbf{i}) y_r, \qquad \psi_r e(\mathbf{i}) = e(s_r{\cdot}\mathbf{i})\psi_r, \qquad y_r y_s = y_s y_r,$$

$$\psi_r\psi_s = \psi_s\psi_r, \qquad \text{if } |r-s|>1,$$
$$\psi_r y_s = y_s \psi_r, \qquad \text{if } s \neq r, r+1,$$

$$\psi_r y_{r+1} e(\mathbf{i}) = (y_r\psi_r + \delta_{i_r i_{r+1}})e(\mathbf{i}), \tag{2.2.2}$$
$$y_{r+1}\psi_r e(\mathbf{i}) = (\psi_r y_r + \delta_{i_r i_{r+1}})e(\mathbf{i}),$$

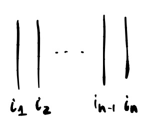
$$\psi_r^2 e(\mathbf{i}) = \begin{cases} (y_{r+1}-y_r)(y_r-y_{r+1})e(\mathbf{i}), & \text{if } i_r \rightleftarrows i_{r+1},\\ (y_r - y_{r+1})e(\mathbf{i}), & \text{if } i_r \to i_{r+1},\\ (y_{r+1}-y_r)e(\mathbf{i}), & \text{if } i_r \leftarrow i_{r+1},\\ 0, & \text{if } i_r = i_{r+1},\\ e(\mathbf{i}), & \text{otherwise,}\end{cases} \tag{2.2.3}$$

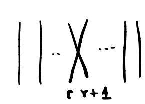
and $(\psi_r\psi_{r+1}\psi_r - \psi_{r+1}\psi_r\psi_{r+1})e(\mathbf{i})$ *is equal to*

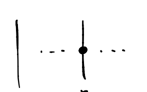
$$\begin{cases} (y_r + y_{r+2} - 2y_{r+1})e(\mathbf{i}), & \text{if } i_{r+2} = i_r \rightleftarrows i_{r+1},\\ -e(\mathbf{i}), & \text{if } i_{r+2} = i_r \to i_{r+1},\\ e(\mathbf{i}), & \text{if } i_{r+2} = i_r \leftarrow i_{r+1},\\ 0, & \text{otherwise,}\end{cases} \tag{2.2.4}$$

for $\mathbf{i},\mathbf{j} \in I^\beta$ *and all admissible* r *and* s.

$\longleftrightarrow$

$e(\underline{i}) =$  deg 0

$\psi_r =$  $\deg \psi_r e(\underline{i}) = -c_{i_r, i_{r+1}}$

$y_r =$  deg 2

In fact, one can generalize the definition to any quiver by means of a matrix with entries in $\mathbb{Z}[u,v]$ denoted $Q = (Q_{ij})$ with $Q_{ii} = 0$, $Q_{ij} \neq 0$, $Q_{ij}(u,v) = Q_{ji}(v,u)$, $\deg Q_{ij} = r$, where $i \xrightarrow{r} j$

Then the relations are the same except

$$\psi_r^2 e(\mathbf{i}) = Q_{i_r,i_{r+1}}(y_r, y_{r+1})e(\mathbf{i})$$

$$(\psi_r\psi_{r+1}\psi_r - \psi_{r+1}\psi_r\psi_{r+1})e(\mathbf{i}) = \begin{cases} \frac{Q_{i_r,i_{r+1}}(y_r,y_{r+1}) - Q_{i_r,i_{r+1}}(y_r,y_{r+1})}{y_{r+2}-y_r}, & \text{if } i_{r+2} = i_r, \\ 0, & \text{otherwise.} \end{cases}$$

One recovers the (first) Khovanov-Lauda definition setting $Q_{ij} = u^{-c_{ij}} + v^{-c_{ji}}$

We stick to $\Gamma = \Gamma_e$ for the rest of the talk.

4. Representation theory of R_n^Λ, semisimple case

Recall that $\mathcal{H}_n^\Lambda$ was semisimple iff writing $\Lambda = (a_1, a_2, a_3, \ldots, a_n)$ in the basis of fundamental weights, no consecutive n-string contains more than a 1: "far apart".

Mathas defines $\alpha_{i,n} = \alpha_i + \ldots + \alpha_{i+n-1}$ and requires $\langle \Lambda, \alpha_{i,n} \rangle \leq 1 \quad \forall i$.

We make the same assumption here. A posteriori, this implies R_n^Λ is ss, but we do not assume so.

