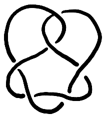
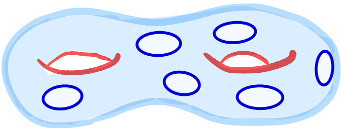
# Reminder:

- Knots and links:



- Surfaces

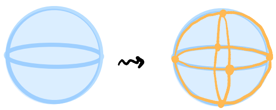


$\rightsquigarrow$ genus $g = 2$

$\rightsquigarrow$ # bdary components $b = 6$

- Euler characteristic:

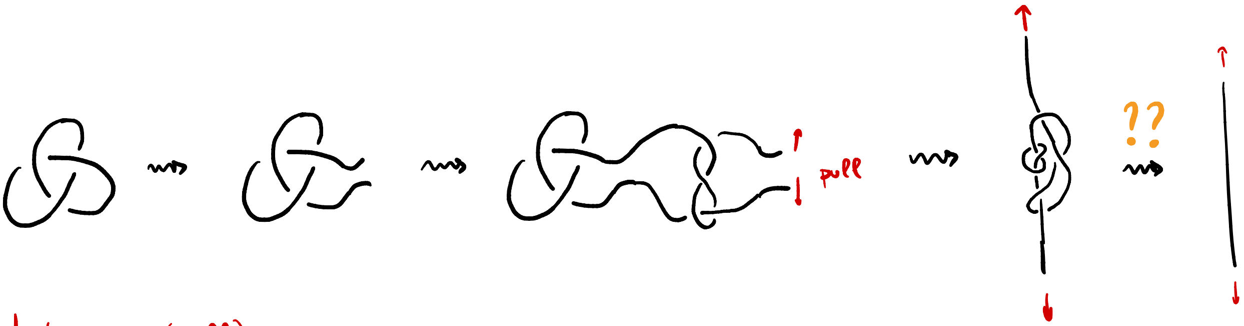
1. Find a triangulation



2. $\chi = V - E + F$

- $\chi(T_{g,b}) = 2 - 2g - b$

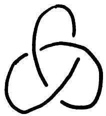
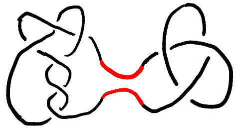
**New Question:** Can you untie a knot by knotting it more?




**Intuition (poll)**

Any ideas?

