Reminder of what we've done so far:

a biased, not fully faithful geometric representation of mathematics

arrows indicate how to take the quotient

by axel sarlin and innokentij zotov

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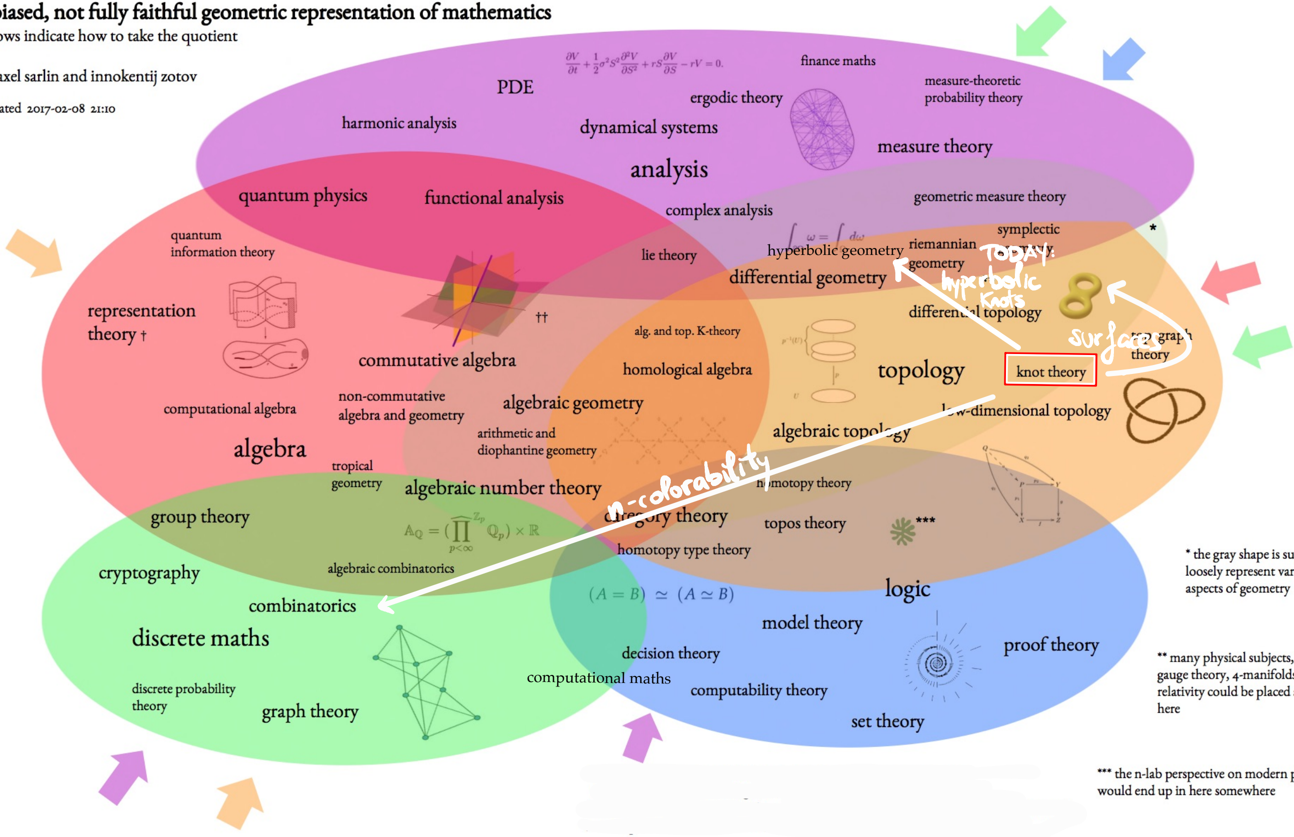


* the gray shape is supposed to loosely represent various aspects of geometry

** many physical subjects, such as gauge theory, 4-manifolds and relativity could be placed around here

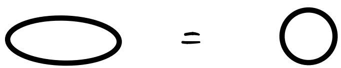
*** the n-lab perspective on modern physics would end up in here somewhere

Next two classes: we explore <u>geometry</u>

Topology

study spaces

- Distances don't matter:



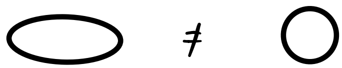
- Study things like:

  orientability,
  connectedness,
  holes ...