Recall that if $\mathcal{Z} = F$ is a field, $\{ \text{Irreps of } \mathcal{H}_n^\Lambda \text{ (ss)} \} \longleftrightarrow \{ S^\lambda : \lambda \ \ell\text{-multipartition of } n \}$

So we hope that the following holds.

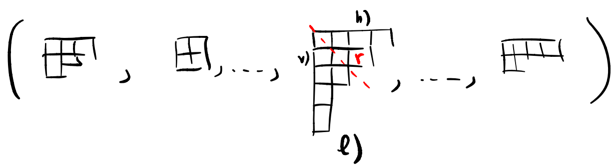
Proposition: Let $\mathcal{Z} = F$, $e > n$ and $\langle \Lambda, \alpha_{i,n} \rangle \leq 1 \ \forall i$. Then for each $\lambda \overset{\ell}{\vdash} n$ there is a unique simple graded R_n^Λ-module S^λ with homogeneous basis $\{ \varphi_{\underline{t}} : \underline{t} \in \mathrm{Std}(\lambda) \}$.

Furthermore $\deg \varphi_{\underline{t}} = 0 \ \forall \underline{t}$ and the action is given by:

$$\varphi_{\underline{t}} \cdot e(\underline{i}) = \delta_{\underline{i}, \underline{i}^{\underline{t}}} \varphi_{\underline{t}}, \quad \varphi_{\underline{t}} y_r = 0, \quad \varphi_{\underline{t}} \cdot \psi_r = \varphi_{s_r \cdot \underline{t}} \quad \overset{\text{if } s_r \cdot \underline{t} \text{ not standard}}{= 0}$$

Here, given $\underline{t} \in \mathrm{Std}(\lambda)$, $\underline{i}^{\underline{t}} = (c_1(\underline{t}) \ (\mathrm{mod}\ e), \ldots, c_n(\underline{t}) \ (\mathrm{mod}\ e))$ "residue sequence of $\underline{t}$"

In turn, $c_r(\underline{t})$ is given by: $\underline{t} =$



$$\Rightarrow c_r(\underline{t}) = \kappa_\ell + h - v$$

We denote the set of all such sequences by I_Λ^n.

Combinatorial Lemma: Assume again $e > n$ and $\langle \Lambda, \alpha_{i,n} \rangle \leq 1 \ \forall i$. Then:

a) For any $\underline{t}, \underline{s} \in \mathrm{Std}(\mathcal{P}_n^\Lambda)$, $\underline{t} = \underline{s}$ iff $\underline{i}^{\underline{t}} = \underline{i}^{\underline{s}}$.

b) If $\underline{i} \in I_\Lambda^n$, then $i_r \neq i_{r+1}$.

c) If $\underline{i} \in I_\Lambda^n$ has $i_{r+1} = i_r \pm 1$ for some r, then $s_r \cdot \underline{i} \notin I_\Lambda^n$.

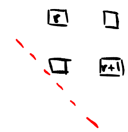
d) If $\underline{i} \in I_\Lambda^n$, then $(i, i\pm 1, i)$ is not a (consecutive) subsequence of $\underline{i}$.

Proof: a) Induction on n. For $n=1$, $\mathcal{P}_n^\Lambda = \{ \underbrace{(\emptyset,\dots,\emptyset,\underset{j}{\square},\emptyset,\dots,\emptyset)}_{c_1 = \kappa_j} : j=1,\dots,\ell\}$ $\Rightarrow$ $I^1_\Lambda = \{\kappa_j : \kappa_1 = 1,\dots,\ell\}$ all different!

Assume $\mathfrak{t} \in \mathrm{Std}(\lambda)$ is the unique tableau in $\mathrm{Std}(\mathcal{P}^\Lambda_m)$ with a given $\mathbf{i}^{\mathfrak t} \in I^m_\Lambda$.

and take $\mathbf{i}^{\mathfrak t} \cup \{i_{m+1}\} \in \mathrm{Std}(\mathcal{P}^\Lambda_{m+1})$. Then $c_{m+1}(\mathfrak t) = \kappa_\ell + h - v \equiv i_{m+1} \mod e$ $\Rightarrow$ $\exists!$ "addable" node with a given content.

κ_ℓ's are far apart

b) If $i_r = i_{r+1}$, then r and $r+1$ lie on the same partition inside λ. Now we have  , impossible.

c) Similarly, if two consecutive contents differ by 1, they must lie in the same partition inside λ.

By definition of the content, r and $r+1$ are in the same column/row, making $s_r\cdot\mathfrak t$ not standard.

d) Similar.