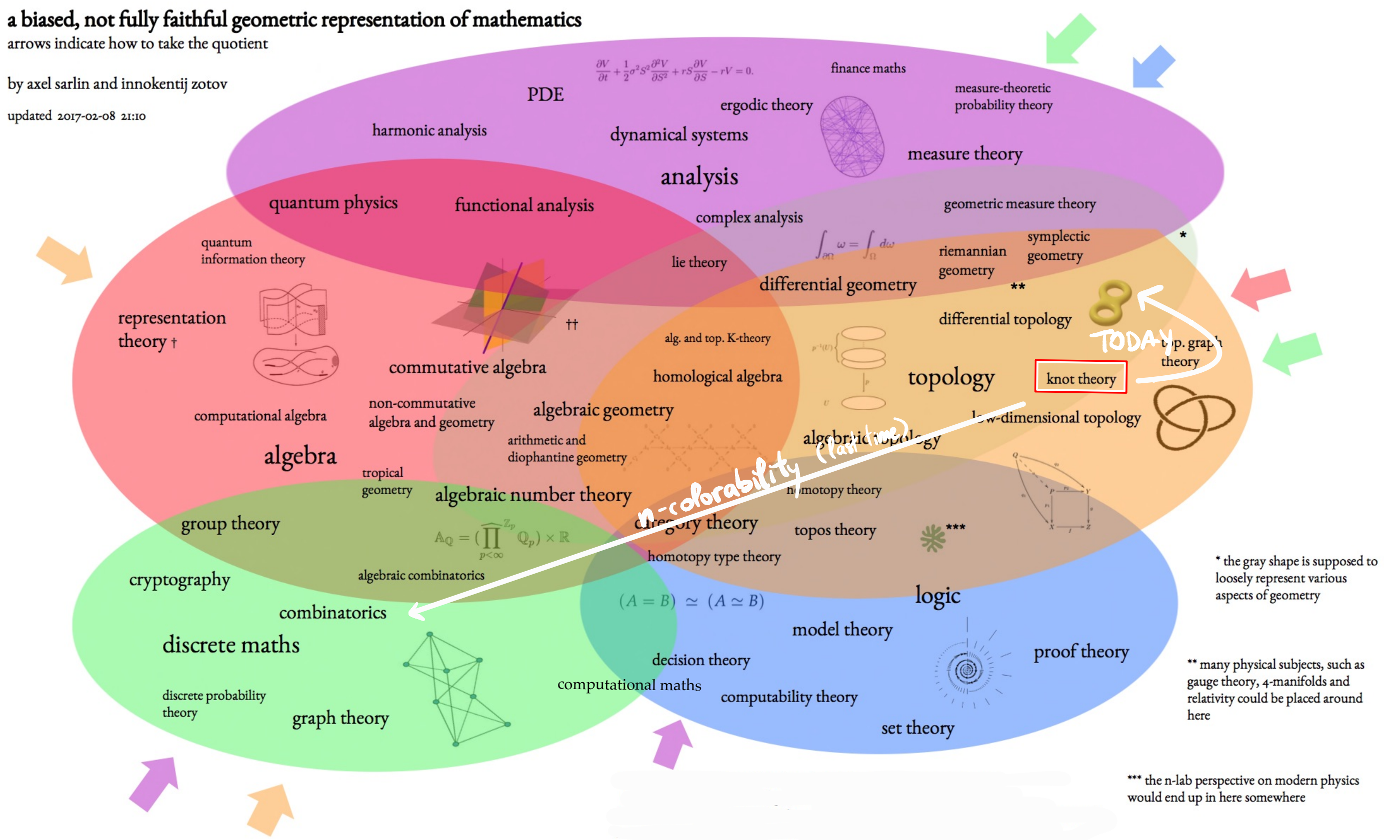
## Mathematical formulation:

Define  #  := 

Are there two (proper) knots $K_1, K_2$ such that

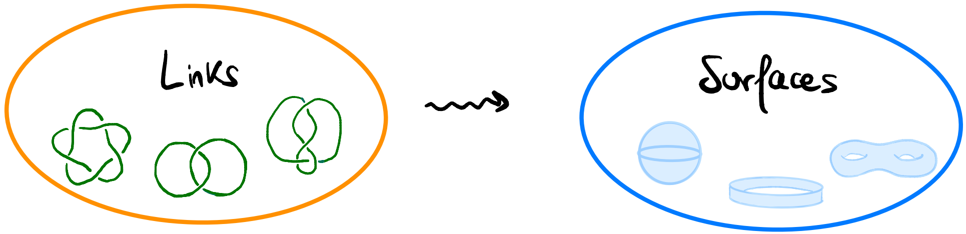
$$K_1 \# K_2 = \text{unknot?}$$

Our tool will come from the topology of surfaces



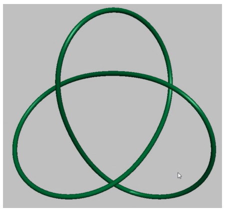
# 7. Seifert surfaces

Back to our goal:

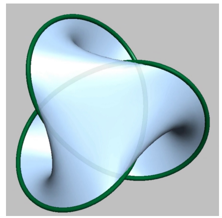



Idea (1930s): [Animation]

**Question:**



$\rightsquigarrow$ ?? $\rightsquigarrow$

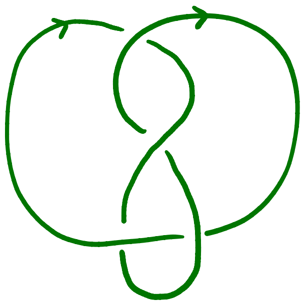


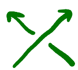
1934, Seifert:

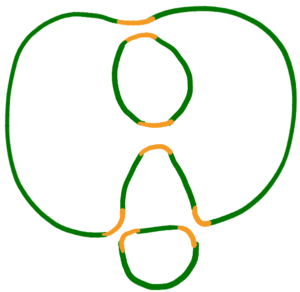
: "Here is an algorithm"

## How does it work?

Step 1: Choose a diagram and orientation for the link:

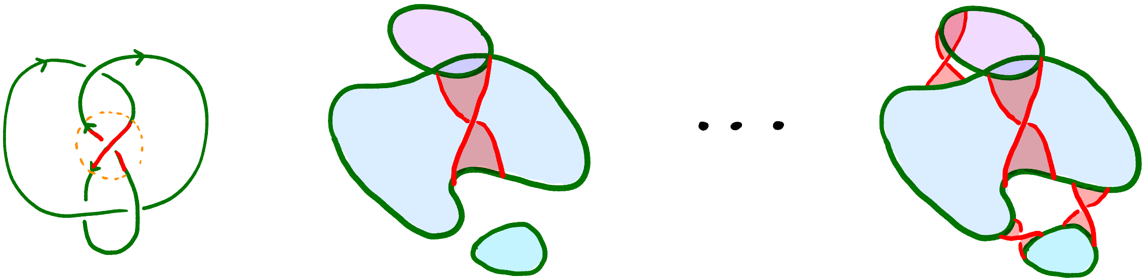


Step 2: smooth out every crossing:  and  become 

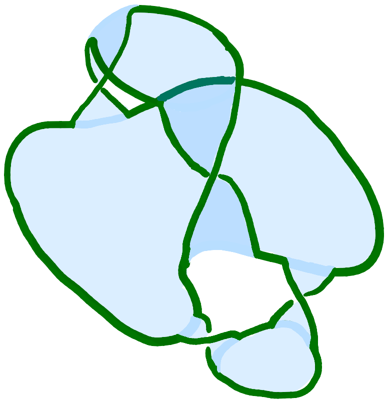


Step 3: make every resultant circle into a disk, and place the disks at different heights

Step 4: place twisted bands where there used to be crossings:



Result:



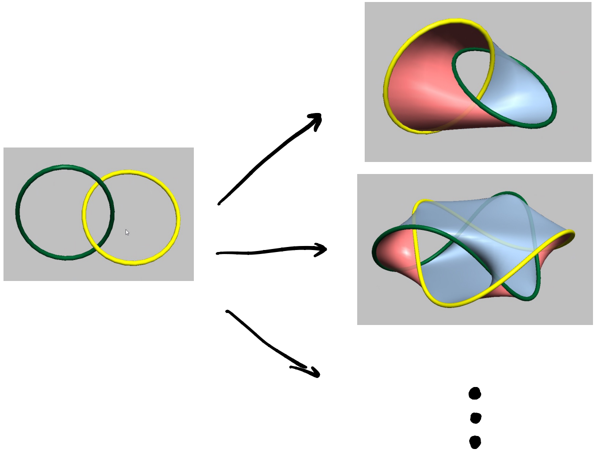
Remark: the result is always orientable.

<u>Definition</u>: a surface whose boundary components form a link L is called a Seifert surface.

Example: the Seifert algorithm gives a Seifert surface

Warning: These are not unique!

The theory of surfaces gives us a new invariant:



genus

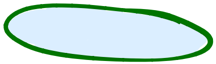
0

1

**Definition**: the genus of a link $g(L)$ is the minimum genus of a Seifert surface for $L$.

[Seifert surfaces software]

## Examples:

- Unknot:  =  = $T_{0,1}$ $\quad$ 0 is obviously minimum.
- Trefoil:  $\rightsquigarrow$  = $T_{1,1}$ $\quad$ so $g(\text{trefoil}) \leq 1$.

