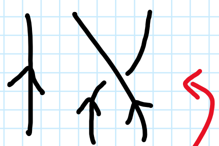
Motivation 1

Sunday, January 30, 2022 11:18 AM

RT-invariants ('88)

- Given $U_q(\mathfrak{g})$, $V \in U_q(\mathfrak{g})$-mod, T a tangle



$$V \otimes V \otimes V$$
$$\uparrow \quad f(T) \text{ linear map (intertwiner)}$$
$$V \otimes V \otimes V$$

R-matrix

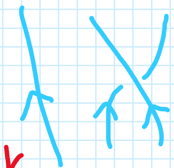
- When $T =$ link, $f(T): \mathbb{C}(q) \longrightarrow \mathbb{C}(q)$
  $\rightsquigarrow$ polynomial in $\mathbb{Z}[q, q^{-1}]$
- For $\mathfrak{g} = \mathfrak{sl}_2$, $V = V_1 = \mathbb{C}(q)^2 \rightsquigarrow$ Jones poly

Lusztig's Canonical Bases ('90)

- K_0(certain cat of perverse sheaves) $\overset{\text{bialg}}{\simeq} U_q^-(\mathfrak{g})$
  $L(\mathfrak{g})$
  $[\text{simple}] \longleftrightarrow$ canonical basis

Crane-Frenkel Categorification ('94)

- Use canonical basis to categorify RT-invariants to construct 4-manifold invariants



K_0 $\mathcal{C}_{V\otimes V\otimes V}$ - category

$\uparrow \quad \hat{f}(T)$ triangulated functor

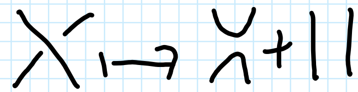
K_0 $\mathcal{C}_{V\otimes V\otimes V}$ - category

BFK - Category $\mathcal{O}$ ('99)

- For $\mathfrak{g} = \mathfrak{sl}_2$, $V = V_1$, they showed

$$K_0\left(\bigoplus_{k=0}^{n} \mathcal{O}_{\mu}^{k,n-k}(\mathfrak{gl}_n)\right) \simeq V_1^{\otimes n}\big|_{q=1}$$

(parabolic cat $\mathcal{O}$)

-  $\rightsquigarrow$ projective functors

$\rightsquigarrow \hat{f}(T): D^b(\mathcal{O}^n) \longrightarrow D^b(\mathcal{O}^m)$

- Conjecture: this gives tangle invariant
- Stroppel '05: Yes, + upgrade to $q \neq 1$

Motivation 2

Sunday, January 30, 2022 11:18 AM

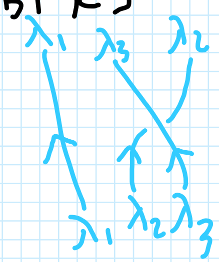
Khovanov homology

- forget Category $\mathcal{O}$, canonical bases for $\mathfrak{g}=\mathfrak{sl}_2$, $V=V_1$, $\exists$ combinatorial approach to categorifying Jones poly
- very computable v.s. Category $\mathcal{O}$ methods

Q: What about other RT-invariants?

A: Webster '17 constructs using KLRW-alg

- method is sort of a hybrid b/t Category $\mathcal{O}$ and combinatorics, follows outline of [BFK]



$$D^b(T^{\underline{\mu}}\text{-mod}) \uparrow D^b(T^{\underline{\lambda}}\text{-mod})$$

$T^{\underline{\lambda}}$ = KLRW algebra is defined using diagrams)

- To understand KLRW algs, understand KLR algebras first.

Q: What is a KLR algebra?

A: Categorification of quantum groups

i.e. $K^{\oplus}(\text{Proj of } KLR(\mathfrak{g})\text{-alg}) \overset{\text{bialg}}{\cong} \overline{U_q^-(\mathfrak{g})}$

- We saw this before with Lusztig's canonical basis, so what is difference?