Proof of the Proposition:

Recall we set

$$\varphi_{\mathfrak t}\cdot e(\mathbf i) = \delta_{\mathbf i, \mathbf i^{\mathfrak t}}\varphi_{\mathfrak t}, \quad \varphi_{\mathfrak t}\, y_r = 0, \quad \varphi_{\mathfrak t}\cdot\psi_r = \varphi_{\mathfrak t(r,r+1)}.$$

(Discuss:)

$e(\mathbf i)e(\mathbf j) = \delta_{\mathbf{ij}}e(\mathbf i)$, ✓ $\mathfrak t \leftrightarrow \mathbf i^{\mathfrak t}$ $\quad \sum_{\mathbf i\in I^\beta} e(\mathbf i) = 1$, ✓ $\mathfrak t\leftrightarrow \mathbf i^{\mathfrak t}$

$y_r e(\mathbf i) = e(\mathbf i) y_r$, ✓ 0 $\quad \psi_r e(\mathbf i) = e(s_r\cdot\mathbf i)\psi_r$, ✓ def of ψ_r $\quad y_r y_s = y_s y_r$, ✓ 0

$\psi_r\psi_s = \psi_s\psi_r$, ✓ def $\qquad$ *if* $|r-s|>1$,

$\psi_r y_s = y_s\psi_r$, ✓ 0 $\qquad$ *if* $s\neq r,r+1$,

(2.2.2) $\quad \psi_r y_{r+1} e(\mathbf i) = (y_r\psi_r + \delta_{i_r i_{r+1}})e(\mathbf i)$, ✓ 0 + $i_r \neq i_{r+1}$

$y_{r+1}\psi_r e(\mathbf i) = (\psi_r y_r + \delta_{i_r i_{r+1}})e(\mathbf i)$,

(2.2.3)
$$\psi_r^2 e(\mathbf i) = \begin{cases} (y_{r+1}-y_r)(y_r-y_{r+1})e(\mathbf i), & \text{if } i_r \rightleftarrows i_{r+1}, \\ (y_r - y_{r+1})e(\mathbf i), & \text{if } i_r \to i_{r+1}, \\ (y_{r+1}-y_r)e(\mathbf i), & \text{if } i_r \leftarrow i_{r+1}, \\ 0, & \text{if } i_r = i_{r+1}, \quad ✓ \text{ (never)} \\ e(\mathbf i), & \text{otherwise}, \quad ✓ \text{ (def)} \end{cases}$$

RHS $=0$ $\quad$ LHS $=0$ c)

and $(\psi_r\psi_{r+1}\psi_r - \psi_{r+1}\psi_r\psi_{r+1})e(\mathbf i)$ *is equal to*

(2.2.4)
$$\begin{cases} (y_r + y_{r+2} - 2y_{r+1})e(\mathbf i), & \text{if } i_{r+2} = i_r \rightleftarrows i_{r+1}, \\ -e(\mathbf i), & \text{if } i_{r+2} = i_r \to i_{r+1}, \\ e(\mathbf i), & \text{if } i_{r+2} = i_r \leftarrow i_{r+1}, \\ 0, & \text{otherwise}, \quad ✓ \text{ braid for } s_r, s_{r+1} \end{cases}$$

(never) by d)

for $\mathbf i, \mathbf j \in I^\beta$ *and all admissible* r *and* s.

Now b) implies $c_{i_r, i_{r+1}} = 0$ so $\deg e(\mathbf i^{\mathfrak t})\cdot\psi_r = 0$ $\Rightarrow$ Set $\deg(\varphi_{\mathfrak t}) = 0$ for all $\mathfrak t$.

This shows that S^λ is a graded $R^\Lambda_n(\mathcal P_e)$-module.

Proof of simplicity: let $M \subseteq S^{\lambda}$ be nonzero and take a nonzero $m = \sum_{\underline{t}'} \mu_{\underline{t}'} \psi_{\underline{t}'}$. Choose a nonzero summand $\mu_{\underline{t}} \psi_{\underline{t}}$.

Then $m\, e(\underline{t}) = \mu_{\underline{t}} \psi_{\underline{t}} \Rightarrow \psi_{\underline{t}} \in M$. Now for any other $\underline{s} \in \mathrm{Std}(\lambda)$ $\exists\, w \in S_n$ s.t. $w \cdot \underline{t} = \underline{s}$. So $\psi_{\underline{s}} = \psi_{\underline{t}} \cdot \psi_w \in M$.