**Observation:** If $g(K) = 0$, then $K$ = unknot. Proof: $g(K) = 0$ means that $K$ is the boundary of $T_{0,1}$ = . But  is clearly unknotted.

**Consequence:** $g(\text{trefoil}) = 1$.

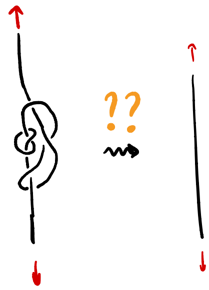
Rmk: Sage gives the genus of a link: `L.genus()`

First task: Euler characteristic of Seifert surfaces.

# 8. Applications: no-untying and knot primality

Recall our original question:

Can I untie a knot by knotting it more?




**Crucial Theorem:** $g(K_1 \# K_2) = g(K_1) + g(K_2)$

**Corollary:** If $K_1$ and $K_2$ are knots but $K_1, K_2 \neq$ unknot, then $K_1 \# K_2 \neq$ unknot.

Proof: $K_1 \neq \bigcirc \Rightarrow g(K_1) \geq 1.$

$K_2 \neq \bigcirc \Rightarrow g(K_2) \geq 1.$

Therefore $g(K_1 \# K_2) \underset{\text{Thm}}{=} g(K_1) + g(K_2) \geq 2 > 0 = g(\bigcirc)$ $\square$

**Definition:** A knot $K$ is prime if whenever $K = K_1 \# K_2$, then either $K_1$ = unknot or $K_2$ = unknot.

Examples:

 $\neq K_1 \# K_2$ prime

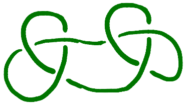
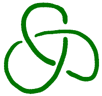
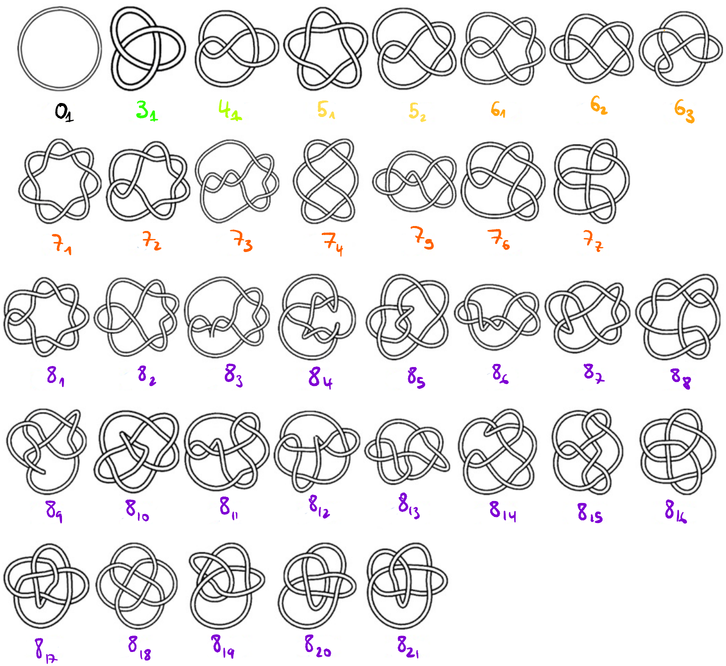
 =  #  not prime

TABLE OF PRIME KNOTS UP TO 8 CROSSINGS




Rmk: Sage has the prime knots stored up to $\sim 14$ crossings.

No one has classified the primes with $\geq 17$ crossings.

**Crucial Theorem:** $g(K_1 \# K_2) = g(K_1) + g(K_2)$

**Corollary:** Every knot can be written as a sum of prime knots.

Proof: Let $K$ be a knot. If $g(K) = 1$, then $K$ is prime (you prove this in the exercises)

This proves the claim for knots of genus $1$.