Differential Geometry

study spaces

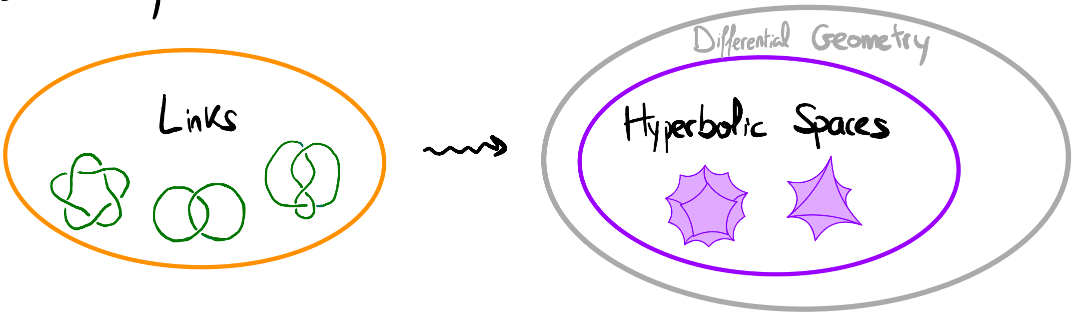
- Distances matter



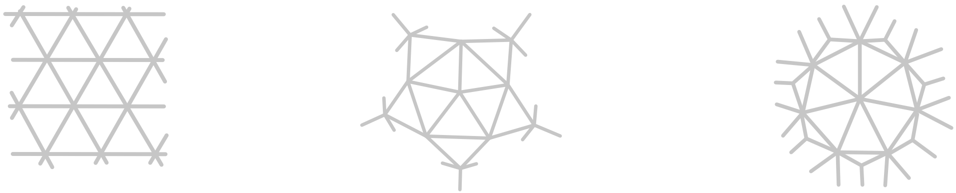
- Study things like:

  length, area, volume, angles
  curvature ...

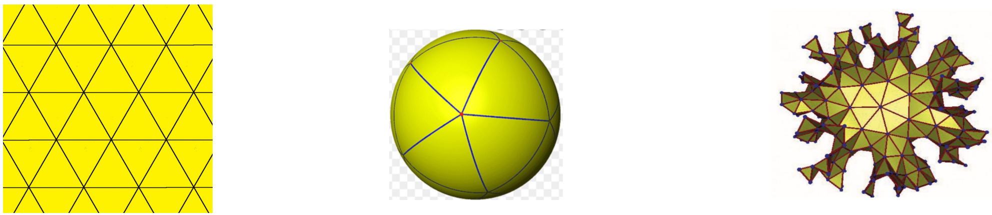
9. Hyperbolic Geometry



We first focus on 2D geometries. 2D space can be "curved locally" in three ways:



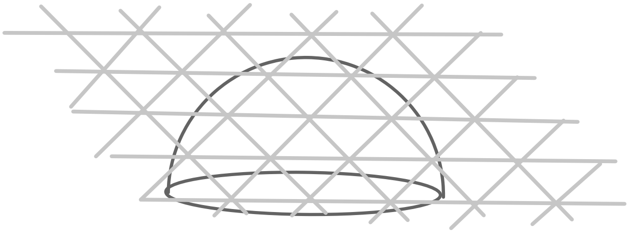
curvature = 0 | curvature > 0 | curvature < 0



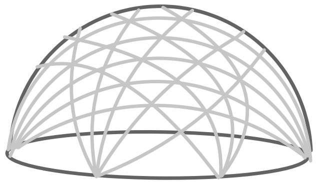
[Watch CodeParade video]

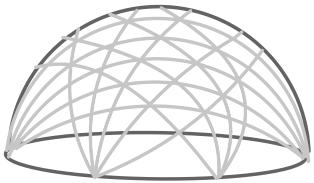
We will work in hyperbolic space with a projection onto the disk.

Let's see what this means for the curvature $= 0$ plane:

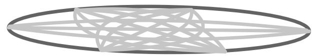


$\rightsquigarrow$

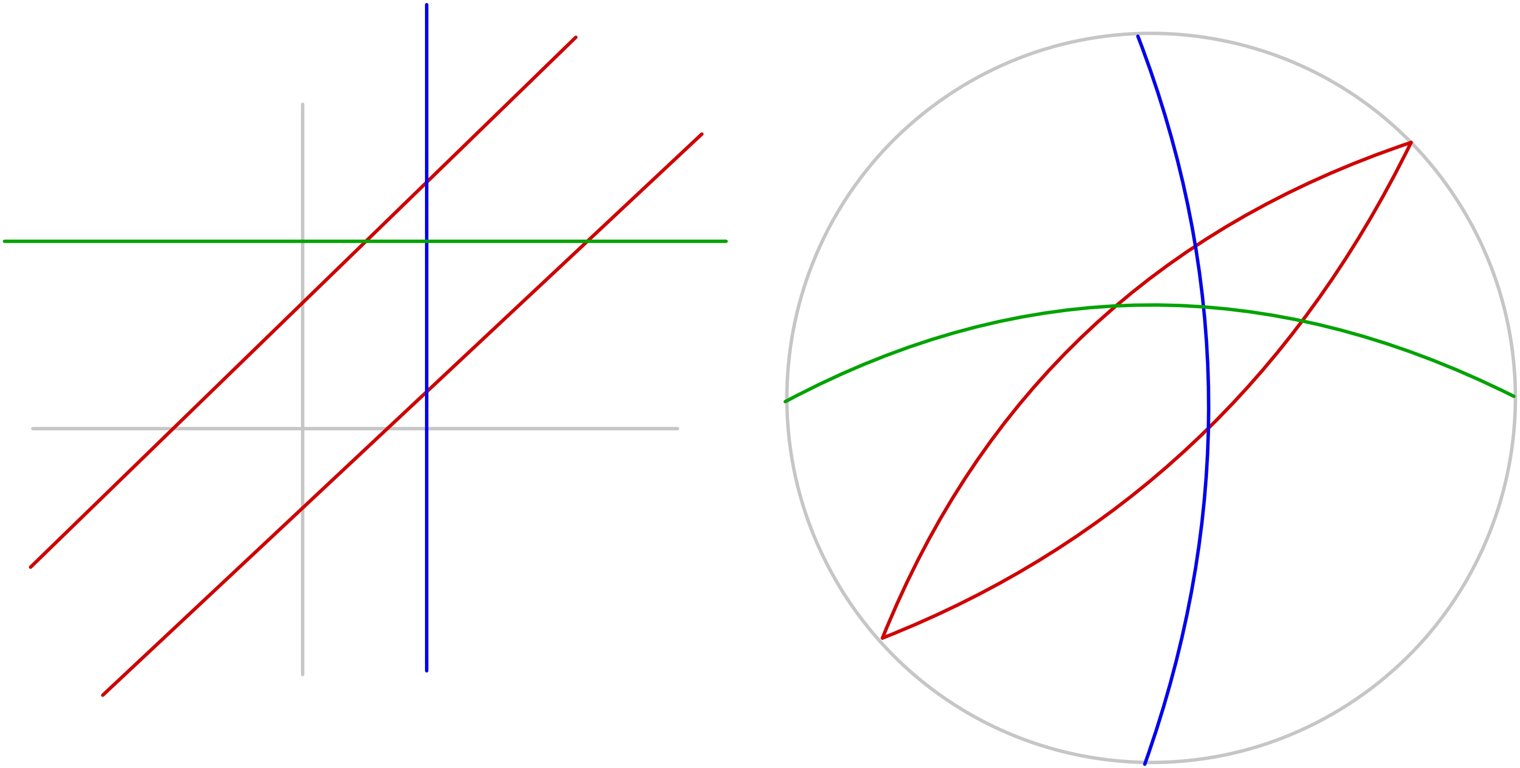




$\rightsquigarrow$



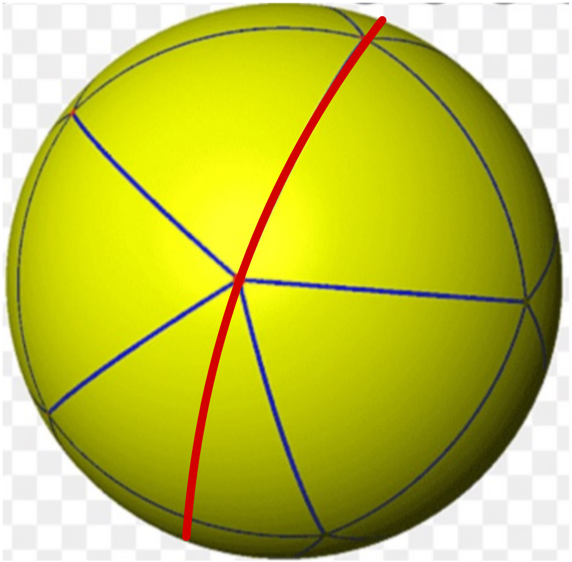
Disk model for the **plane**

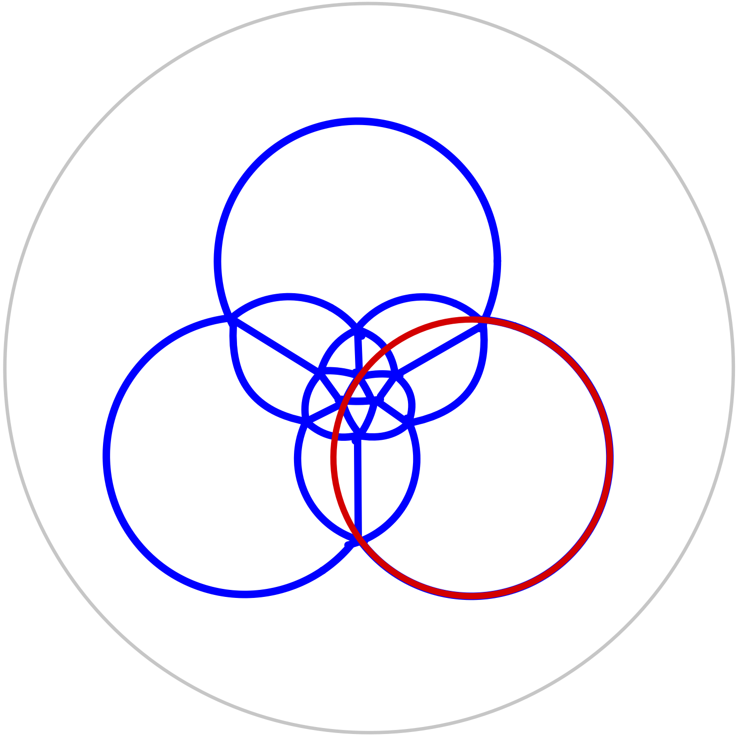


Straight line $\rightsquigarrow$ Converges at infinity

