8. Alternative geometries: Poincare disk model

Topology

study spaces

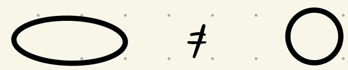
- Distances don't matter:

- Study things like:
  orientability,
  connectedness,
  holes ...

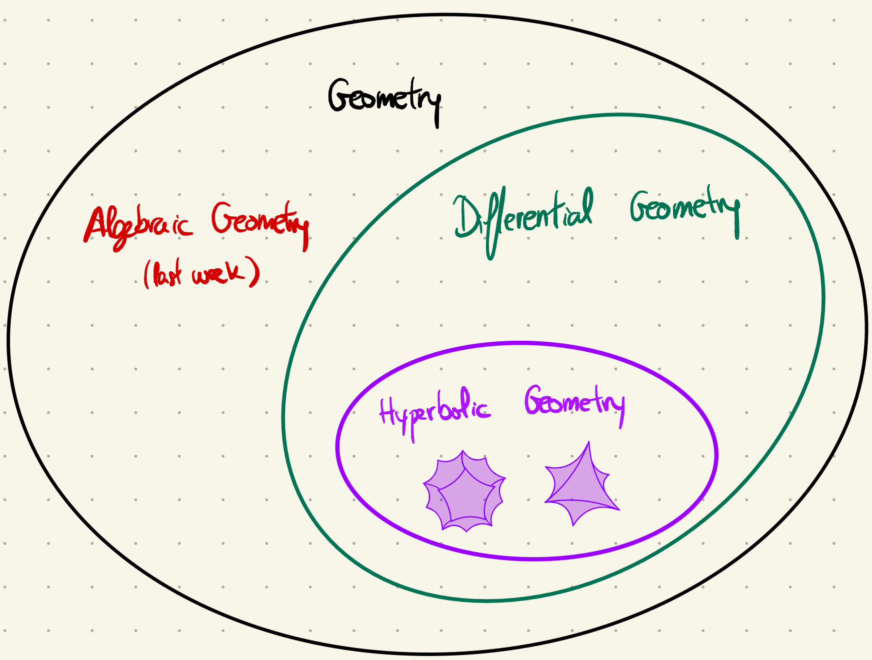
Differential Geometry

study spaces

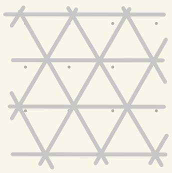
- Distances matter



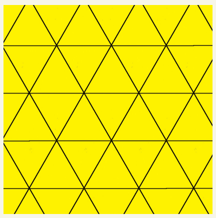
- Study things like:
  length, area, volume, angles
  curvature ...

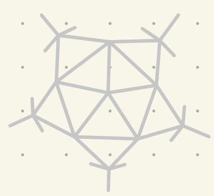


We will focus on 2D geometries. 2D space can be "curved locally" in three ways:

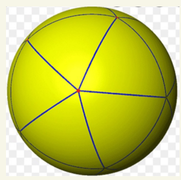


curvature = 0

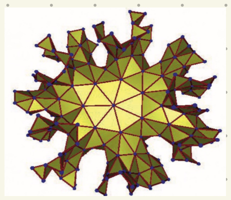




curvature > 0



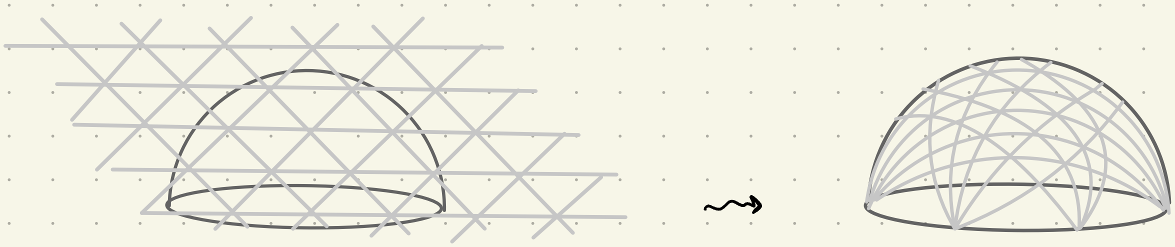
curvature < 0



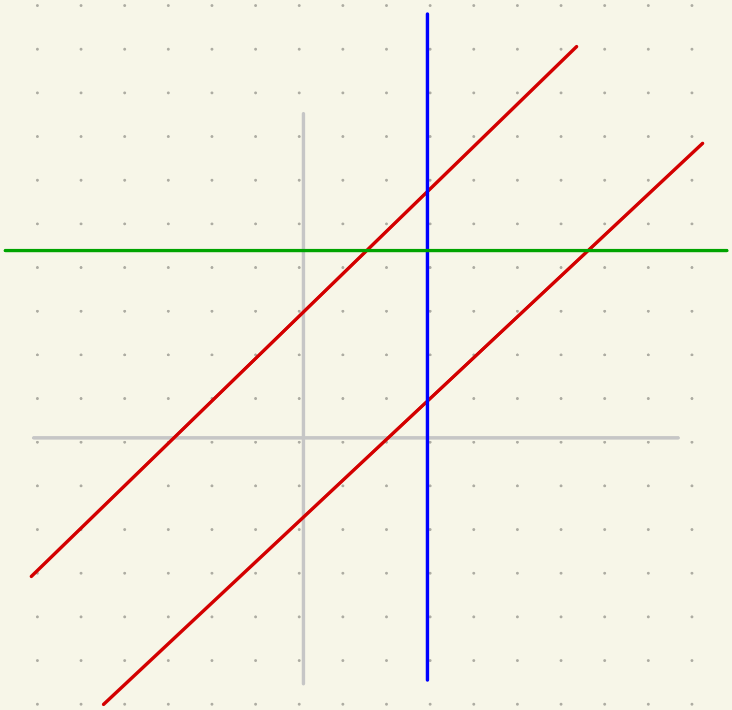
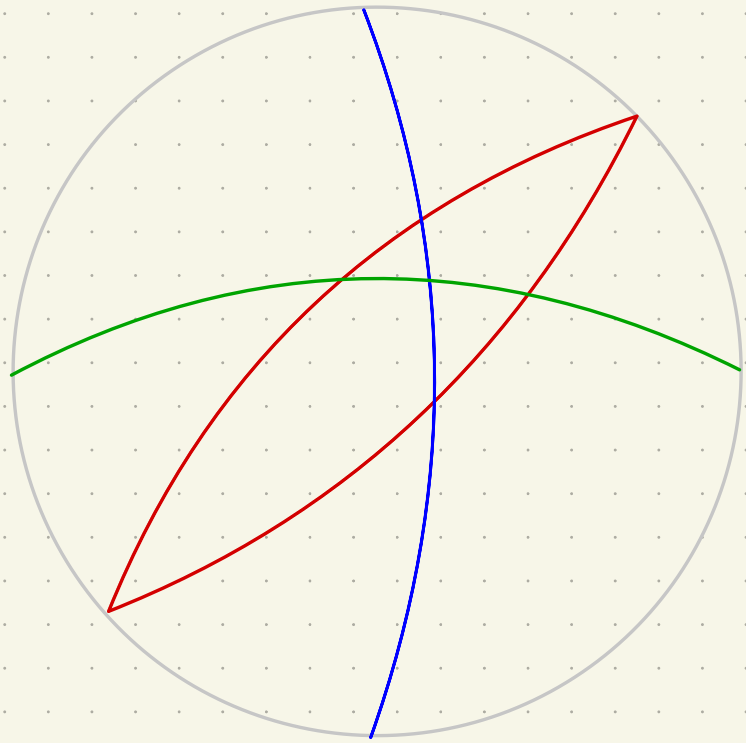
[Watch CodeParade video]

We will work in hyperbolic space with a projection onto the disk.

