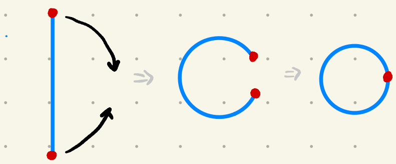
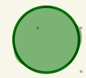
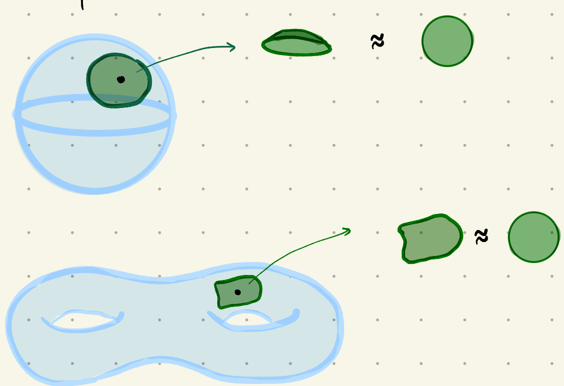
4. Equivalence relations: how to glue in math

We have been talking about gluing spaces such as

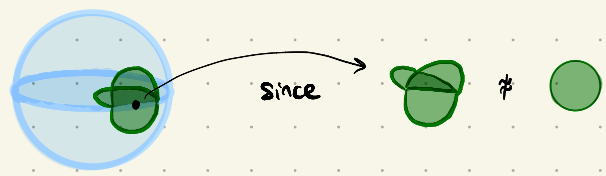


Recall: a surface without boundary is a subset of 3D space such that arand every point there is a "disk": a copy of $B^2 = \{x^2+y^2 \le 1\}$ = 

Examples:

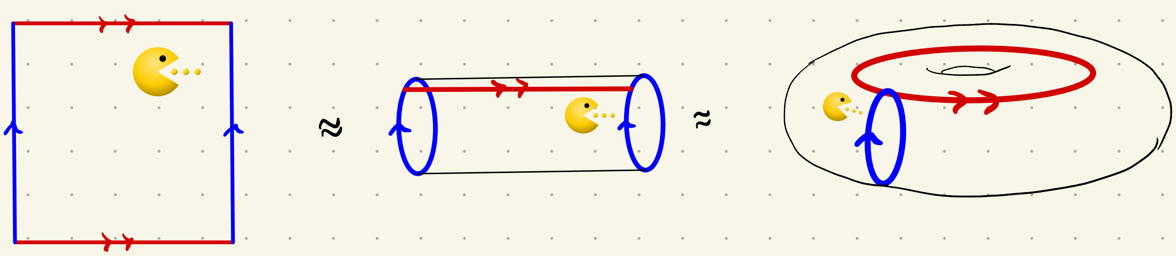


Nonexamples:



Goal: We want to "glue" surfaces together

For instance, take the square with "identified sides"



Question: how does one make this mathematically rigorous?

The language of sets

A set is a collection of objects without order or repetitions.

Examples: $\{1,2,3\}$, $\mathbb{R}$, $\{\text{knots in } \mathbb{R}^3\}$, ...

↳ finite ↳ Infinite

- Element in a set: $1 \in \{1,2,3\}$ means "1 is an element of the set $\{1,2,3\}$"

  ↳ Negation: elephant $\notin \{1,2,3\}$

- Equality: two sets are equal iff they have the same elements.

  $$\{1,2,3\} = \{2,1,3,2\}$$

  ↳ repetitions are "ignored"

- Empty set: $\emptyset = \{\ \}$

- Subsets: $A \subseteq B$ A is a subset of A if every element in A is in B

  $$\{1,2\} \subseteq \{1,2,3\} \quad \text{but} \quad \{1,7\} \not\subseteq \{1,2,3\}$$

- Union: $A \cup B = \{\text{elements in } A \text{ or in } B\}$

  $$\{1,2\} \cup \{2,3,4\} = \{1,2,3,4\}$$

- Intersection: $A \cap B = \{\text{elements in } A \text{ and } B\}$

  $$\{1,2\} \cap \{2,3,4\} = \{2\}$$

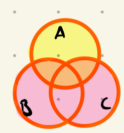
- Set defined by condition: $\{x \in \mathbb{R} \mid x > 0\} = \{\text{positive real numbers}\}$

  $$\{n \in \mathbb{Z} \mid n \text{ is even}\} = \{0, \pm 2, \pm 4, \dots\}$$

- Product of sets: $A \times B = \{(a,b) \mid a \in A, b \in B\}$. Example: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

Remark: A set may contain other sets, for instance $\{1, \{1,2\}\}$ is a valid set.

Question: prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$



Equivalence relations

Very often we want to say two elements in math are not equal, but "equivalent" in some sense.

Examples:

- Days of the week: March 25 $\neq$ March 18 but they were both Saturdays
- Two spaces are homeomorphic but not equal:

- Two sets which have the same size but are not equal.

Definition: a relation $\sim$ on a set X is a subset of $X \times X$. It is an equivalence relation if, additionally,

- (Reflexivity): For all $x \in X$, $x \sim x$
- (Symmetry): For all $x, y \in X$, $x \sim y \Leftrightarrow y \sim x$.
- (Transitivity): For all $x, y, z \in X$, $x \sim y$, $y \sim z \Rightarrow x \sim z$.

