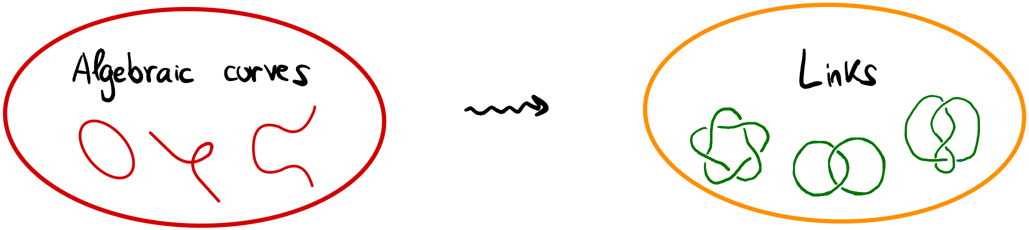
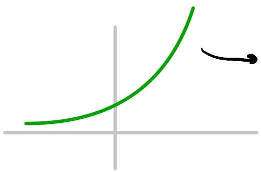
11. Algebraic Geometry



Differential Geometry

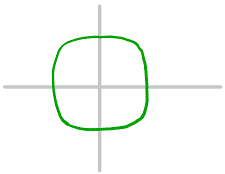
study spaces given by "differentiable" equations

Example: $y = e^x$



differentiable curve

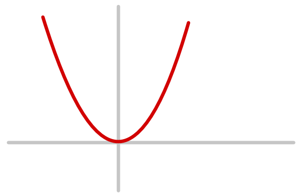
$e^{x^2} + e^{y^2} = 5$



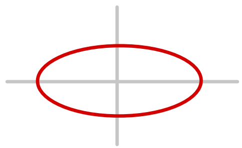
Algebraic Geometry

study spaces given by polynomial equations

$y = x^2$



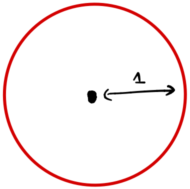
$x^2 + 3y^2 = 1$



Geometry

Polynomials

Algebraic Geometry



$$\longleftrightarrow$$

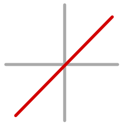
$$x^2 + y^2 - 1$$

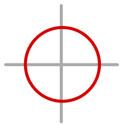
Algebraic curves

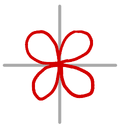
Definition: An algebraic curve consists of a set of points in the plane satisfying a polynomial equation:

$$C = \{(x,y) : p(x,y) = 0\}$$

Examples:

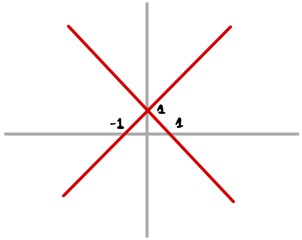
- Line $C = \{(x,y) : y - x = 0\}$ =  deg 1

- Circle $C = \{(x,y) : x^2+y^2-1 = 0\}$ =  deg 2

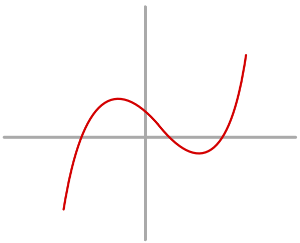
- Flower $C = \{(x,y) : (x^2+y^2)^3 - 4x^2y^2 = 0\}$ =  deg 6

Degree of a curve: degree of the polynomial

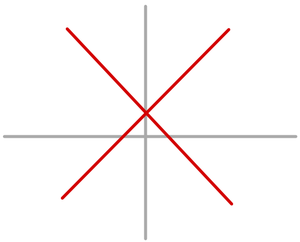
Question: can you find a polynomial $p(x,y)$ such that $C = \{(x,y) : p(x,y) = 0\}$ is the following?



Irreducible curve: a curve that does not break up into the union of "simpler" curves.



irreducible



reducible

Algebraic varieties

More generally, you can consider polynomials in more variables, e.g. 3, and their vanishing locus:

$$V(x-y,\ x^2+y^2+z^2-1) = \left\{ (x,y,z) : \begin{array}{l} x-y=0 \\ x^2+y^2+z^2-1=0 \end{array} \right\}$$

[GeoGebra]

Dimensions: "Generically" we have

$V(f) \subset \mathbb{R}^3$ has dimension 2 (surface)

$V(f,g) \subset \mathbb{R}^3$ has dimension 1 (curve)

$V(f,g,h) \subset \mathbb{R}^3$ has dimension 0 (set of points)

$V(f) \subset \mathbb{R}^2$ has dimension 1

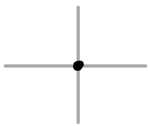
$V(f,g) \subset \mathbb{R}^2$ has dimension 0.

12. Singularities of complex algebraic curves

Undesirable property:

An algebraic curve doesn't always look like a curve!

$x^2+y^2=0$



Even worse: $x^2+y^2+1=0$??

Solution: use complex numbers! A complex number is one of the form $a+bi$ where $a,b \in \mathbb{R}$ and $i=\sqrt{-1}$.