Similarly, the plane with positive curvature has a disk model:

"spherical"



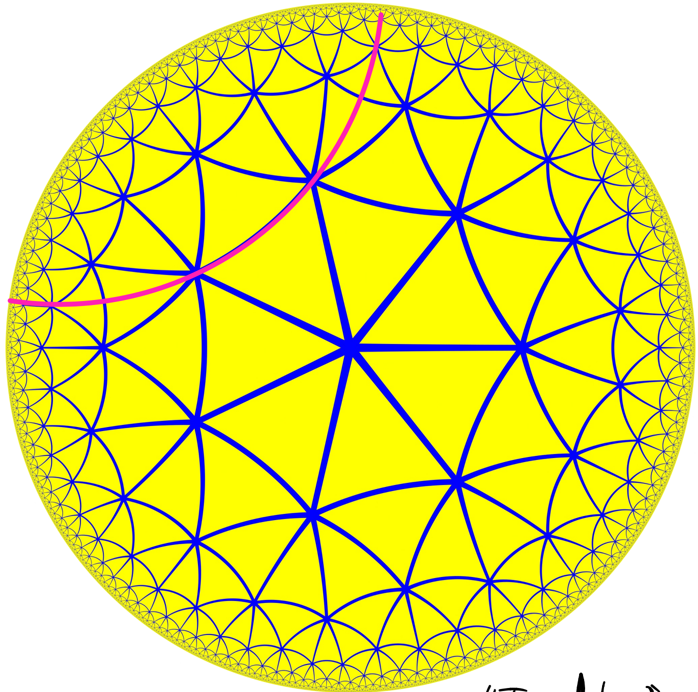


Straight line ⇝ Converges <u>inside</u> the disk

Similarly, the plane with negative curvature has a disk model:
"hyperbolic"

Hyperbolic plane

$\rightsquigarrow$



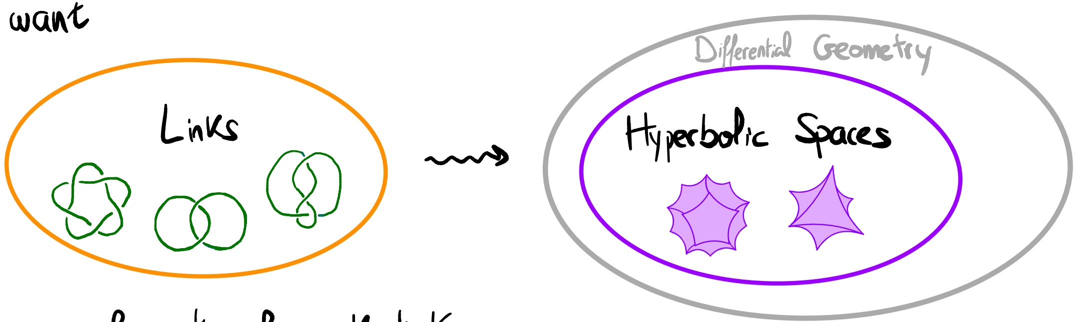
"Tesselation"