Let's see what this means for the curvature = 0 plane:



Disk model for the plane

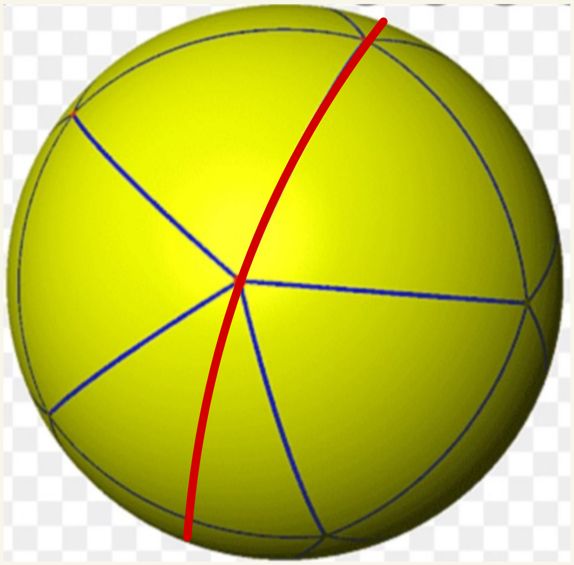
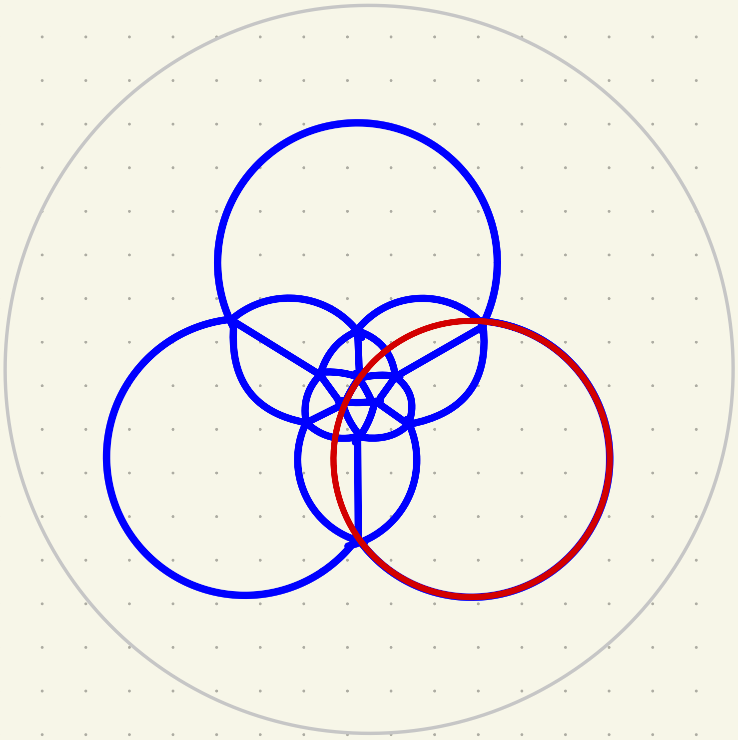
 

Straight line ⇝ Converges at infinity

(Like last week, this is really the <u>projective</u> plane)

Similarly, the plane with <u>positive curvature</u> has a disk model:

"spherical"

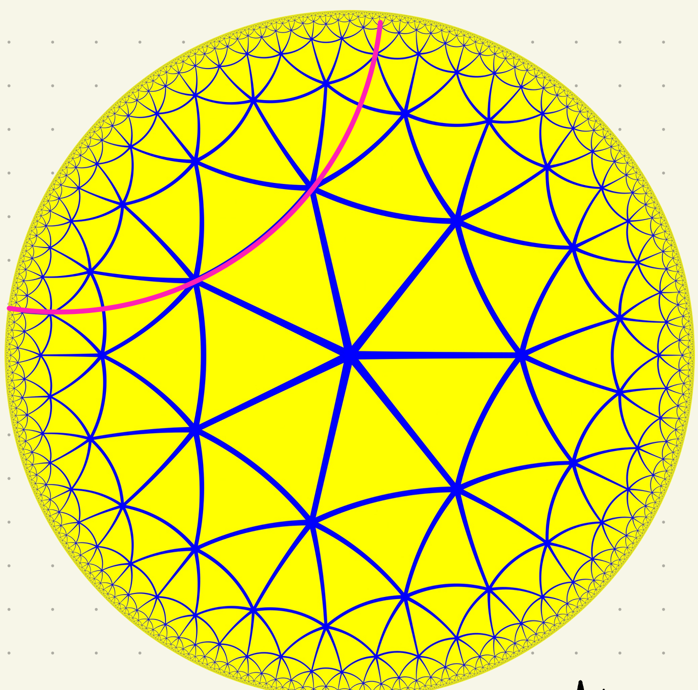
 ⇝ 

Straight line ⇝ Converges <u>inside</u> the disk

Similarly, the plane with negative curvature has a disk model:

"hyperbolic"

Hyperbolic plane

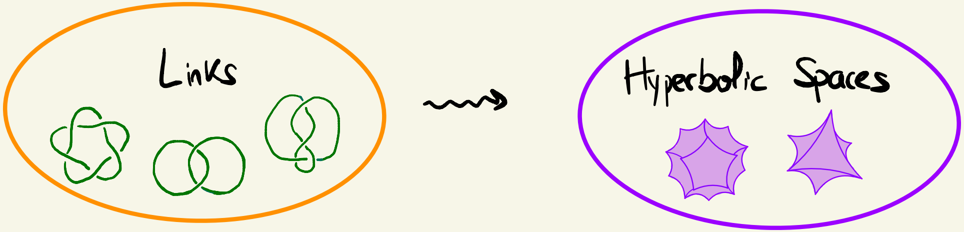


"Tesselation"

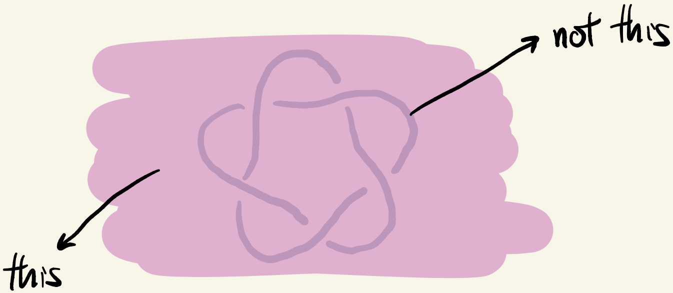
Straight line ⇝ Converges outside the disk

On Hyperbolic Knot Theory

Recall we want



Consider the <u>complement</u> of a Knot K:



Call it C_K for complement.

We say that C_K is hyperbolic if "it can be made to look like $\mathbb{H}^3$ locally".

Theorem (Thurston, 1978): All knots can be classified into:

- Torus knots
- Satellite knots

} Contrived, "easy"

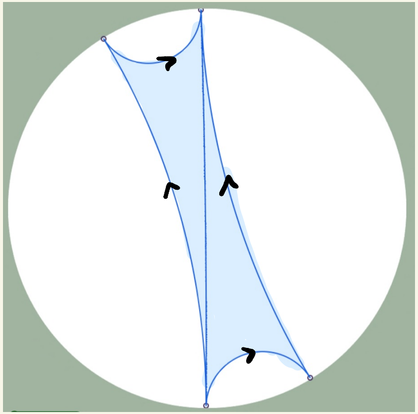
- Hyperbolic knots
  (knots with C_L hyperbolic)

} Most knots!

Moreover, the hyperbolic structure, if it exists, it is unique.

How to build hyperbolic 2D spaces using ideal polygons:

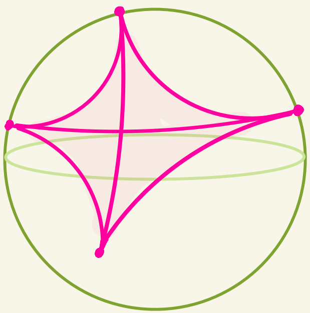
Identify "portals"



Hyperbolic torus

A polygon is "ideal" if its vertices lie in the ideal boundary.

How to build hyperbolic 3D spaces using ideal polyhedra:

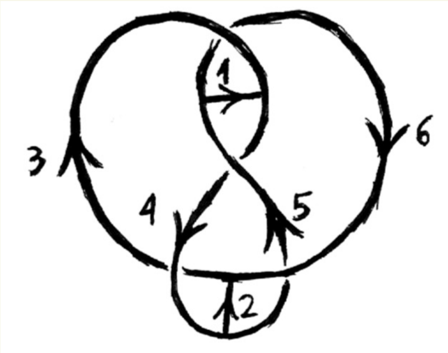


Hyperbolic 3D-space is constructed in a similar way to the 2D version: lines are given by circular arcs.