Hence $M = S^{\lambda}$. □

Next, we show that R_n^{Λ} is graded cellular. We need a lemma:

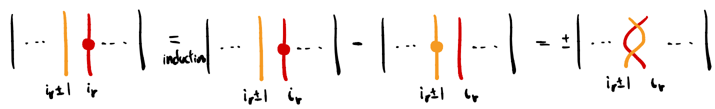
Vanishing lemma: The following vanish:

a) The "idempotent generators" $e(\underline{i})$ whenever $\underline{i} \notin I^n_{\Lambda}$.

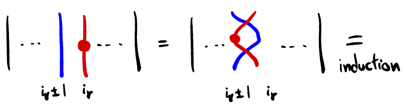
b) The "dot" generators y_r

Proof of a) is omitted, it comes down to a detailed study of the set I^n_{Λ} à la Okounkov-Vershik.

b) Since $\sum_{\underline{i} \in I^n} e(\underline{i}) = 1$, by a) it suffices to show that y_r kills $e(\underline{i})$ for $\underline{i}_{\downarrow r} \in I^r_{\Lambda}$. (Induct on r).

• If $i_{r-1} = i_r \pm 1$: $y_r e(\underline{i}) =$ 

this bit is $e(s_r \cdot \underline{i})$, which is 0 by a) and the previous lemma.

• If $i_{r-1} \neq i_r$: $y_r e(\underline{i}) =$ 

$\overset{\text{induction}}{=} 0$.

(The fact that these are the only two cases also follows from the proof of a))

We are ready to prove the cellularity theorem.

Define the elements $e_{\underline{s},\underline{t}} = \psi_{d(\underline{s})^{-1}}\, e(\underline{i}^{\lambda})\, \psi_{d(\underline{t})}$, where $\underline{i}^{\lambda} = \underline{i}^{\underline{t}^{\lambda}}$, and $\underline{t}^{\lambda} \in \mathrm{Std}(\lambda)$ is the tableau filled-in in order (e.g. $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 5 \end{smallmatrix} , \begin{smallmatrix} 6 & 7 \\ 8 \end{smallmatrix} \right)$).

Theorem: Let $e > n$ and Λ s.t. $\langle \Lambda, \alpha_{i,n} \rangle \leq 1$ $\forall i$. Then R_n^{Λ} is graded cellular with graded cellular basis

$$\{ e_{\underline{s}\underline{t}} : \underline{s}, \underline{t} \in \mathrm{Std}(\lambda) \}, \text{ and } \deg(e_{\underline{s},\underline{t}}) = 0.$$

Proof: By the Vanishing Lemma, $R_n^{\Lambda}(\Gamma_e)$ is spanned by ψ_r's and $e(\underline{i})$'s.

Also, since

and $(\psi_r\psi_{r+1}\psi_r - \psi_{r+1}\psi_r\psi_{r+1})e(\mathbf{i})$ *is equal to*

$$(2.2.4) \qquad \begin{cases} (y_r + y_{r+2} - 2y_{r+1})e(\mathbf{i}), & \text{if } i_{r+2} = i_r \rightleftarrows i_{r+1}, \\ -e(\mathbf{i}), & \text{if } i_{r+2} = i_r \to i_{r+1}, \\ e(\mathbf{i}), & \text{if } i_{r+2} = i_r \leftarrow i_{r+1}, \\ 0, & \text{otherwise,} \end{cases}$$

$\overset{\text{CL.d)}}{=} 0$, the ψ_r's satisfy the braid relations.

Furthermore, the relation $\psi_r e(\mathbf{i}) = e(s_r \cdot \mathbf{i}) \psi_r$, allows us to write every product as $\psi_u e(\underline{i}) = \psi_u e(\underline{i}^{\lambda}) \psi_{d(\underline{t})}$ for $u \in S_n$.

Next, note that the elements of the form $e_{\underline{s},\underline{t}} = \psi_{d^{-1}(\underline{s})}\,(e(\underline{i}^{\lambda}))\,\psi_{d(\underline{t})}$ already span, since

$$\psi_u e(\underline{i}^{\lambda})\psi_w = \sum_{\underline{j}\in I^n} e(\underline{j})\psi_u e(\underline{i}^{\lambda})\psi_w = \sum_{\underline{j}\in I^n_\Lambda} \underbrace{\delta_{u\cdot\underline{j},\,\underline{i}^{\lambda}}}_{1 \text{ iff } u = d^{-1}(\underline{t}^{\lambda})} e(\underline{i}^{\lambda})\psi_w$$