Straight line $\rightsquigarrow$ Converges outside the disk

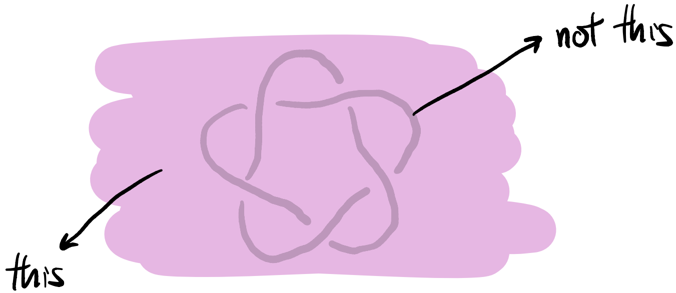
First task: go over to NonEuclid to explore the hyperbolic plane (see Exercises)

10. Hyperbolic Knot Theory

Recall we want



Consider the <u>complement</u> of a Knot K:



Call it C_K for complement.

We say that C_K is hyperbolic if "it can be made to look like $\mathbb{H}^3$ locally".

Theorem (Thurston, 1978): All knots can be classified into:

- Torus Knots
- Satellite Knots

} Contrived, "easy"

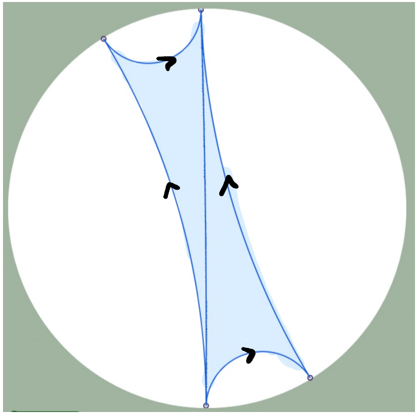
- Hyperbolic Knots (knots with C_L hyperbolic)

} Most knots!

Moreover, the hyperbolic structure, if it exists, it is unique.

How to build hyperbolic 2D spaces using ideal polygons:

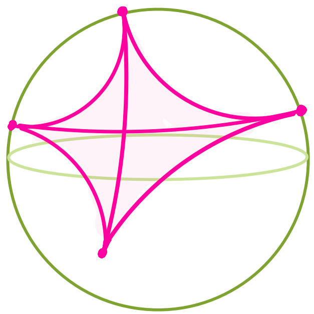
Identify "portals"



Hyperbolic torus minus a point.

A polygon is "ideal" if its vertices lie in the ideal boundary.

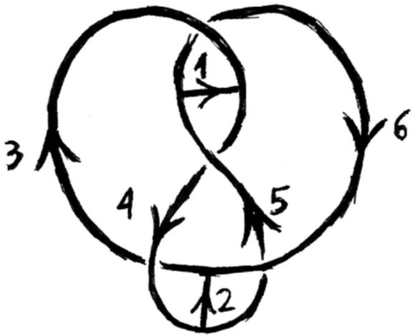
How to build hyperbolic 3D spaces using ideal polyhedra:



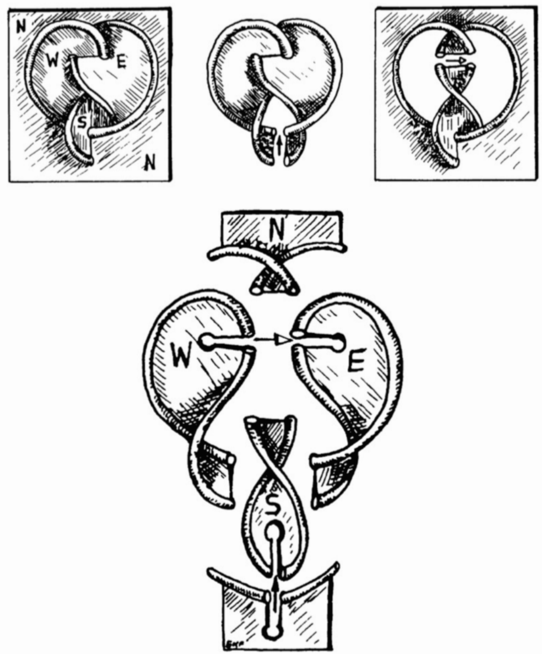
Hyperbolic 3D-space is constructed in a similar way to the 2D version: lines are given by circular arcs.