$(\overset{\bullet}{\vec{L}}(\mathfrak{g}), \otimes)$	$(\overset{\bullet}{KLR}(\mathfrak{g})\text{-alg}, \otimes)$
- obj ✓	- obj ✓
- gen morph ?	- gen morph ✓
- relations ??	- relations ✓
- Cat of V_λ ???	- Cat of V_λ ✓

$\mathrm{Ext}(\bigoplus_{\text{simple}} S, \bigoplus_{\text{simple}} S) = $ KLR algebra

Outline Remarks 1

Sunday, January 30, 2022 11:18 AM

Rmrk:
monoidal functor out of $(\dot{KLR}(\mathfrak{g}), \otimes)$ = 2-rep or categorical action of $U_q(\mathfrak{g})$

( ) = ?, main ref is [KL09]

(1) One example of a KLR-algebra is nilHecke alg. [KL'09] use many well-known facts about nilHecke alg w/o proofs, but b/c this example is so important, I thought I would give outlines of how to prove these facts. For example, similar computations occur in computations of equivariant Borel-Moore homology of Affine Grassmanian and Affine Flag Varieties

(2) (b) and (c) (the parts w/ proj and simples) is pretty fundamental stuff when categorifying stuff. Also, here you get to see how much simpler working with diagrams is when proving (d), the analogous result for $\dot{\mathcal{U}}(\mathfrak{g})$ requires the deepest theorem in geo rep theory.

(3) If you never categorified something in your life, now's a good chance, as the argument is fairly clean.

Main reference is [Mat14], but skip around a lot

(4) Topic is so important that I decided to have a whole talk instead of the 1-page summary in [Mat14]

Outline Remarks 2

Sunday, January 30, 2022 11:18 AM

- A cellular alg = finite/weyl group analogue of a highest weight cat (aka cat $\mathcal{O}$)
- In short, gives you systematic way to construct irr rep / any field
- reference is a different book entirely

(5) Q: what is a cyclotomic Hecke alg?

A: Here is 1-def

$v_i \in \mathbb{C}(q)$

$$H_d^{cyc} = \frac{H_d^{aff}}{(x_1 - v_1) \cdots (x_1 - v_\ell)}$$

Lem: $\{\text{f.d rep of } H_d^{aff}\} = \{\text{rep of } H_d^{cyc}\}$

Pf: V f.d. $\Rightarrow x_1 \circlearrowright V$ has minimal poly

Rem $\ell = 1$, $v_1 = 1$, recover H_d^{fin}, finite Hecke alg

(much of the material in (5) is a gen of rep of H_d^{fin} at roots of unity)

(6) analogue of (5) but for KLR_d^{cyc}

(7) $H_d^{cyc} \simeq KLR_d^{cyc}$

- RHS is graded, so LHS inherits grading

A grading on LHS really is not obvious!

Ex: H_d^{fin}, naive approach = "just give gen $\{T_s\}_{s \in S}$ a grading"

Problem: $T_s^2 - (v + v^{-1}) T_s - 1 = 0$ ← has to be deg 0

$\Rightarrow \deg(T_s) = 0$ if relation is homogeneous

Outline Remarks 3

Sunday, January 30, 2022 11:18 AM

Applications: (1) Upgrade Ariki's Cat Theorem

Rough statement Let char $K = 0$

$[S_K^\lambda : D_K^\mu] \longleftrightarrow$ controlled by canonical basis of $U(\widehat{sl}_e)$

Specht, irr of H_d^{cyc}

BK – Cat theorem

$[S_K^\lambda : D_K^\mu]_q \longleftrightarrow$ controlled by canonical basis of $U_q(\widehat{sl}_e)$

Remark: Very similar to KL-conj

controlled by KL basis of $H_d^{fin}(\mathfrak{g}) \longleftrightarrow [M_\lambda : L_\mu]$

Verma, simple of $U(\mathfrak{g})$

Slogan: Decomp numbers are controlled by a canonical bases on opposite side of Schur-Weyl duality

(2) James Conjecture: Let char $F = p$

$[S_F^\lambda : D_F^\mu] = ??$

- Can factor problem

$$\begin{array}{ccc} S_K^\lambda & \xrightarrow{\text{mod } p} & =???\text{ of } D_F^\mu \\ \text{Ariki}\downarrow & & \uparrow \text{mod } p \\ H_d^{fin}\big|_{q=e^{2\pi i/p}} & \overset{BK}{\underset{\sim}{\longleftrightarrow}} & KLR(\widehat{sl}_p) \end{array}$$

-decomp/char 0

James conjecture = mod p map is trivial

Williamson '14: Nope

- aka mod rep theory is hard!