KLR Algebras: Properties

The main source for this is [L].

1. $Z(R(\nu))$

We work with a fixed $\Gamma = (I, E_\Gamma)$ and $\nu = \sum \nu_i i \in \mathbb{N}[I]$, $|\nu| = m$

Recall that

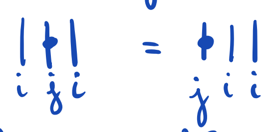
- $Seq(\nu)$ = sequences in I^m with $i \in I$ occurring ν_i times
- $R = R(\nu) = \bigoplus_{\vec{i}, \vec{j} \in Seq(\nu)} {}_{\vec{j}}R(\nu)_{\vec{i}}$ (top $= \vec{j}$, bottom $= \vec{i}$)

For $\vec{i} \in Seq(\nu)$ we have $Pol(\nu, \vec{i}) \subseteq {}_{\vec{i}}R_{\vec{i}}$ where $Pol(\nu, \vec{i}) \cong \mathbb{Z}[x_{1,\vec{i}}, \dots, x_{m,\vec{i}}]$ contains diagrams with dots but no crossings

(Recall that $x_{k,\vec{i}} =$ )

Let $Pol(\nu) = \prod_{\vec{i} \in Seq(\nu)} Pol(\nu, \vec{i})$.

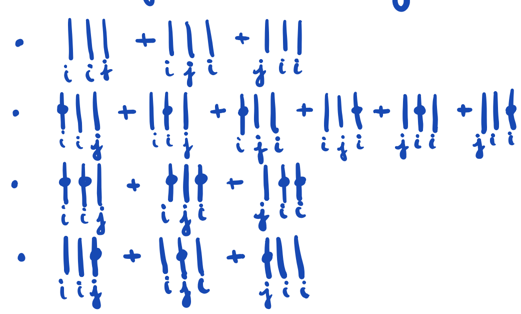
S_m acts on such diagrams by permuting strands

ex. $(1\,2)$ 

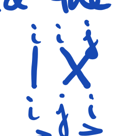
We define $Sym(\nu) := Pol(\nu)^{S_m}$

For example, for $\Gamma =$ , $\nu = 2i + j$

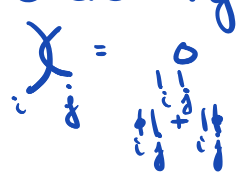
$Sym(\nu)$ is generated by

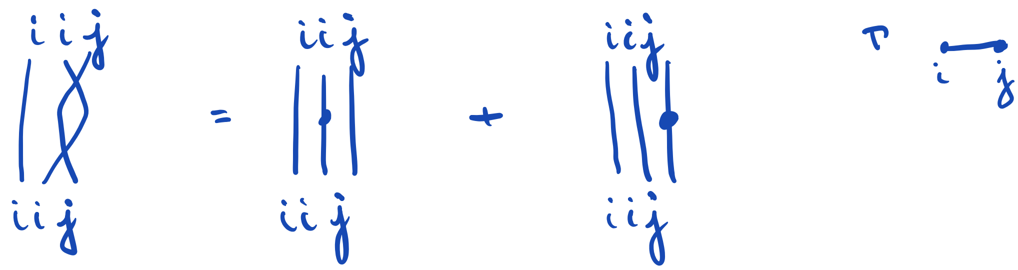


Let $\vec{i}, \vec{j} \in Seq(\nu)$. We define ${}_{\vec{j}}1_{\vec{i}}$ as the diagram D with $top(D) = \vec{j}$, $bot(D) = \vec{i}$, and the fewest possible crossings.

<u>ex</u>: ${}_{iij}1_{iji} =$ 

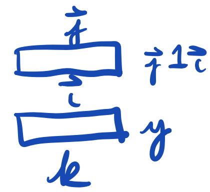
Notice that for $\vec{i}, \vec{j} \in Seq(\nu)$, ${}_{\vec{i}}1_{\vec{j}}\,{}_{\vec{j}}1_{\vec{i}} \in {}_{\vec{i}}R_{\vec{i}}$ is injective, as all bigons can be removed using relation 2.3, giving ${}_{\vec{i}}1_{\vec{j}}\,{}_{\vec{j}}1_{\vec{i}}$ as a linear combination of diagrams with no crossings, only dots