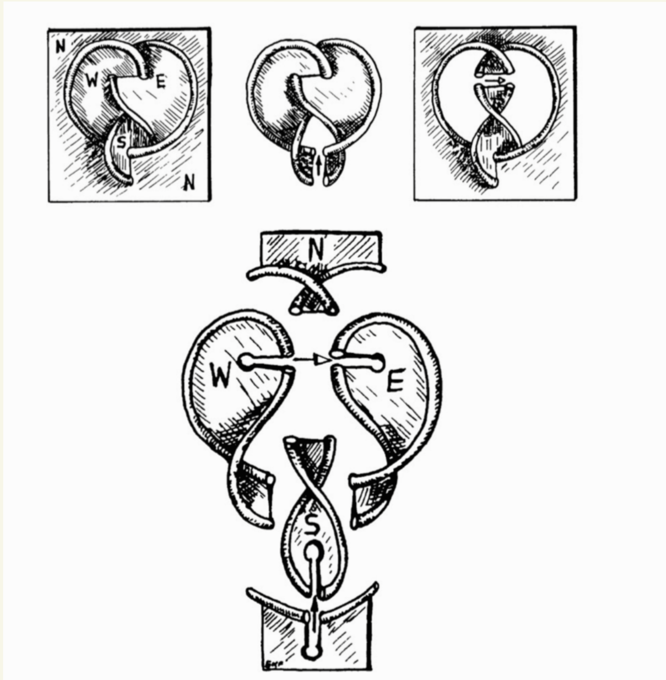
↓

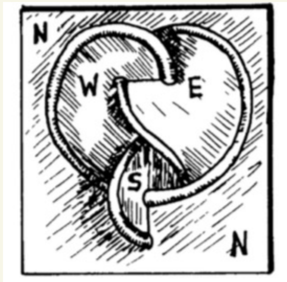
Here we can identify <u>faces</u> to form a closed 3D space.

The complement of the figure eight knot is hyperbolic.

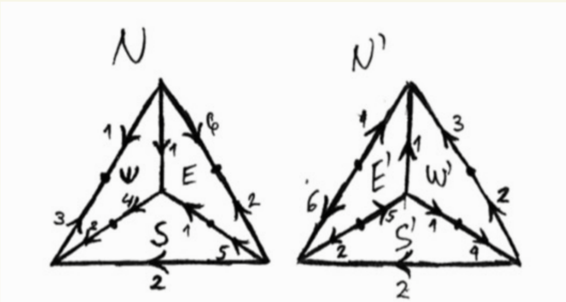


Introduce 2D patches bounding the complement

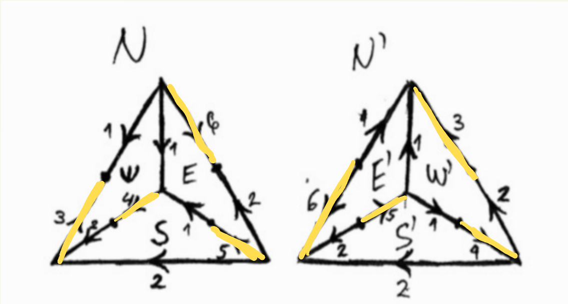


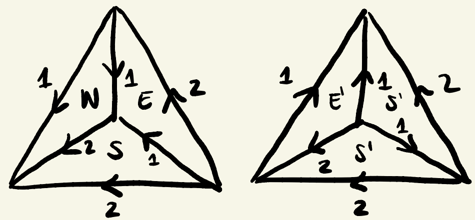
These patches divide 3D-space into two:  (front and back).

So we can "fill in" 2 tetrahedra (front and back) with faces N, S, E, W
N', S', E', W'

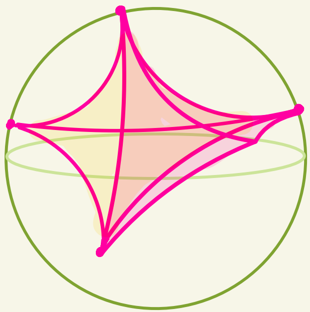
and identifications: 

Finally, shrink the edges coming from the knot (3, 4, 5, 6) :





We finally obtain the complement of the knot as the "Dirichlet domain" :



Facets and edges are identified accordingly.

