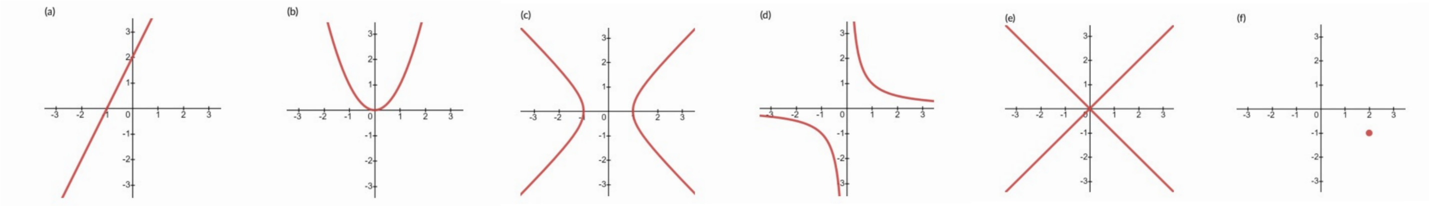
11. Algebraic geometry

1. Express the following subsets as algebraic curves in $\mathbb{R}^2$



2. Show how to factor the polynomial $x^3-x^2y+xy^2-y^3-x+y$ by examining the graph of the algebraic curve $V(x^3-x^2y+xy^2-y^3-x+y)$ on Desmos.

3. Find the intersection of the varieties $V((x-1)^2+y^2-1)$ and $V((x+1)^2+y^2-4)$. (Use Desmos to gain intuition, but find the two points algebraically).

4. (Silly) What is $V(f,g)$ in terms of $V(f)$ and $V(g)$? What are the varieties $V(0)$ and $V(1)$?

5. Describe the following varieties in $\mathbb{R}^3$. Then check your answers with GeoGebra.

   a) $V(x^2-y^2)$

   b) $V(x^2+y^2)$

   c) $V(x^2+y^2-z)$

   d) $V(xz, yz)$

6. Is $\{(n,0) : n \in \mathbb{Z}\} \subset \mathbb{R}^2$ a variety? Why or why not?

7. An ideal is a subset $I \subset \{\text{polynomials in } x, y\}$ which is closed under addition, subtraction and for any polynomial p and $f \in I$, $pf \in I$. The ideal generated by a set S is $(S) = \{p_1f_1 + p_2f_2 + \dots + p_ef_e : p_i \text{ are arbitrary polynomials}, f_j \text{ are polynomials in } I\}$

   - Prove that (S) is an ideal
   - Prove that $V(f_1, \dots, f_n) = V((\{f_1, \dots, f_n\}))$

8. If V and W are varieties, prove the following. (Feel free to use 7.)

   a) $V \cap W$ is a variety

   b) $V \cup W$ is a variety.

12. Singularities of complex algebraic curves

0. Simplify the following complex numbers. Feel free to double check in Sage.

   - $(1+i)^2$
   - $3+2i+i\cdot(5-2i)$
   - $(3+4i)(3-4i)$

1. Factor the following polynomials over $\mathbb{C}$:

   a) x^2+2x+2

   b) x^4-1

   c) x^2+y^2

2. Find the singularities of the following curves

   a) $\{3x^2y+2xy=0\}$

   b) $\{xy^2+x+y=0\}$

   c) $\{x^2+y^2-1=0\}$

3. Recall that complex curves are 2-dimensional. Consider the complex line $x+y=1$.

   a) Substitute $x=a+bi$ and $y=c+di$, and find the equations that a,b,c,d must satisfy so that (x,y) lies in the complex curve.

   b) Solve for d to obtain a single equation in a,b,c.

   c) Use the Sage code below to plot this surface in $\mathbb{R}^3$.

```
var('x y z')
implicit_plot3d(x^2+y^2+z^2==4, (x,-3,3), (y,-3,3), (z,-3,3))
```

4. Repeat what you did in 3 for the complex curve $x^2+y^2=1$.

5. (Harder) Prove that the set in 4 is a surface that can be deformed into a cylinder.