Group Axioms:

- Closure: for all $x,y \in G$ $x * y \in G$
- Associativity: for all $x,y,z \in G$ $(x*y)*z = x*(y*z)$
- Identity element: there is $e \in G$ s.t. $ex = x$ and $xe = x$ for all $x \in G$
- Inverses: for all $x \in G$ there is $x^{-1} \in G$ s.t. $xx^{-1} = e$ and $x^{-1}x = e$

13. Group Theory

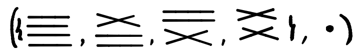
1. Decide whether the following sets and operations form groups:

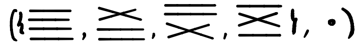
a) $(\mathbb{R}, +)$

b) $([-1,1], +)$

c) $((0,\infty), \cdot)$

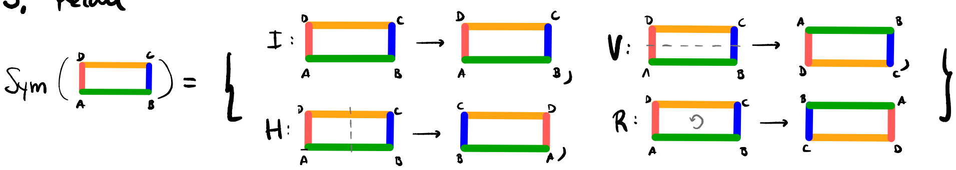
d) $([1,\infty), \cdot)$

e) 

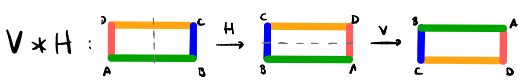
f) 

2. Let $C_2 \times C_2$ be the group with set $\{(0,0),(0,1),(1,0),(1,1)\}$ and operation given by addition modulo 2, so for example $(1,0)+(1,1) = (2,1) = (0,1)$. Write down the operation table and prove that $C_2 \times C_2$ is indeed a group.

3. Recall



with multiplication given by successive application of the maps, for instance



(Notice that we apply the transformations from right to left)

therefore $V * H = R$.

Write down the operation table for Sym(▭). Do you see any similarities with that in 2?

4. a) Let $G = \langle a \mid a^7 = 1 \rangle$. Find the 7 elements of G.

b) Let $G = \langle a, b \mid a^3 = 1,\ b^2 = 1,\ ba = a^2 b \rangle$. Prove that $abab = 1$, $(ab)^{-1} = ba^2$, and simplify ba^2b.

c) Let $G = \langle a, b \mid a^4 = 1,\ b^2 = 1,\ ba = a^3 b \rangle$. Find the 8 different elements in G.

5. How many elements does Sym(△) have? (These are the symmetries of an equilateral triangle)

Find a presentation of Sym(△) using 2 generators. You may check your answer using the following Sage code:

```
F.<a,b>=FreeGroup(2)
rels=[a^2,b^2,a*b*a]
G=F/rels
S=SymmetricGroup(3)
G.is_isomorphic(S)
```

(This returns false because the relations are wrong)

(You may use G.order() to find the number of elements of G)

6. (Harder): Find presentations for:

a) The group of symmetries of a regular polygon. (2 generators)

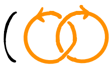
b) The group of permutations of $1, 2, \ldots, n$. ($n-1$ generators)

14. The knot group

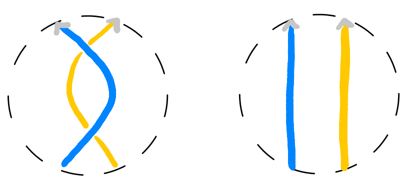
1. Compute the following knot groups, using the given diagrams.

   a) π()

   b) π()

   c) π()

2. Consider link diagrams D_1 and D_2 which only differ in the region inside the dashed circles below. Prove that both D_1 and D_2 will have equivalent relations:



3. (Challenge) Prove that $\pi(L)$ is infinite for any link L.

4. (Challenge) Find two knots K_1, K_2 that are different but such that $\pi(K_1) = \pi(K_2)$.