In particular, $\mathrm{rk}_{\mathcal{Z}} R^{\Lambda}_n \leq \sum_{\lambda\in\mathcal{P}^{\Lambda}_n} |\mathrm{Std}(\lambda)|^2 \underset{\text{combinatorial fact}}{=} \ell^n n!$

On the other hand, let $K = \overline{\mathrm{Frac}(\mathcal{Z})}$, and write $R^{\Lambda}_n(K) = R^{\Lambda}_n \otimes_{\mathcal{Z}} K$. Let $J(R^{\Lambda}_n(K))$ be the Jacobson radical. By the Proposition, we constructed Specht modules S^{λ}. These are pairwise nonisomorphic ungraded modules (look at the action of $e(\underline{i}^{\lambda})$) and so by Fitting $S^{\lambda} \cong S^{\mu}\langle d\rangle \Rightarrow \lambda = \mu$, $d = 0$. By Wedderburn,

$$\ell^n n! \geq \dim R^{n}_{\Lambda}(K)/J(R^n_\Lambda(K)) \geq \sum_{\lambda\in\mathcal{P}_n} (\dim S^{\lambda})^2 = \sum_{\lambda\in\mathcal{P}_n} |\mathrm{Std}(\lambda)|^2 = \ell^n n!$$

Therefore we have equalities, hence semisimplicity and the $e_{\underline{s},\underline{t}}$ form a basis over $\mathcal{Z}$!

Furthermore, $e_{\underline{s},\underline{t}}\cdot e_{\underline{u},\underline{w}} = \delta_{\underline{t},\underline{u}}\, e_{\underline{s},\underline{w}}$ so R^{Λ}_n is a sum of matrix rings of sizes $|\mathrm{Std}(\lambda)|$. Cellularity follows.

Regarding the grading, we need to show $\deg(e_{\underline{s},\underline{t}}) = 0$. This is easy: a typical element $\psi_r e(\underline{i})$, $\underline{i}\in I^{\Lambda}_n$ has $i_r \not\!\!- i_{r+1}$ (Combinatorial Lemma), so $\deg(\psi_r e(\underline{i})) = -c_{i_r, i_{r+1}} = 0$ ◻

To conclude, using the machinery of last time one shows:

1.6.7. **Theorem** (Hu-Mathas [57]). *Suppose that $\mathcal{Z} = K$ is a field and that $\mathscr{H}_n$ is content separated and that $\{ f_{\mathsf{st}} \mid (\mathsf{s},\mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}^{\Lambda}_n)\}$ is a seminormal basis of $\mathscr{H}_n$. Then $\{f_{\mathsf{st}}\}$ is a cellular basis of $\mathscr{H}_n$ and there exists a unique seminormal coefficient system $\boldsymbol{\alpha}$ such that*

$$f_{\mathsf{st}}T_r = \alpha_r(\mathsf{t})f_{\mathsf{sv}} + \frac{1+(v-v^{-1})c^{\mathcal{Z}}_{r+1}(\mathsf{t})}{c^{\mathcal{Z}}_{r+1}(\mathsf{t}) - c^{\mathcal{Z}}_{r}(\mathsf{t})} f_{\mathsf{st}},$$

where $\mathsf{v} = \mathsf{t}(r, r+1)$. Moreover, if $\mathsf{s}\in\mathrm{Std}(\boldsymbol{\lambda})$ then $F_{\mathsf{s}} = \frac{1}{\gamma_{\mathsf{s}}} f_{\mathsf{ss}}$ is a primitive idempotent and $\underline{S}^{\boldsymbol{\lambda}} \cong F_{\mathsf{s}}\mathscr{H}_n$ is irreducible for all $\boldsymbol{\lambda}\in\mathcal{P}^{\Lambda}_n$.

Corollary: The map $\Theta : R^{\Lambda}_n(\Gamma_e) \longrightarrow \mathcal{H}^{\Lambda}_n$ defined by

$$e(\underline{i}^{\underline{s}}) \mapsto F_{\underline{s}}$$

$$\psi_r e(\underline{i}^{\underline{s}}) \mapsto \frac{1}{\alpha_r(\underline{s})}\left(T_r - \frac{c^{\mathcal{Z}}_{r+1}(\underline{s}) - c^{\mathcal{Z}}_{r}(\underline{s})}{1 + (v-v^{-1})c^{\mathcal{Z}}_{r+1}(\underline{s})}\right) F_{\underline{s}}$$

is an isomorphism.

Proof: direct check.