Denote $\mathbb{C} = \{\text{complex numbers}\}$, $\mathbb{C}^2 = \{\text{pairs of complex numbers } (x,y)\}$

Definition: A complex algebraic curve is a subset of $\mathbb{C}^2$ of the form $C = \{(x,y) \in \mathbb{C}^2 : P(x,y) = 0\}$

Fact: Complex algebraic curves do look like curves always!

Caveat: they are harder to draw since $\mathbb{C}^2$ is a 4D space $\Rightarrow$ Draw "the real points"!

Singularities

Let P be a polynomial $P(x,y) = a_{00} + a_{10}x + a_{01}y + a_{11}xy + a_{12}xy^2 + a_{21}x^2y + \dots$

Define $\frac{\partial x^a y^b}{\partial x} = a x^{a-1} y^b$ $\quad \frac{\partial x^a y^b}{\partial y} = b y^{b-1} x^a$, $\quad \frac{\partial(\text{constant})}{\partial x} = 0 = \frac{\partial(\text{constant})}{\partial y}$

and $\frac{\partial P}{\partial x} = \frac{\partial a_{00}}{\partial x} + \frac{\partial(a_{10}x)}{\partial x} + \frac{\partial a_{01}y}{\partial x} + \quad a_{11}xy + a_{12}xy^2 + a_{21}x^2y + \dots$

Example: $P = 2x^3y^2 + 3x^7y^9 \Rightarrow \frac{\partial P}{\partial x} = 6x^2y^2 + 21x^6y^9$, $\quad \frac{\partial P}{\partial y} = 4x^3y + 27x^7y^8$

Definition: A singularity of an algebraic curve is a point (x_0, y_0) on $C = \{(x,y) : P(x,y) = 0\}$

such that $\frac{\partial P}{\partial x}(x_0,y_0) = 0 = \frac{\partial P}{\partial y}(x_0,y_0)$

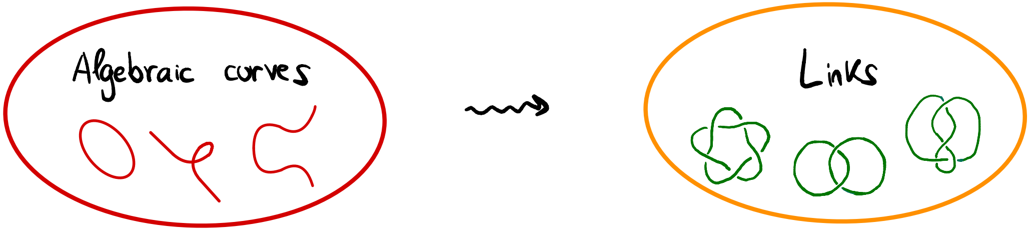
Example: Take $C = \{(x,y) : y^2 - x^3 + 3x - 2 = 0\}$. Then to find its singularities, we equate

$\frac{\partial P}{\partial x} = 0 \Rightarrow -3x^2 + 3 = 0 \Rightarrow x^2 = 1$ ↗ $x=1$ ↘ $x=-1$ $\quad (x_0,y_0) = (1,0)$ on C? $\quad 0^2 - 1^3 + 3\cdot 1 - 2 = 0$ Yes!

$\frac{\partial P}{\partial y} = 0 \Rightarrow 2y = 0 \Rightarrow y = 0$ $\quad (x_0,y_0) = (-1,0)$ on C? $\quad 0^2 + 1^3 - 3 - 2 = -4$ No!

[Desmos]

A knot from a singularity



Take $C = \{ y^2 - x^3 = 0 \}$

Singularities: $\begin{cases} -3x^2 = 0 \\ 2y = 0 \end{cases} \Rightarrow (x_0, y_0) = (0,0)$ on C ✓

Now write $x = a+bi$, $y = c+di$ and substitute:

$$y^2 - x^3 = (c+di)^2 - (a+bi)^3$$

$$= c^2 - d^2 + 2cdi - a^3 - 3a^2bi + 3ab^2 + b^3 i$$

$$= 0 \begin{cases} c^2 - d^2 - a^3 + 3ab^2 = 0 \\ 2cd - 3a^2b + b^3 = 0 \end{cases}$$

This gives a surface inside $\mathbb{R}^4$

let's take a slice of this surface near the singularity: let $S^3 = \{ a^2+b^2+c^2+d^2 = 1 \}$

Intersecting the two varieties, we get

$$\mathcal{U} = \begin{cases} c^2-d^2-a^3+3ab^2 = 0 \\ 2cd-3a^2b+b^3 = 0 \\ \underbrace{a^2+b^2+c^2+d^2 = 0}_{S^3} \end{cases}$$

dimension 1

Recall that $S^1 = \{x^2+y^2=1\} = \mathbb{R} + \text{a point}$

$S^2 = \{x^2+y^2+z^2=1\} = \mathbb{R} + \text{a point}$

$\rightsquigarrow S^3 = \mathbb{R}^3 + \text{a point}.$

Ignore the point and view $\mathcal{U}$ inside $\mathbb{R}^3$. This is a knot!

In fact $c^2-d^2-a^3+3ab^2=0$ defines a torus in $\mathbb{R}^3$

$2cd-3a^2b+b^3=0$ defines a different torus in $\mathbb{R}^3$

[See animations]