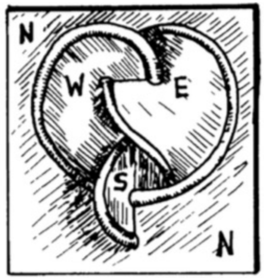
Here we can identify opposite faces to form a closed 3D space.

The complement of the figure eight knot is hyperbolic.



Introduce 2D patches bounding the complement

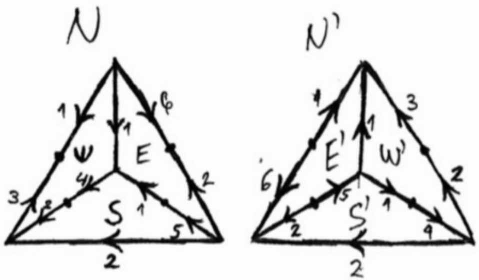


These patches divide 3D-space into two: 

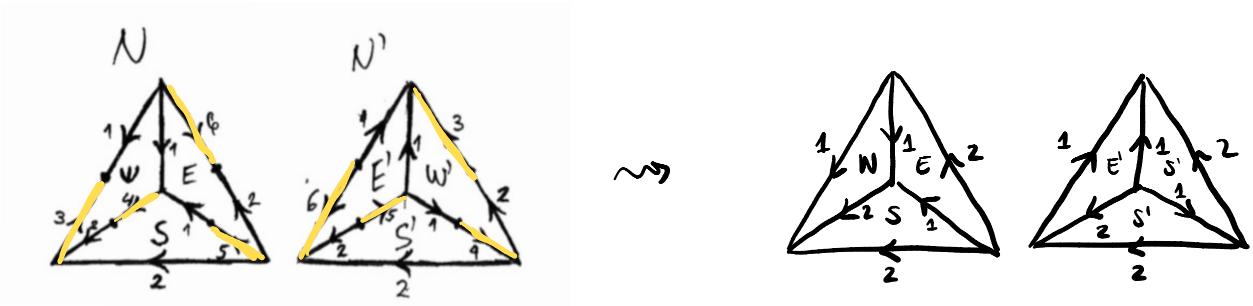
(front and back).

So we can "fill in" 2 tetrahedra (front and back) with faces N, S, E, W
N', S', E', W'

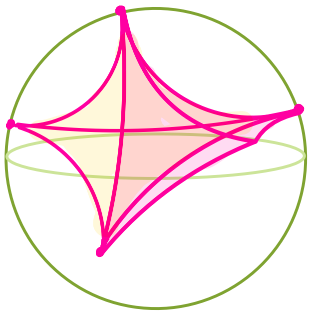
and identifications:



Finally, shrink the edges coming from the knot (3, 4, 5, 6):



We finally obtain the complement of the knot as the "Dirichlet domain with portals":



Facets and edges are identified accordingly.

Question: with these "portals" in place, how would this space look like from the inside?

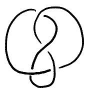
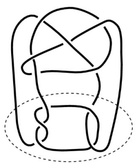
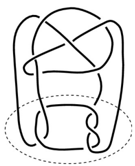
Hyperbolic volume

Just like triangles in $\mathbb{H}^2$ have an area determined by their shape, tetrahedra in $\mathbb{H}^3$ have a volume determined by their shape (but not so straightforward).
The maximum it can be is $-8\int_0^{\frac{\pi}{4}} \log(2\sin t)\,dt \approx 1.5911\ldots$

Since the hyperbolic structure of C_K is unique, its volume <u>only</u> depends on the knot.

$\rightsquigarrow$ can use it to distinguish knots.

Examples:

- $\mathrm{vol}\Big($  $\Big) = -6\int_0^{\frac{\pi}{3}} \log(2\sin t)\,dt \approx 2.02988\ldots$ $\leftarrow$ smallest possible
- $\mathrm{vol}\Big($  $\Big) = 4.40083\ldots$
-  ,  have the same volume

Volume conjecture: $\lim\limits_{N\to\infty} \frac{2\pi \log|\langle K\rangle_N(\sqrt{1})|}{N} = \mathrm{vol}(K)$. Bridge combinatorics $\leftrightarrow$ hyperbolic geometry.