(Recall relation 2.3,  $i = j$; $i \cdot j = 0$; $i \cdot j = -1$)

ex: 

Since multiplication by $\vec{i}1_{\vec{j}}\vec{j}1_{\vec{i}}$ is injective, multiplication by $\vec{j}1_{\vec{i}}$ must be too, ie the map

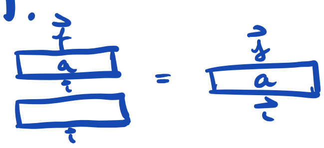
$$\vec{i}R_{\vec{k}} \longrightarrow \vec{j}R_{\vec{k}}$$
$$y \longmapsto \vec{j}1_{\vec{i}}\,y$$

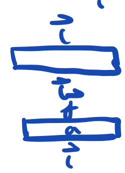


is injective.

Thm: $Z(R(\nu)) = Sym(\nu)$

Pf: Let $z \in Z(R(\nu))$.

Since $a1_{\vec{i}} = a$, 

and $1_{\vec{i}}a = 0$,  $= \begin{cases} 0 & \vec{j} \neq \vec{i} \\ a & \vec{j} = \vec{i} \end{cases}$

we must have $z \in \bigoplus_{\vec{i} \in Seq(\nu)} \vec{i}R_{\vec{i}}$

So, for $z_{\vec{i}} := 1_{\vec{i}}\,z = z1_{\vec{i}}$ we can write

$$z = \sum_{\vec{i} \in Seq(\nu)} z_{\vec{i}}$$

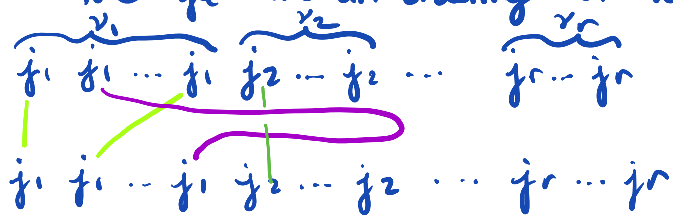
Consider $\vec{j} = j_1^{\nu_1} \cdots j_r^{\nu_r}$ where $j_1 \dots j_r$ is some ordering of vertices occurring in ν.

If $z_{\vec{j}} = 0$, then for $\vec{i} \in Seq(\nu)$

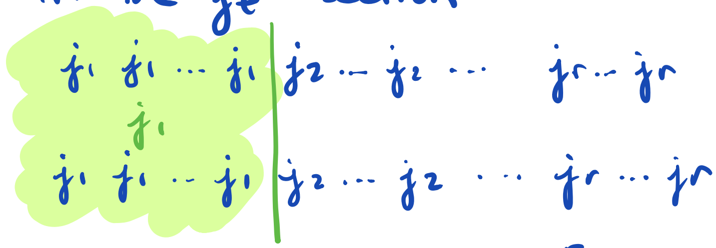
$$0 = z_{\vec{j}}(\vec{j}1_{\vec{i}}) = (z1_{\vec{j}})(\vec{j}1_{\vec{i}}) = z(\vec{j}1_{\vec{i}}) = \vec{j}1_{\vec{i}}\,z = \vec{j}1_{\vec{i}}\,z_{\vec{i}}$$

Since multiplication by $\vec{j}1_{\vec{i}}$ is injective, we must have $z_{\vec{i}} = 0\ \forall \vec{i}$, so $z = 0$.

Consider an element of ${}_{\vec{j}}R_{\vec{j}}$ where $\vec{j} = j_1^{\nu_1} \cdots j_r^{\nu_r}$ where the j_t are an ordering of vertices that occur in ν.



Any strand connected to a j_t must eventually return to the j_a section, meaning that any crossings over strands labelled j_s for $s \neq t$ are double crossings, and so can be removed via relation 2.3 until all strands labelled j_t are entirely in the j_t section



So, we have that ${}_{\vec{j}}R_{\vec{j}} \cong \bigotimes_{t=1}^{r} NH_{\nu_t}$.