Example: the relation on $\mathbb{Z}$ given by $x \sim y \Leftrightarrow x-y$ is a multiple of 3 is an equivalence relation.

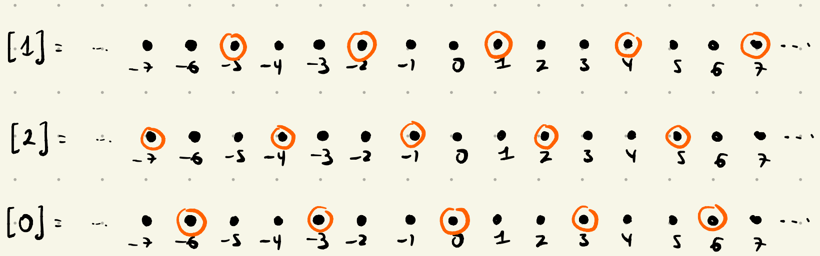
- Reflexivity: $x - x = 0 = 3 \cdot 0$, so $x \sim x$. ✓
- Symmetry: $x \sim y \Rightarrow x - y = 3 \cdot n \Rightarrow y - x = 3 \cdot (-n) \Rightarrow y \sim x$ ✓
- Transitivity: $x - y = 3 \cdot n_1$, $y - z = 3 \cdot n_2 \Rightarrow x - z = x - y + y - z = 3 \cdot (n_1 + n_2) \Rightarrow x \sim z$ ✓

Equivalence classes

Given an equivalence relation, the equivalence class of an element $x \in X$ is the set $[x] = \{ z \in X \quad z \sim x \}$

Example: using $\sim$ as above, $[1] = \{ n \in \mathbb{Z} \mid n - 1 \text{ is a multiple of } 3 \}$

Picture:



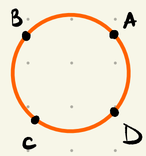
The set of equivalence classes forms the quotient by the relation $\sim$. In this case,

$$\mathbb{Z}/\sim \; = \{ \cdots, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, \cdots \}/\sim$$

$$\cong \{ [0], [1], [2] \}$$

Quotients allow us to glue !

Example: Take the circle X and choose points A, B, C, D on X

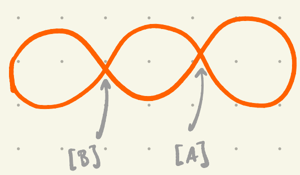


Define the equivalence relation on X: $\begin{cases} A \sim D \\ B \sim C \end{cases}$

Then the quotient $X/\sim$ is the set of equivalence classes:

- $[p]$ for $p \neq A, B, C, D$
- $[A] = \{A, D\} = [D]$
- $[B] = \{B, C\} = [C]$

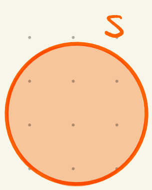
In other words, the quotient space is in bijection (and this bijection is a homeomorphism) with

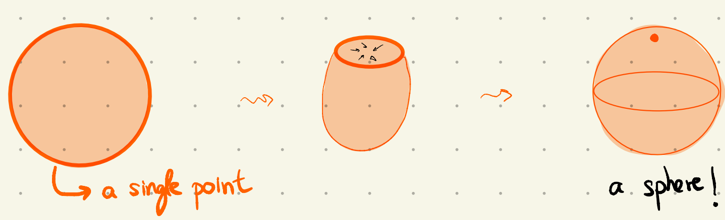


Example: Take X = unit disc $= \{ (x,y) \mid x^2+y^2 \leq 1 \}$

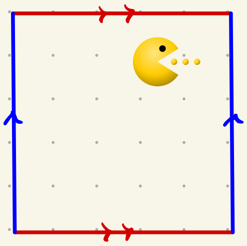
S = unit circle $= \{ (x,y) \mid x^2+y^2 = 1 \}$

Define the equivalence relation $\sim$ on X by: $\begin{cases} x \sim x & \text{for all } x \\ x \sim y & \text{for all } x, y \in S \end{cases}$ "glue all the points of S together"

$X =$ 

$X/\sim \; =$ 

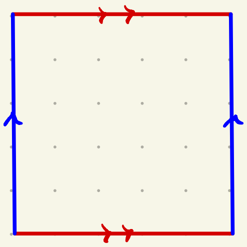
Finally, we can make sense of the pacman question:

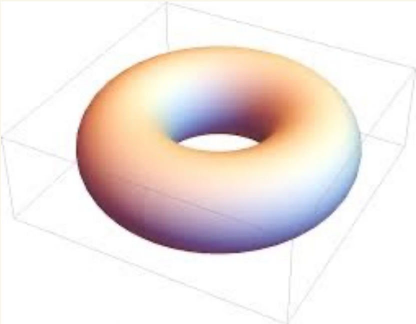


$$= \quad [0,1]\times[0,1]/\sim \quad , \text{ where } \quad \begin{array}{l}(x,0)\sim(x,1)\\(0,y)\sim(1,y)\end{array}$$

Furthermore, this is homeomorphic to the usual torus in $\mathbb{R}^3$, given by

$$[0,1]\times[0,1]/\sim \longrightarrow \mathbb{R}^3$$





$$(x,y) \longmapsto (\cos(x)\cos(y),\ \cos(x)\sin(y),\ \sin(x))$$