If $g(K) = 2$, then either $K = K_1 \# K_2$ for $K_1, K_2 \neq$ unknot, in which case $g(K_1) = g(K_2) = 1$ and each of $K_1, K_2$ are prime. Otherwise $K$ itself is prime.

by the theorem

This proves the claim for knots of genus $2$.

If $g(K) = 3$, then either $K = K_1 \# K_2$ for $K_1, K_2 \neq$ unknot, in which case $g(K_1), g(K_2) \leq 2$ and each of $K_1, K_2$ are prime. Otherwise $K$ itself is prime.

... $\square$

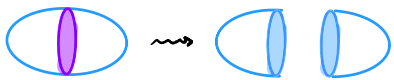
This is called a "proof by induction".

**Remark:** Seifert surfaces can be used to show uniqueness.

# Proof of the crucial theorem

Warning: this is a step harder than what we've done so far, it's supposed to give you a taste!

Definition: Let $S$ be a surface with a disk $D$ bounding a circle in $S$. Then "performing surgery along $D$" is the action of replacing $S$ by a surface $S'$ where we cut the surface $S$ at the circle and cap off the remaining "tubes":



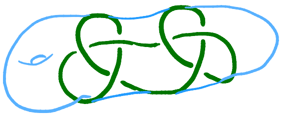
Lemma: If performing surgery on a surface $S$ results in a connected surface $S'$, then $g(S') = g(S) - 1$.

Proof of the lemma: Take a triangulation for $S$ s.t. the disk is one of the triangles. Then after surgery:

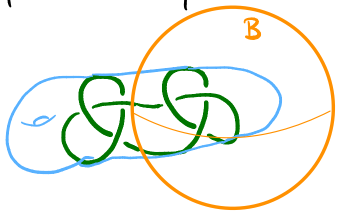
$V \rightsquigarrow V+3$, $E \rightsquigarrow E+3$, $F \rightsquigarrow F+2$. So $\chi \rightsquigarrow \chi + 2$. But $\chi = 2-2g$ so $g \rightsquigarrow g-1$. $\square$

Proof of the theorem: It suffices to show $g(K_1 \# K_2) \geq g(K_1) + g(K_2)$.

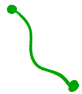
Take a Seifert surface $S$ for $K_1 \# K_2$ of minimal genus:



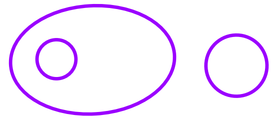
Next, take a sphere that separates the two summed knots:



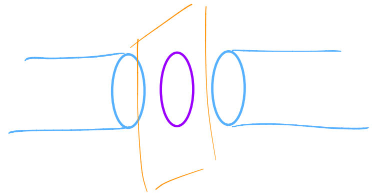

The intersection between B and S consists of:



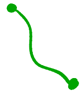
+ some circles

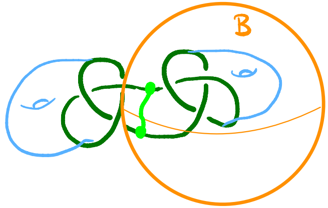


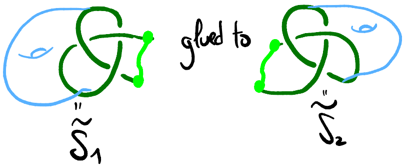
Take an innermost circle. Then perform surgery along it:



The resulting surface is a Seifert surface for $K_1 \# K_2$. If it is connected, then by the Lemma, its genus is lower than $g(S)$, impossible. It follows that it is disconnected and therefore only one of the components bounds $K_1 \# K_2$, and we can remove the other one.

This procedure removes innermost circles one at a time, so eventually we get a surface $\tilde{S}$ whose intersection with $B$ is just  :




Next, notice that $\tilde{S}$ = 


Now $g(K_1 \# K_2) = g(\tilde{S}) = g(\tilde{S}_1) + g(\tilde{S}_2) \geqslant g(K_1) + g(K_2)$ as desired. $\square$

Your second task: • Investigate the genus of knots

• Prove the infinitude of prime knots.