Since the centre of NH_d is symmetric polynomials in d variables, we obtain

$$Z({}_{\vec{j}}R_{\vec{j}}) \cong Z\left(\bigotimes_{t=1}^{r} NH_{\nu_t}\right) \cong \bigotimes_{t=1}^{r} \mathbb{Z}[x_1, \dots, x_{\nu_t}]^{S_{\nu_t}}$$

Now, assume $z_{\vec{j}} \neq 0$ for $z \in Z(R(\nu))$, and let $c_{\vec{j}} := \bigoplus_{\tau \in seq(\nu)} w(\tau)(z_{\vec{j}}) \in Sym(\nu)$ where $w(\tau) \in S_m$ satisfies $w(\tau)(\vec{j}) = \vec{\tau}$. Then $z - c_{\vec{j}} \in Z(R(\nu))$ is such that $(z - c_{\vec{j}})_{\vec{j}} = 0 \in {}_{\vec{j}}R_{\vec{j}}$

By our previous result, this ${}_{\vec{j}}R_{\vec{j}}$ component being zero implies $z - c_{\vec{j}} = 0$, ie $z = c_{\vec{j}}$, so $Z(R(\nu)) \subset Sym(\nu)$. Since we know $Sym(\nu) \subset Z(R(\nu))$ we have equality.

□

We have $Sym(\nu) \subset Pol(\nu) \subset R(\nu)$, so $R(\nu)$ is finite free over $Sym(\nu)$.

- we assume from now on that all modules are left-graded and finitely generated

2. Bijection between Simples and Indecomposable Projectives

Any simple $R(\nu)$-module is finite dimensional. Let $Sym^+(\nu)$ be the unique maximal ideal of $Sym(\nu)$ and let S be a simple (graded) $R(\nu)$-module.

Prop: $Sym^+(\nu) \cdot S = 0$

Pf: Since S is simple $Sym^+(\nu) \cdot S$ is either S or 0. Since S is finite dimensional it has an element of some highest degree. $Sym^+(\nu)$ contains elements of positive degree, so $Sym^+(\nu) \cdot S$ will have elements of highest degree strictly larger than S, and so must be 0. $\square$

So, S is a graded module over $R(\nu)/Sym^+(\nu)R(\nu)$. There are finitely many simple modules in $R(\nu)$-mod, up to isomorphism and grading shift. From each such equivalence class we choose a representative S_b. Any simple $R(\nu)$ module is isomorphic to a unique $S_b\{a\}$. Each S_b has a unique indecomposable projective cover P_b, and any indecomposable object in $R(\nu)$-pmod is isomorphic to some (unique) $P_b\{a\}$.

Claim: There is a bijection between indecomposable projectives of $R(\nu)$ and simples of $R(\nu)$.

Pf: Let P, P' be indecomposable projectives such that

$$\begin{tikzcd} P \arrow[dr, "\pi_1"] \arrow[d, bend right, dashed, "f"'] & \\ P' \arrow[r, "\pi_2"'] \arrow[u, bend right, dashed, "g"'] & S \end{tikzcd}$$

Since they are projective, there exist maps $f: P \to P'$ and $g: P' \to P$ such that $\pi_2 \circ f = \pi_1$ and $\pi_1 \circ g = \pi_2$. Then $g \circ f \in End(P)$. A corollary to Fitting's lemma (see [1]) gives us that $g \circ f$ must be either nilpotent or invertible. If $(g \circ f)^n = 0$ for some n, we would have $\pi_1 = \pi_1 (g \circ f)^n = 0$, a contradiction. Therefore, $g \circ f$ is invertible. Since P, P' are indecomposable we have $P \cong P'$. Therefore if S has a projective indecomposable cover, it is unique.

Such a cover must exist since $R(\nu)$ is a finite dimensional algebra (see [2]). $\square$

3. The "Bar" Involution

For $P \in R(\nu)$-pmod (the category of projective objects) we define

$$\overline{P} := HOM(P, R(\nu))^{\psi} = \bigoplus_{a \in \mathbb{Z}} Hom(M, R\{a\})$$

(all graded $R(\nu)$-module morphisms).

Recall that $P_{\vec{i}} := \bigoplus_{\vec{j} \in \deg(\nu)} \vec{j} R(\nu) \vec{i}$ is a left graded projective.