Question : with these "portals" in place, how would this space look like from the inside ? (See exercises)

Exercises: Hyperbolic Geometry

These exercises must be carried out using NonEuclid (link on the website). Do as many as you can.

For each exercise, select "Clear All" under "Select Measurement or Modification"

1. Lines

(a) Two lines are parallel if they don't intersect inside the disk. Construct three lines (using "Line")

Call them l_1, l_2, l_3. They must satisfy:

- l_1, l_2 intersect at a single point.
- l_3 is parallel to both l_1 and l_2.

(b) Construct two lines (using "Line"). Construct their intersection point (using "Intersection point" and its instructions.)

Next, measure the 4 angles at the point of intersection (see the instructions under "Measure angle")

Finally, move the points you used to define the lines in order to make the 4 angles $90°$.

2. Perpendicular bisector.

(a) Draw a line segment, with endpoints A and B.

(b) Draw a circle with center A and passing through B.
Draw a circle with center B and passing through A.

(c) Draw a line through the intersection points of the 2 circles.

(d) Check that the line and segment are perpendicular (measure the angle).

(More exercises on the next page).

3. Triangles

(a) Construct a triangle (using line segments) with angles adding up to $100°$.

Then one with angles adding up to $<05°$.

(b) Construct a triangle and draw perpendicular bisectors to each side (Feel free to hide the auxiliary circles). By moving the points around, convince yourselves that the three bisectors intersect in a single point. Finally find the circle that passes through the vertices of the triangle.

(c) Construct a triangle $\widehat{ABC}$. Choose a point D on the segment BC. Finally measure the triangles ("Measure triangles") $\widehat{ABC}$, $\widehat{ABD}$, $\widehat{ADC}$. Let $\mathcal{A}(\text{triangle}) = 180° - \text{angle sum}$.

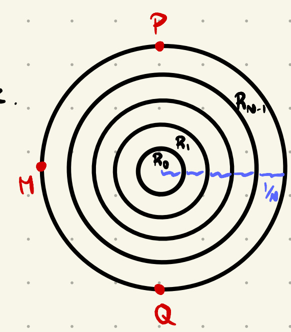
Check that $\mathcal{A}(\widehat{ABC}) = \mathcal{A}(\widehat{ABD}) + \mathcal{A}(\widehat{ADC})$. Call $\mathcal{A}$ the "area" of the triangle.

4. Areas

(a) Draw two triangles with equal side lengths (in different parts of the disk). Check that their angles are equal. Is this true in Euclidean geometry?

(b) Draw two triangles with equal angles (in different parts of the disk). Check that their side lengths are equal. Is this true in Euclidean geometry?

(c) Conclude from this and 3.c) that the notion of area we defined is reasonable.



5. (Harder) Towards differential geometry.

Consider the unit disk $\{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$ divided into annular regions $R_0, \dots, R_{N-1}$ each of width $\frac{1}{N}$.

Define the norm $\|(x,y)\| = \sqrt{x^2+y^2}$. Define a distance on each R_k given by $d(p,q) = \dfrac{\|p-q\|}{1-\left(\frac{k}{N}\right)^2}$.

Sketch what you imagine is the shortest path between P and Q. Do the same for P and M.

How would you formalize this setup "as $N \to \infty$". For instance, how would you compute the length of a path?

(Extra) Hyperbolic Knot Theory

1. Go over to Hyperrogue to experience the hyperbolic plane "from a local perspective".
2. Go over to Hypernom to experience hyperbolic 3D space "from a local perspective".

   (Use the arrow keys + WASD)

3. Open SnapPy (install it if you haven't yet) and input: M = Manifold()

The link editor will pop up. Draw any link and click on Tools > Send to SnapPy.

Verify that your link is hyperbolic by computing its volume via M.volume().

(If it isn't, it will return 0). Then explore the hyperbolic geometry of the complement of your link using

- M.dirichlet_domain().view()
- M.inside_view() (Use the arrow keys + WASD)