<u>Claim</u>: $\overline{P_{\vec{i}}} = P_{\vec{i}}$

<u>Pf</u>:

$$\begin{aligned}
\overline{P_{\vec{i}}} &= HOM(P_{\vec{i}}, R(\nu))^{\psi} \\
&= HOM(R(\nu) 1_{\vec{i}}, R(\nu))^{\psi} \\
&= (1_{\vec{i}} R(\nu))^{\psi} \\
&= \Big(\bigoplus_{\vec{j} \in \deg(\nu)} \vec{i} R(\nu) \vec{j}\Big)^{\psi} \\
&= \bigoplus_{\vec{j} \in \deg(\nu)} \vec{j} R(\nu) \vec{i} \\
&= P_{\vec{i}}
\end{aligned}$$

$\square$

Let $K_0(R(\nu))$ be the Grothendieck group of $R(\nu)$-pmod, and let $G_0(R(\nu))$ be the Grothendieck group of $R(\nu)$-fmod (finite dimensional graded $R(\nu)$-modules).

We define a $\mathbb{Z}(q, q^{-1})$ bilinear pairing

$$(\ , \ ) : K_0(R(\nu)) \times G_0(R(\nu)) \longrightarrow \mathbb{Z}[q, q^{-1}]$$

by $(P, M) = \mathrm{gdim}_{\mathbb{K}}(P^{\psi} \otimes M)$

<u>Claim</u>: $([P_b], [S_a]) = \delta_{ba}$

"<u>Pf</u>": $([P_b], [S_a]) = Hom(P_b, S_a) = \delta_{ab}$

(see pg 14 of [3]) $\square$

Let $Seqd(\nu) = \{ i_1^{(n_1)} \ldots i_r^{(n_r)} \mid n_k \in \mathbb{N}, \sum_{a=1}^{r} n_a i_a = \nu \}$

<u>ex</u>: $Seqd(2i+j) = \{ iij, iji, jii, i^{(2)}j, ji^{(2)} \}$

For $\vec{i} \in Seqd(\nu)$, $e_{\vec{i}} := e_{i_1, n_1} \otimes \cdots \otimes e_{i_r, n_r}$ where the $e_{i,n}$ are minimal idempotents in the nilHecke rings.

Let $(\vec{i})! := [n_1]! \ldots [n_r]!$ (where $[m]! = \dfrac{q^m - q^{-m}}{q - q^{-1}}$)

and let $\hat{\vec{i}} = i_1 \dots i_1 i_2 \dots i_2 \dots i_r \dots i_r$. Note that $\hat{\vec{i}} = \vec{i}$ iff $\vec{i} \in \mathrm{Seq}(\nu)$. $\langle \vec{i} \rangle = \sum_{k=1}^{r} \frac{n_k(n_k-1)}{2}$

For $\vec{i} \in \mathrm{Seqd}(\nu)$ define $P_{\vec{i}} := R(\nu)\psi(1_{(\vec{i})})\{-\langle \vec{i} \rangle\}$ and ${}_{\vec{i}}P = 1_{\vec{i}} R(\nu)\{-\langle \vec{i} \rangle\}$ (left and right graded projectives, respectively). Then we have

$$P_{\hat{\vec{i}}} \cong P_{\vec{i}}^{(\vec{i})!} \quad \text{and} \quad {}_{\hat{\vec{i}}}P = {}_{\vec{i}}P^{(\vec{i})!}$$

<u>ex</u>: $P_{ii} \cong P_{i^{(2)}}^{[2]!} = P_{i^{(2)}}^{q+q^{-1}} = P_{i^{(2)}}\{1\} \oplus P_{i^{(2)}}\{-1\}$

4. Quantum Serre Relations

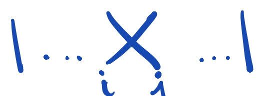
Let $\dots \vec{i}_0 \dots, \dots \vec{j}_0 \dots$ denote sequences that only differ at the positions of $\vec{i}_0, \vec{j}_0$

<u>Claim</u>: There are isomorphisms of graded projective right $R(\nu)$-modules

a) $\dots ij \dots P \cong \dots ji \dots P$ if $i \cdot j = 0$

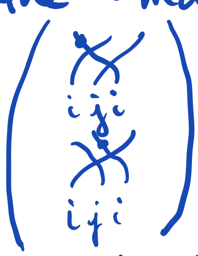
b) $\dots iji \dots P \cong \dots i^{(2)}j \dots \oplus \dots ji^{(2)} \dots P$ iff $i \cdot j = -1$

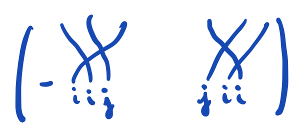
Note: There are similar equivalences for left modules. Both the statements and proof follow from the right case via ψ.

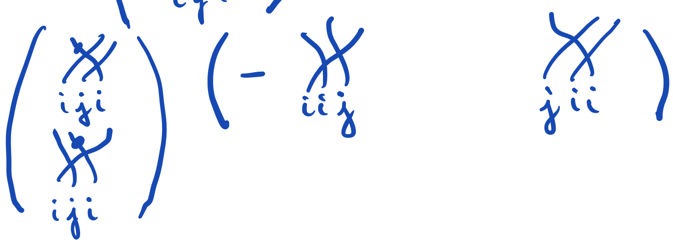
<u>Pf</u>: a) Multiplication by 

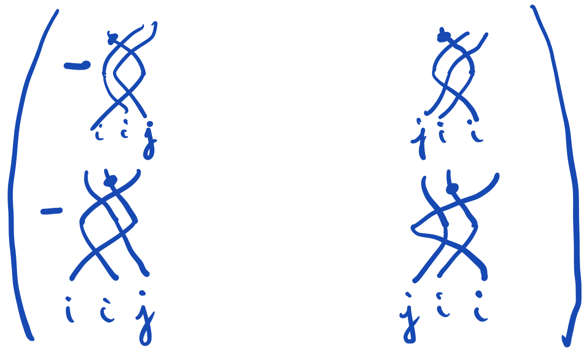
is a grading-preserving iso between $\dots ij \dots P$ and $\dots ji \dots P$ for $i \cdot j = 0$

b) Let $B_0 : \dots iji \dots P \to \dots i^{(2)}j \dots P \oplus \dots ji^{(2)} \dots P$
and $B_1 : \dots i^{(2)}j \dots P \oplus \dots ji^{(2)} \dots P$
be the grading-preserving maps given by the matrices

$B_0 =$ 

and $B_1 =$ 

Then $B_0 B_1 =$ 

$=$ 

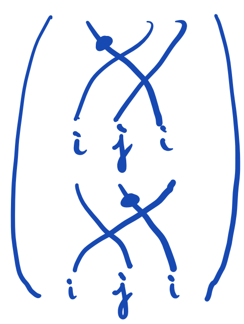
$\overset{(2.3)}{=}$ 

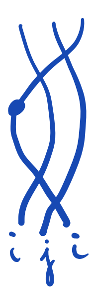
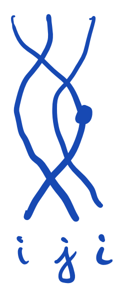
$\overset{(2.3,\ 2.5)}{=}$ 

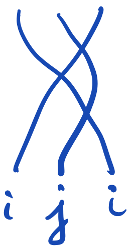
$$= \begin{pmatrix} 1_{i^{(2)}j} & 0 \\ 0 & 1_{ji^{(2)}} \end{pmatrix}$$

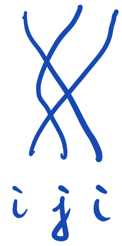
↳ $1_{(\tau)}$ is the identity on the projective ${}_{(\tau)}P$
so the last matrix is the identity matrix we need.

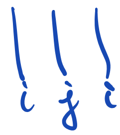
On the other side

$B_1 B_0 =$  

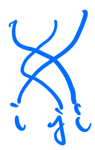
$= -$  $+$ 

$\overset{(2.3)}{=}$ − 

+ 

$\overset{(2.8)}{=}$ 

(where 2.8 in [1] is precisely



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so B_0, B_1 are isomorphisms. □

[1] M. Khovanov and A.D. Lauda. "A diagrammatic approach to categorification of quantum groups. I". In: *Represent. Theory* 13 (2009), pp. 309–347

[2] T. Leinster. "The bijection between projective indecomposable and simple modules". 2014. arXiv: 1410.3671v1.

[3] D.J. Benson. *Representations and Cohomology: (Basic Representation Theory of Finite Groups and Associative Algebras)*, Vol. 1. Cambridge University Press, 2010.